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ON THE BOHR COMPACTIFICATION

By

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This paper contains some rather easy observations on the Bohr compactification of a topological group and might be considered as a supplement to the work ((1)) . Its excuse is that several of the results, while trivial to prove in the present context, have earlier been labourously verified by methods of functional analysis or by deep-lying structure theorems, sometimes under much too strong conditions. In this connection we strongly want to point out the importance of treating the Bohr compactification as the solution of a particular "universal problem", even if other modes of definition are possible (cf. theorem 2).

The paper is divided in two sections. In 1. we establish a connection between the Bohr compactification of an arbitrary topological group and the dual of its discrete character group together with some consequences of this. We also observe that the operation of forming Bohr compactifications commutes with that of forming quotients and products but not in general with that of taking subgroups, hence neither with the forming of projective limits. In 2. we extend the notion of Bohr compactification to arbitrary topological-algebraical systems and consider in particular the Bohr compactifications of topological rings.

1. By the Bohr compactification of a topological group G is meant an ordered pair $(\rho, \hat{G})$ consisting of a compact group $\hat{G}$ and a continuous representation ρ of G onto a dense subgroup of $\hat{G}$, such that:

Given a continuous representation ξ of G onto a dense subgroup of some compact group H, there exists a (necessarily uniquely determined) continuous representation $\hat{\xi}$ of $\hat{G}$ onto H such that $\xi = \hat{\xi} \circ \rho$.

The definition is conveniently memorized by the following commutative diagram

$$\begin{array}{ccc} \hat{G} & & \\ \rho\uparrow & \searrow\hat{\xi} & \\ G & \xrightarrow{\ \xi\ } & H \end{array}$$

Usually we drop designations of arrows if no confusion arise. Also, by abuse of language, we speak of $\hat{G}$ rather than $(\rho,\hat{G})$ as the Bohr compactification of G. Ordered pairs (ξ,H) defined as above are briefly termed <u>compact representations</u> of G.

It is obvious that the Bohr compactification of a topological group G is determined up to a canonical isomorphism (algebraic and topologic), leaving the images of G invariant, by the preceeding definition. Clearly $(\rho,\hat{G})$ can be realized as the (separated) completion of G with respect to the finest uniform structure $\mathcal{U}$ such that

a) $\mathcal{U}$ is precompact.

b) $\mathcal{U}$ is compatible with the group structure, i.e. the mappings $x \longrightarrow x^{-1}$ and $(x,y) \longrightarrow xy$ are uniformly continuous.

c) $\mathcal{U}$ defines a topology coarser than the initial topology on G.

This uniform structure and its associated topology are called the Bohr structure and the Bohr topology of G, written $\mathcal{U}_B$ and $\mathcal{T}_B$, respectively. Note that the one determines the other ((1, p. 128, prop. 1)). In ((1)) an explicit construction of $\mathcal{T}_B$ in terms of the initial group topology were given. (To prove the existence of $\hat{G}$, however, no construction is required once realized that G is determined by a structure $\mathcal{U}$ as above. Indeed, the family of uniform structures on G satisfying a), b), c) is non-empty since it contains the coarsest uniform structure on G. Hence its supremum is well defined and easily seen to satisfy a), b), c). This structure must then be $\mathcal{U}_B$.)

The Bohr compactification of an abelian group is clearly an abelian group. By means of the commutator subgroup of a group the category of topological groups is retracted onto the category of abelian topological groups. Moreover, this transmission commutes with the operation of taking Bohr compactifications. Indeed, if $[G]$ denotes the commutator of the topological group G then the quotient group $G/[G]$ together with the canonical surjection $G \longrightarrow G/[G]$ is characterized up to canonical isomorphism by the universal factorization property

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & A \\ \downarrow & \nearrow_{[\varphi]} & \\ G/[G] & & \end{array}$$

A being any abelian topological group and φ any continuous representation of G into A. Using this definition and the definition of $\hat{G}$, an ordinary diagram chasing shows that $\widehat{G/[G]}$ and $\hat{G}/[\hat{G}]$ are identical (cf. theorem 7). In particular $[\hat{G}]$ is closed hence compact, since $\widehat{G/[G]}$ is Hausdorff. Hence we have as an amusing application of the definition of Bohr compactifications:

T h e o r e m 1 . The commutator of a compact group is compact.

