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ON REPRESENTATION OF BANACH-SPACES

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1. BOOLEAN ALGEBRAS OF PROJECTIONS

The purpose of the present paper is to sketch the structural influence which the existence of a spectral-operator has on the underlying Banach-space. We are going to represent the Banach-space as a function space, closed under complex conjugation, and with limitations on its topology. Under more restricted conditions we also obtain a representation as an L^p-space, where the measure is finite.

We give first a couple of definitions:

D e f i n i t i o n 1 . A Boolean algebra (B.A.) of projections $\mathcal{B}$ in a Banach-space X is said to be (σ-) c o m p l e t e if

(1) $\mathcal{B}$ is (σ-) complete as an abstract lattice,

(2) for every family (sequence)

$\{P_\alpha\} \subseteq \mathcal{B}$; $\alpha \in I$ (= some index set) we have:

$$[\overline{P_\alpha(X)}] = (\bigvee_\alpha P)(X)$$

i.e. the least closed manifold containing all $P_\alpha(X)$ is equal to the range of the supremum of all the P 's . For all alternate definition, see ((1)). The characterisation above is given in ((6)) . For general information about spectral-measures and spectral-operators we refer to Dunford ((2)) and ((3)) . Very little of it will be needed here. We point out that if $\mathcal{B}$ is the range of a spectral-measure P , which is the resolution of the identity of some spectral-operator, then $\mathcal{B}$ is σ-complete. It is, however, always possible to consider a σ-complete B.A. of projections as the range of a countably additive spectral-measure P , defined on the field of Baire-sets in the Stone-representation of $\mathcal{B}$.

D e f i n i t i o n 2 . A B.A. of projections $\mathcal{B}$ is of c y c l i c t y p e if there is an element $x_o \in X$, such that the linear manifold spanned by all Px_o ; $P \in \mathcal{B}$, is dense in X. We say then that x_o is a c y c l i c element of X. Our basic tool is the following theorem of Bade ((1)) :

T h e o r e m 1. If $\mathcal{B}$ is a σ-complete B.A. of projections in a Banach-space X, and $x_o \in X$ is an arbitrary element, then there exists a linear functional $x_o^* \in X^*$ so that

(i) $\langle Px_o, x_o^* \rangle \geq 0 \qquad P \in \mathcal{B}$

(ii) $\langle Px_o, x_o^* \rangle = 0 \implies Px_o = 0$

We outline the argument of a new and simplified proof of this theorem, taking time to develop the terminology and concepts needed in the sequel. (For details, see ((6)).) We do this through two lemmas. However, first we note that we may assume, without loss of generality, that $\mathcal{B}$ is of cyclic type. This is a relatively simple consequence of the Hahn-Banach-theorem. Furthermore, we know that we are allowed to consider $\mathcal{B}$ as the range of a countably additive spectral-measure P, defined on some Boolean algebra of sets, Σ. Hence, every element $x^* \in X^*$ determines a scalar measure μ_x on Σ defined by:

$$\mu_{x^*}(\alpha) = \langle P_\alpha x_o, x^* \rangle \quad ; \quad \alpha \in \Sigma .$$

D e f i n i t i o n 3. A linear functional $x^* \in X^*$ is r e a l if the corresponding measure is real. It is p o s i t i v e if the measure is positive, and then we write $x^* \geq 0$.

Let $|\mu|$ denote the total variation of a measure μ on Σ, and let $\mu \gg \nu$ indicate that ν is absolutely continuous with respect to μ.

L e m m a 1 . Every $x^* \in X^*$ has a decomposition

$$x^* = x_1^* - x_2^* + ix_3^* - ix_4^* \quad ; \quad x_i^* \in X^*$$

where each $x_i^* \geqslant 0$, $i = 1, \ldots, 4$ and $\| x_i^* \| \leqslant K \cdot \| x^* \|$ for a positive constant K , independent of x^* . If $\tilde{x}^* = \sum_{i=1}^{4} x_i^*$ we have

$$\tilde{x}^* \geqslant 0 , \mu_{\tilde{x}^*} >> |\mu_{x^*}| \text{ and } \mu_{\tilde{x}^*}(\alpha) \geqslant |\mu_{x^*}|(\alpha) \quad ; \quad \alpha \in \Sigma .$$

