ON REPRESENTATION OF BANACH-SPACES

By

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1. BOOLEAN ALGEBRAS OF PROJECTIONS

The purpose of the present paper is to sketch the structural influence which the existence of a spectral-operator has on the underlying Banach-space. We are going to represent the Banach-space as a function space, closed under complex conjugation, and with limitations on its topology. Under more restricted conditions we also obtain a representation as an $L^p$-space, where the measure is finite.

We give first a couple of definitions:

**Definition 1.** A Boolean algebra (B.A.) of projections $\mathcal{B}$ in a Banach-space $X$ is said to be $(\sigma-)$ complete if

1. $\mathcal{B}$ is $(\sigma-)$ complete as an abstract lattice,
2. for every family (sequence)
   \[
   \{P_\lambda\} \subseteq \mathcal{B}; \quad \lambda \in I \quad (= \text{some index set}) \quad \text{we have:}
   \]
   \[
   \overline{\bigcup_{\lambda \in I} P_\lambda(X)} = (\bigvee \lambda)(X)
   \]

i.e. the least closed manifold containing all $P_\lambda(X)$ is equal to the range of the supremum of all the $P$'s. For all alternate definition, see ((1)).

The characterisation above is given in ((6)). For general information about spectral-measures and spectral-operators we refer to Dunford ((2)) and ((3)). Very little of it will be needed here. We point out that if $\mathcal{B}$ is the range of a spectral-measure $P$, which is the resolution of the identity of some spectral-operator, then $\mathcal{B}$ is $\sigma$-complete. It is, however, always possible to consider a $\sigma$-complete B.A. of projections as the range of a countably additive spectral-measure $P$, defined on the field of Baire-sets in the Stone-representation of $\mathcal{B}$.

**Definition 2.** A B.A. of projections $\mathcal{B}$ is of
cyclic type if there is an element \( x_0 \in X \), such that the linear manifold spanned by all \( P x_0 ; P \in \mathcal{B} \), is dense in \( X \). We say then that \( x_0 \) is a cyclic element of \( X \). Our basic tool is the following theorem of Bade ((1)):

**Theorem 1.** If \( \mathcal{B} \) is a \( \sigma \)-complete B.A. of projections in a Banach-space \( X \), and \( x_0 \in X \) is an arbitrary element, then there exists a linear functional \( x_0^* \in X^* \) so that

(i) \( \langle P x_0, x_0^* \rangle \geq 0 \) \( \quad \forall P \in \mathcal{B} \)

(ii) \( \langle P x_0, x_0^* \rangle = 0 \implies P x_0 = 0 \)

We outline the argument of a new and simplified proof of this theorem, taking time to develop the terminology and concepts needed in the sequel. (For details, see ((6)).) We do this through two lemmas. However, first we note that we may assume, without loss of generality, that \( \mathcal{B} \) is of cyclic type. This is a relatively simple consequence of the Hahn-Banach-theorem. Furthermore, we know that we are allowed to consider \( \mathcal{B} \) as the range of a countably additive spectral-measure \( P \), defined on some Boolean algebra of sets, \( \Sigma \). Hence, every element \( x^* \in X^* \) determines a scalar measure \( \mu_{x^*} \) on \( \Sigma \), defined by:

\[
\mu_{x^*}(\alpha) = \langle P x_0, x_0^* \rangle ; \quad \alpha \in \Sigma.
\]

**Definition 3.** A linear functional \( x^* \in X^* \) is real if the corresponding measure is real. It is positive if the measure is positive, and then we write \( x^* \geq 0 \).