Consider the commutative diagram

$$(1)\qquad \begin{array}{ccccccccc} & & [G] & \xrightarrow{\hat{i}} & \hat{G} & \longrightarrow & \widehat{G/[G]} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle\rho} & & \uparrow & & \\ 0 & \longrightarrow & [G] & \xrightarrow{i} & G & \longrightarrow & G/[G] & \longrightarrow & 0 \end{array}$$

Clearly $\hat{i}(\widehat{[G]}) = [\hat{G}]$, since $\rho(G)$ dense in $\hat{G}$ implies $\rho(i([G]))$

dense in $[\hat{G}]$ and since $\hat{i}(\widehat{[G]}) = \overline{\rho(i([G]))}$. This shows that the diagram (1) is e x a c t in the horisontal lines. Here and in the following we use the term exact in the t o p o l o g i c as well as in the algebraic sense. The kernel of one map is equal to the image of the preceeding, and all maps are s t r i c t morphisms in the sense of ((3)) .

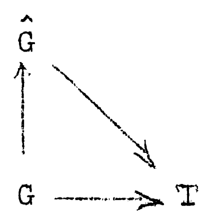
It is natural to conjecture that the map $\widehat{[G]} \longrightarrow \hat{G}$ is one-to-one, i.e. that $\widehat{[G]} = [\hat{G}]$. However, it is not obvious that this should be true. In fact $\widehat{[G]}$ is equal to $[\hat{G}]$ if and only if the Bohr topology on $[G]$ considered as a topological group coalesce with the topology induced from the Bohr topology on G . It is true that the former is always finer than the latter (since $\widehat{[G]} \longrightarrow \hat{G}$ is continuous), so it is enough to establish the converse. In any case we have not been able to prove the result.

Anzai and Kakutani has shown, using structure theory and functional analysis techniques, that if G is a locally compact abelian group, then $\hat{G}$ coalesces with the dual of its discrete dual ((2)) .

In his book ((7)) Rudin takes this theorem as a definition of the Bohr compactification of a locally compact abelian group. As we shall see, however, the characterization is in fact valid for a l l abelian groups. This has also been observed by E. Alfsen. The key point in his proof is that on a compact group the characters separate points. The following proof, however, is by far the shortest one:

Let G be any abelian topological group and ch G its character group (i.e. the group of continuous representations of G into the circle group T), always equipped with the compact-open topology if nothing is said to be contrary. Direct use of the definition gives the following diagram

(2)



showing that the character groups of G and $\hat{G}$ are algebraically isomorphic – or equivalently – isomorphic as d i s c r e t e topological group (in particular every character on G is almost periodic since it can be lifted to $\hat{G}$). But from

$$ch_d G = ch_d \hat{G}$$

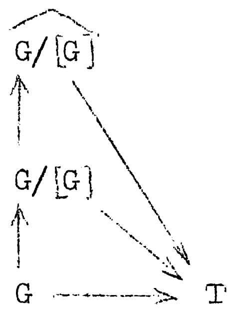
we get

$$ch\ ch_d G = \hat{G}$$

by the duality theorem for discrete/compact abelian groups. Hence the result.

If G is not abelian we may of course factorize the characters via $G/[G]$, exchanging (2) with

(2')



and getting

$$ch\ ch_d G = \widehat{G/[G]}$$

We note this as

T h e o r e m 2 . If G is a topological group, then

$$\hat{G}/[\hat{G}] = \widehat{G/[G]} = ch\ ch_d G \quad .$$

C o r o l l a r y 1 . $\hat{G}/[\hat{G}] \subset T^{ch\ G}$.

From theorem 2 we get immediately the following result due to Hewitt and Zuckerman ((6)) . The proof is modelled after ((7)) .

C o r o l l a r y 2 . Let G be a topological group. For any $\varphi_1, \varphi_2, \ldots, \varphi_n \in \text{ch } G$, for any $\varepsilon > 0$ and for any representation

$$\Phi : \text{ch } G \longrightarrow T$$

there exists an element x of G such that

$$|\hat{x}(\varphi_i) - \Phi(\varphi_i)| < \varepsilon \qquad i = 1, 2, \ldots, n$$

where $\hat{x} : \text{ch } G \longrightarrow T$ is defined by $\hat{x}(\varphi) = \varphi(x)$.