For a proof of this, and other unproved statements in the text, we refer to ((6)) . It should be noted that we always can multiply $\tilde{x}^*$ with a scalar without destroying its dominating properties. We will therefore assume that $\tilde{x}^*$ has norm equal to one. The above result now enables us to prove the key lemma needed to obtain Bade's theorem:

L e m m a 2 . For every $\varepsilon > 0$ there is a finite set

$\{x_1^*, \ldots, x_n^*\} \subseteq K$ (= the unit sphere in X^*) and a $\delta > 0$

such that

$$\langle P_\alpha x_0, \tilde{x}_i^* \rangle \quad ; \quad i = 1, 2, \ldots, n \quad ; \quad \alpha \in \Sigma$$

$$\Longrightarrow |\langle P_\alpha x_0, x^* \rangle| < \varepsilon \qquad x^* \in K$$

(The idea of the proof is essentially the same as in ((4)) , ch. IV. 9.2). The proof runs as follows: We assume the statement to be wrong, obtaining thereby the following sequences:

$$\{x_n^*\} \subseteq K \quad ; \quad \{\alpha_n\} \subseteq \Sigma$$

so that

$$\text{(1a)} \qquad \langle P_{\alpha_n} x_o, \tilde{x}_i^* \rangle < \frac{1}{2^n} \quad ; \quad i = 1, 2, \ldots, n$$

$$\text{(1b)} \qquad |\langle P_{\alpha_n} x_o, x_{n+1}^* \rangle| \geq \varepsilon$$

K is compact in the weak X-topology. Thus we can pick out a weakly convergent subsequence, which also is denoted by $\{x_n^*\}$. The $\lim\limits_{n\to\infty} \langle P_\alpha x_o, x_n^* \rangle$ will then exist for all $\alpha \in \Sigma$. If $y^* = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \tilde{x}^*$, then $y^* \in K$. Furthermore, y^* and x_n^* ; $n = 1, 2, \ldots,$ will determine measures μ and μ_n, respectively, and $\mu_n << \mu$ for all n according to the definition of y^*. Applying the Vitali-Hahn-Saks theorem, we obtain that $\mu(\alpha) \longrightarrow 0 \Rightarrow \mu_n(\alpha) \longrightarrow 0$ uniformly in $n = 1, 2, \ldots$. Finally we prove by means of (1a) that $m \longrightarrow \infty \Rightarrow \mu(\alpha_m) \longrightarrow 0$, which thereby contradicts (1b), and hence the lemma is true. Q.e.d.

The rest is now simple. In fact, suppose $\varepsilon > 0$ is given, and let $\{x_1^*, \ldots, x_n^*\}$ be as stated in Lemma 2. We then put

$$x^*(\varepsilon) = \frac{\sum\limits_{k=1}^{n} \tilde{x}_k^*}{\left\| \sum\limits_{k=1}^{n} \tilde{x}_k^* \right\|} \in K$$

The functional x_o^* defined by:

$$x_o^{\star} = \sum_{n=1}^{\infty} \frac{1}{2^n} x^{\star}(\frac{1}{n})$$

will then fulfil the properties (i) & (ii) in Theorem 1.

2. REPRESENTATION OF BANACH-SPACES

We now assume that $\mathcal{B}$ is the range of a countably additive spectral-measure P in X. P can, for instance, be the resolution of the identity of a spectral-operator A in X. We also assume that $\mathcal{B}$ is of cyclic type, so that the operator A will have simple spectrum. This also implies that the scalar measure $\mu(\alpha) = \langle P_{\alpha} x_o, x_o^{\star} \rangle$; $\alpha \in \Sigma$, will dominate the operator-valued measure P, and hence every measure μ_x defined by $\mu_x(\alpha) = \langle P_{\alpha} x, x_o^{\star} \rangle$; $\alpha \in \Sigma$, $x \in X$. This, together with the fact that every $x \in X$ has a continuous decomposition like the functionals, (Lemma 1), makes it possible to prove the following:

L e m m a 3. There is a one to one linear and continuous map of X onto a dense subspace of $L^1_{\mu}(S)$. The map is given explicitly by $x \to \frac{d\mu_x}{d\mu}$.