Let \( |\mu| \) denote the total variation of a measure \( \mu \) on \( \Sigma \), and let \( \mu \gg \nu \) indicate that \( \nu \) is absolutely continuous with respect to \( \mu \).
Lemma 1. Every \( x^* \in X^* \) has a decomposition

\[
x^* = x_1^* - x_2^* + ix_3^* - ix_4^* ; \quad x_i^* \in K^*
\]

where each \( x_i^* \geq 0 \), \( i = 1, \ldots, 4 \) and \( \|x_1^*\| \leq K\|x^*\| \) for a positive constant \( K \), independent of \( x^* \). If \( \tilde{x}^* = \sum_{i=1}^{4} x_i^* \) we have

\[
\tilde{x}^* \geq 0, \quad \mu_{\tilde{x}}^* > 1/\mu_{x^*}^* \quad \text{and} \quad \mu_{x^*}^*(\alpha) \geq 1/\mu_{x^*}^*(\alpha) ; \quad \alpha \in \Sigma^* .
\]

For a proof of this, and other unproved statements in the text, we refer to ((6)). It should be noted that we always can multiply \( x^* \) with a scalar without destroying its dominating properties. We will therefore assume that \( \tilde{x}^* \) has norm equal to one. The above result now enables us to prove the key lemma needed to obtain Bade's theorem:

Lemma 2. For every \( \varepsilon > 0 \) there is a finite set

\[
\{ x_1^*, \ldots, x_n^* \} \subseteq K \quad (= \text{the unit sphere in } X^* ) \quad \text{and} \quad \delta > 0
\]

such that

\[
\left\langle P_{x^*} x_0, \tilde{x}_i^* \right\rangle \quad ; \quad i = 1, 2, \ldots, n ; \quad \alpha \in \Sigma
\]

\[
\Rightarrow |\left\langle P_{x^*} x_0, x^* \right\rangle| < \varepsilon \quad x^* \in K
\]

(The idea of the proof is essentially the same as in ((4)), ch. IV. 9.2). The proof runs as follows: We assume the statement to be wrong, obtaining thereby the following sequences:
\[ \{ x_n^* \} \subseteq K \; ; \; \{ x_n \} \subseteq \Sigma \]

so that

\[ \langle P_{x_n} x_0, x_i^* \rangle < \frac{1}{2^n} \; ; \; i = 1, 2, \ldots, n \]

\[ \langle P_{x_n} x_0, x_{n+1}^* \rangle \geq \varepsilon \]

\( K \) is compact in the weak \( X \)-topology. Thus we can pick out a weakly convergent subsequence, which also is denoted by \( \{ x_n^* \} \). The limit \( \langle P_{x_0} x_0, x_n^* \rangle \)

will then exist for all \( x \in \Sigma \). If

\[ x^* = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n^* \]

then \( x^* \in K \).

Furthermore, \( y^* \) and \( x_n^* \); \( n = 1, 2, \ldots \), will determine measures \( \mu \) and \( \mu_n \), respectively, and \( \mu_n \ll \mu \) for all \( n \) according to the definition of \( y^* \). Applying the Vitali-Hahn-Saks theorem, we obtain that

\[ \mu(x) \rightarrow \infty \Rightarrow \mu_n(x) \rightarrow 0 \quad \text{uniformly in} \quad n = 1, 2, \ldots \]

Finally, we prove by means of (1a) that \( m \rightarrow \infty \Rightarrow \mu(x_m) \rightarrow 0 \), which thereby contradicts (1b), and hence the lemma is true. Q.e.d.

The rest is now simple. In fact, suppose \( \varepsilon > 0 \) is given, and let

\[ \{ x_1^*, \ldots, x_n^* \} \]

be as stated in Lemma 2. We then put

\[ x^*(\varepsilon) = \frac{\sum_{k=1}^{n} x_k^*}{\| \sum_{k=1}^{n} x_k^* \|} \subseteq K \]

The functional \( x_0^* \) defined by:
\[ x_0^* = \sum_{n=1}^{\infty} \frac{1}{2^n} x^*(\frac{1}{n}) \]

will then fulfill the properties (i) & (ii) in Theorem 1.

2. REPRESENTATION OF BANACH-SPACES

We now assume that \( \mathcal{B} \) is the range of a countably additive spectral-measure \( P \) in \( X \). \( P \) can, for instance, be the resolution of the identity of a spectral-operator \( A \) in \( X \). We also assume that \( \mathcal{B} \) is of cyclic type, so that the operator \( A \) will have simple spectrum. This also implies that the scalar measure \( \mu(\omega) = \langle Px, x_0^* \rangle \); \( x \in \mathcal{B} \), will dominate the operator-valued measure \( P \), and hence every measure \( \mu_x \) defined by \( \mu_x(\omega) = \langle Px, x_0^* \rangle \); \( x \in \mathcal{B} \), \( x \in X \). This, together with the fact that every \( x \in X \) has a continuous decomposition like the functionals, (Lemma 1), makes it possible to prove the following:

Lemma 3. There is a one to one linear and continuous map of \( X \) onto a dense subspace of \( L_1^\mu(S) \). The map is given explicitly by \( x \rightarrow \frac{d\mu_x}{d\mu} \).