P r o o f . Clearly $\Phi \in \text{ch ch}_d G = \widehat{G/[G]}$. Since the set of all maps Ψ in $\text{ch ch}_d G$ for which $|\Psi(\varphi_i) - \Phi(\varphi_i)| < \varepsilon$ is open, it must contain an element $\hat{x}$ from the (dense) image of G by the map $G \longrightarrow G/[G] \longrightarrow \widehat{G/[G]}$. Obviously the elements $\hat{x}$ thus obtained operates on ch G as defined.

Since the maps $\hat{x}$ is really characters on ch G , we may also formulate corollary 2 by saying that any representation of ch G into T is the [1)]

Recall that a subset A of a group G is *l e f t r e l a t i v e l y d e n s e* if G is covered by a finite number of left translates xA . A is *i n v a r i a n t* if $xAx^{-1} = A$ for every x in G .

T h e o r e m 3 . Every group G admits a finest precompact uniform structure $\mathcal{U}_M$ compatible with the group operations. If $\mathcal{T}_M$ denotes the associated topology, then $\mathcal{U}_M$ is the common left and right structure of $\mathcal{T}_M$. The system of neighbourhoods of the neutral element for the latter consists of the sets V which admit a sequence of subsets (V_n) such that:

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1) pointwise limit of characters on ch G, i.e. of continuous

i) $V_1^2 \subset V$ and $V_{n+1}^2 \subset V_n$ for $n = 1, 2, \ldots$.

ii) Each V_n is a symmetric invariant and left relatively dense subset containing the neutral element.

G admits a maximally almost periodic group topology if and only if $\mathcal{T}_M$ is Hausdorff.

P r o o f . Let G_{pc} denote the group G equipped with some precompact uniform structure (compatible with the group operations) and G_d the group G with the discrete structure. Since every compact representation of G_{pc} can be factorized via its (separated) completion $(i_{pc}, \tilde{G}_{pc})$, we must have $(i_{pc}, \tilde{G}_{pc})$ equal to $(\rho_{pc}, \hat{G}_{pc})$. This means that the given precompact structure on G (= the inverse image by i_{pc} of the structure on $\tilde{G}_{pc}$) is its own Bohr structure (= the inverse image by ρ_{pc} of the structure on $\hat{G}_{pc}$). Now, since the identity mapping $G_d \longrightarrow G_{pc}$ is uniformly continuous, it is also uniformly continuous with respect to the Bohr structures on G_d and G_{pc} . By the first part of the proof this means that the precompact structure $\mathcal{U}$ on G deduced from the discrete one is finer than the initial precompact structure given G . The fact that $\mathcal{U}_M$ is the common left and right uniformity of its associated topology follows from ((1, prop. 1)) . Finally the characterization of $\mathcal{T}_M$ follows from ((1, theorem 1)) . This proves the first part of the theorem. Suppose $\mathcal{T}_M$ is Hausdorff. Then $\rho_d : G_d \longrightarrow \hat{G}_d$ is injective, being the completion imbedding of G with respect to the Hausdorff structure $\mathcal{U}_M$. It follows that the $\mathcal{U}_M$ -uniformly continuous functions (= the almost periodic functions) on the topological group $(G, \mathcal{T}_M)$ separate points. Hence $\mathcal{T}_M$ is maximally almost periodic. Conversely, suppose that G admits a maximally almost periodic group topology, converting G into a topological group G_{map} . Since $\rho_{map} : G_{map} \longrightarrow \hat{G}_{map}$ is injective, the Bohr structure on G induced from $\hat{G}_{map}$ by ρ_{map} is Hausdorff. But then the finer structure $\mathcal{U}_M$, hence also $\mathcal{T}_M$ must be Hausdorff.