It is not difficult to see that this lemma also holds true for the dual space $X^{\star}$ when the map is given by $x^{\star} \to \frac{d\mu_{x^{\star}}}{d}$. With these results at hand we now can prove:

T h e o r e m 2. If $\hat{y} \in L^1_{\mu}(S)$ is the element corresponding to $y \in X^{\star}$ then for every $x \in X$:

$$\langle x, y^{\star} \rangle = \int_S \hat{x}(s)\hat{y}(s)d\mu$$

($\hat{x} \in L^1_{\mu}(S)$ is the element corresponding to $x \in X$.) We indicate the

From the last theorem we then get:

$$\langle x, y^* \rangle = \int_S \hat{x} \, d\mu_{y^*} \leq \| \hat{x} \|_\infty \cdot K$$

Hence we have:

$$K' \| \hat{x} \|_1 \leq \| x \| \leq K \cdot \| \hat{x} \|_\infty ; \; K, K' > 0$$

for $\hat{x} \in L^\infty_\mu(S)$. This implies that the topology of X also has an upper bound. The X-topology and the L^∞-topology can, however, not coalesce, because the spectral-measure P then will stop being countably additive.

We conclude this paper by mentioning a couple of results in the direction of the Kakutani-representation of an abstract L-space. Doing this, we restrict ourselves to r e a l Banach-spaces, and note that it is then an easy consequence of Lemma 3 that X becomes a vector-lattice with the order inherited from its real representation space. We now ask when the map $X \longrightarrow L^1_\mu(S)$ is a map not only into, but onto $L^1_\mu(S)$. The following conditions turn out to be sufficient:

$$(A_0) \quad \| x^+ + x^- \| = \| x \| \quad \text{where} \quad x = x^+ - x^- \in X_0 \quad \text{and} \quad x^+, x^- \geq 0$$

$$(L_0) \quad x, y \geq 0 \implies \| x + y \| = \| x \| + \| y \| \quad x, y \in X_0$$

Here X_0 is the class of step-elements in X . The same conditions with X_0 replaced with X are denoted by (A) and (L) , respectively.

T h e o r e m 3 . If X is a real Banach-space where (A_0) and (L_0) hold, then there is a measure μ on Σ so that X is isometrically isomorphic to the real space $L^1_\mu(S)$. X is therefore a vector-lattice where (A) and (L) hold, and the congruence $X \to L^1\ (S)$ is also a lattice-isomorphism.

This representation of X is the same as obtained by Kakutani ((5)) . Compared to his proof ours is rather simple. This is, of course, due to the fact that we a s s u m e the existence of a B.A. of projections in X , while the essential content of Kakutani's proof is the construction of a suitable Boolean algebra. We finally study the consequences of a modified axiom (L_o^p) :

$$(L_o^p)\quad x \wedge y = 0 \implies \|x+y\|^p = \|x\|^p + \|y\|^p \qquad x,y \in X_o$$
$$1 \leq p < \infty$$

The corresponding condition with X_o replaced with X is denoted (L^p) .

T h e o r e m 4 . If X is a real Banach-space where (L_o^p) holds, then there is a measure μ on Σ so that X is isometrically isomorphic with the real space $L^p_\mu(S)$. The congruence is also a lattice-isomorphism and (L^p) holds in X .

References

((1)) W.G. Bade: On Boolean algebras of projections and algebras of operators. Trans. Am. Math. Soc. Vol. 80, 1955.

((2)) N. Dunford: Spectral operators. Pac. Journal of Math. Vol. 4, No. 3, 1954.

((3)) N. Dunford: A survey of the theory of spectral operators. Bull. Am. Math. Soc. Vol. 64, 1958.

((4)) Dunford-Schwartz: Linear operators, part I. Intersc. Publ. Inc., New York, 1958.

((5)) S. Kakutani: Concrete representation of abstract (L)-spaces and the mean ergodic theorem. Ann. of Math. (2), 42, 1941.

((6)) J. Aarnes: "Tellbart additive spektralmål av syklisk type og deres strukturelle implikasjoner i Banach-rum."