It is not difficult to see that this lemma also holds true for the dual space \( X^* \) when the map is given by \( x^* \rightarrow \frac{d\mu_x^*}{d\mu} \). With these results at hand we now can prove:

Theorem 2. If \( \hat{y} \in L_1^\mu(S) \) is the element corresponding to \( y \in X^* \) then for every \( x \in X \):

\[ \langle x, \hat{y}^* \rangle = \int_S \hat{x}(s)\hat{y}(s)d\mu \]

(\( \hat{x} \in L_1^\mu(S) \) is the element corresponding to \( x \in X \).) We indicate the
From the last theorem we then get:

$$\langle x, y^* \rangle = \int \hat{x} d\mu y^* \leq \|x\|_\infty \cdot k$$

Hence we have:

$$k^* \|\hat{x}\|_1 \leq \|x\| \leq k \cdot \|\hat{x}\|_\infty \cdot k, k^* > 0$$

for $\hat{x} \in L^\infty_\mu(S)$. This implies that the topology of $X$ also has an upper bound. The $X$-topology and the $L^\infty$-topology can, however, not coalesce, because the spectral-measure $P$ then will stop being countably additive.

We conclude this paper by mentioning a couple of results in the direction of the Kakutani-representation of abstract $L$-space. Doing this, we restrict ourselves to real Banach-spaces, and note that it is then an easy consequence of Lemma 3 that $X$ becomes a vector-lattice with the order inherited from its real representation space. We now ask when the map $X \rightarrow L^1_\mu(S)$ is a map not only into, but onto $L^1_\mu(S)$. The following conditions turn out to be sufficient:

$$(A_0) \|x^+ + x^-\| = \|x\| \quad \text{where} \quad x = x^+ - x^- \in X_0$$

and $x^+, x^- \geq 0$

$$(L_0) x, y > 0 \implies \|x + y\| = \|x\| + \|y\| \quad x, y \in X_0$$

Here $X_0$ is the class of step-elements in $X$. The same conditions with $X_0$ replaced with $X$ are denoted by $(A)$ and $(L)$, respectively.

**Theorem 3.** If $X$ is a real Banach-space where $(A_0)$ and $(L_0)$ hold, then there is a measure $\mu$ on $\Sigma$ so that $X$ is isometric-
ally isomorphic to the real space $L_{\mu}(S)$. $X$ is therefore a vector-lattice where (A) and (L) hold, and the congruence $X \rightarrow L_{\mu}(S)$ is also a lattice-isomorphism.

This representation of $X$ is the same as obtained by Kakutani ((5)). Compared to his proof ours is rather simple. This is, of course, due to the fact that we assume the existence of a B.A. of projections in $X$, while the essential content of Kakutani's proof is the construction of a suitable Boolean algebra. We finally study the consequences of a modified axiom $(L_{o}^{P})$:

$$(L_{o}^{P}) x \wedge y = 0 \Rightarrow \| x + y \|^{P} = \| x \|^{P} + \| y \|^{P} \quad x, y \in X_{o}$$

$1 \leq p < \infty$

The corresponding condition with $X_{o}$ replaced with $X$ is denoted $(L^{P})$.

**Theorem 4.** If $X$ is a real Banach-space where $(L_{o}^{P})$ holds, then there is a measure $\mu$ on $\Sigma$ so that $X$ is isometrically isomorphic with the real space $L_{\mu_{\mu}}^{P}(S)$. The congruence is also a lattice-isomorphism and $(L^{P})$ holds in $X$. 
References


((6)) J. Aarnes: "Tellbart additive spektralmål av syklisk type og deres strukturelle implikasjoner i Banach-rum."