We shall term $\mathcal{U}_M$ and $\mathcal{T}_M$ the Maak structure and the Maak topology of G . The reason for this is the following

T h e o r e m 4 . A complex valued function f on G is uniformly continuous with respect to $\mathcal{U}_M$ if and only if it is almost periodic in the sense of Maak, i.e. if and only if each $\varepsilon > 0$ admits a finite covering (A_i) of G such that $|f(xay) - f(xa'y)| < \varepsilon$ for all x,y whenever a,a' belong to a common A_i .

In fact the result follows from ((1, theorem 1 and theorem 2)). Another corollary is the following (cf. ((1, theorem 3))) :

T h e o r e m 5 . A family of complex valued functions on G is uniformly equicontinuous with respect to $\mathcal{U}_M$ if and only if it is uniformly almost periodic in the sense of Maak.

Clearly Maak's theory of almost periodic functions on groups is included in the theory of almost periodic functions on topological groups. However, theorem 3 tells us that the converse statement is also true, i.e. that the "topologic" theory can always be deduced from the "abstract" one. To make the statement precise we introduce the notion of B-equivalence, saying that two group topologies on a group are B-equivalent whenever the Bohr topology defined by the one coalesce with the Bohr topology defined by the other. The notion of B-equivalence on a group G is obviously an equivalence relation in the set of group topologies on G . Each equivalence class contains exactly one precompact topology (i.e. one topology derived from a (uniquely determined) precompact uniform structure on G compatible with the group operations) which is the common Bohr topology of all members of the class. The collection of equivalence classes thus obtained is organized to a complete lattice $\dot{\mathcal{B}}_B(G)$ if it is equipped with an ordering $\dot{\leq}$ such that $\dot{\mathcal{T}} \dot{\leq} \dot{\mathcal{T}}'$ whenever $\mathcal{T}_B \subset \mathcal{T}'_B$. In fact the collection of precompact group topologies on G form a complete lattice with respect to inclusion.

(It is closed under ordinary supremums of arbitrary families $(\mathcal{T}_B^{(i)})$ and the infimum of such a family is simply the supremum of the family of precompact group topologies coarser than all $\mathcal{T}_B^{(i)}$). According to theorem 3, however, this lattice coalesce with the lattice $\mathcal{B}_M(G)$ of subtopologies of $\mathcal{T}_M(G)$ compatible with the group structure. Each such topology determines has a common left and right uniform structure and hence an algebra of complex valued uniformly continuous functions. In particular $\mathcal{T}_M$ determines the algebra $A_M(G)$ of Maak almost periodic functions on G (theorem 4), and every other algebra constructed in this way is a uniformly closed subalgebra of $A_M(G)$. Conversely, every such is the algebra of exactly one precompact group uniformity, namely the "structure initiale" it defines. In fact any precompact uniform structure is uniquely determined by its algebra of complex valued uniformly continuous functions, and the fact that this particular one is compatible with the group operations is a trivial consequence of the Maak almost periodicity. Finally we note that the collection of uniformly closed subalgebras of $A_M(G)$ form a complete lattice $\mathcal{O\!l}_M(G)$ with respect to inclusion.

Let $\mathcal{T}'$, A and $\dot{\mathcal{T}}$ be arbitrary members of $\mathcal{B}_M(G)$, $\mathcal{O\!l}_M(G)$ and $\dot{\mathcal{B}}_B(G)$, respectively, and write $\mathcal{T}_A$ and $\mathcal{T}_B$ for the inverse image topology defined by A and the unique Bohr topology of $\dot{\mathcal{T}}$. With these notations our considerations can be expressed in the following form

T h e o r e m 6 . There exists a canonical lattice isomorphism between $\mathcal{B}_M(G)$, $\mathcal{O\!l}_M(G)$ and $\dot{\mathcal{B}}_B(G)$ such that if $\mathcal{T}'$, A and $\dot{\mathcal{T}}$ are corresponding members, then $\mathcal{T}' = \mathcal{T}_A = \mathcal{T}_B$ and A is the algebra of complex valued functions on G uniformly continuous with respect to $\mathcal{U}_B$ ($= \mathcal{U}_A = \mathcal{U}'$).

A continuous representation $\varphi : G \longrightarrow G'$ of one topological group into another can always be lifted to a continuous representation $\hat{\varphi} : \hat{G} \longrightarrow \widehat{G'}$ giving the commutative diagram

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\widehat{\varphi}} & \widehat{G'} \\ \uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \rho'} \\ G & \xrightarrow{\varphi} & G' \end{array}$$

Obviously the correspondence $G \longrightarrow \hat{G}$, $\varphi \longrightarrow \widehat{\varphi}$ defines a covariant functor retracting the category of topological groups (and continuous representations) onto the full subcategory of compact groups.

T h e o r e m 7 . The functor $\wedge$ is right exact and commutes with the operation of forming products.

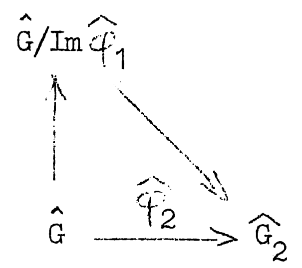
P r o o f : Consider an exact sequence of topological groups

$$0 \longrightarrow G_1 \xrightarrow{\varphi_1} G \xrightarrow{\varphi_2} G_2 \longrightarrow 0$$

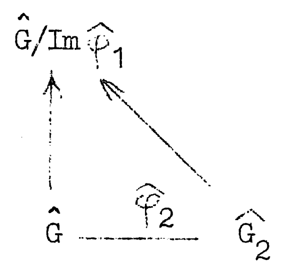
From this sequence we get the commutative diagram

$$\begin{array}{ccccc} & & G/\mathrm{Im}\,\varphi_1 & & \\ & & \uparrow & & \\ \hat{G}_1 & \xrightarrow{\widehat{\varphi}_1} & \hat{G} & \xrightarrow{\widehat{\varphi}_2} & G_2 \\ \uparrow{\scriptstyle \rho_1} & & \uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \rho_2} \\ G_1 & \xrightarrow{\varphi_1} & G & \xrightarrow{\varphi_2} & G_2 \end{array}$$

(Since $\mathrm{Im}\,\varphi_1 = \mathrm{Ker}\,\varphi_2$ is an invariant subgroup of G , $\mathrm{Im}\,\widehat{\varphi}_1$ is by a continuity argument invariant in $\hat{G}$, hence $\hat{G}/\mathrm{Im}\,\widehat{\varphi}_1$ is well defined.) Clearly $\widehat{\varphi}_2$ is surjective and both $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$ are strict morphisms being continuous representations between compact groups. By commutativity $\hat{\varphi}_2$ is constant on $\rho(\mathrm{Ker}\,\varphi_2) = \rho(\mathrm{Im}\,\varphi_1) = \hat{\varphi}_1(\mathrm{Im}\,\rho_1)$, and by continuity it is then constant on $\overline{\hat{\varphi}_1(\mathrm{Im}\,\rho_1)} \supset \hat{\varphi}_1(\overline{\mathrm{Im}\,\rho_1}) = \mathrm{Im}\,\hat{\varphi}_1$. Hence $\hat{\varphi}_2$ give rise to a representation $\hat{G}/\mathrm{Im}\,\hat{\varphi}_1 \longrightarrow \hat{G}_2$ such that



commutes. On the other hand the composite vertical map of (5) is constant on $\mathrm{Im}\,\varphi_1 = \mathrm{Ker}\,\varphi_2$ hence defines a representation $G_2 \longrightarrow \hat{G}/\mathrm{Im}\,\hat{\varphi}_1$, which can be lifted to give a representation $\hat{G}_2 \longrightarrow \hat{G}/\mathrm{Im}\,\hat{\varphi}_1$ making



commutative. Clearly the two representations being surjective are inverse to each other, proving the first part of theorem 7.

Let $(G_i)_{i \in I}$ be a family of topological groups, $\sum' G_i$ its direct sum and $G_j \longrightarrow \sum' G_i$ the canonical imbedding of G_j in $\sum' G_i$. Suppose $\sum' G_i \longrightarrow H$ is a compact representation of $\sum' G_i$. We get commutative diagrams

$$\begin{array}{ccccc} \widehat{G}_j & & & & \\ \uparrow & & & \searrow & \\ G_j & \longrightarrow & \sum' G_i & \longrightarrow & H \end{array} ,$$

hence a compact representation $\sum' \widehat{G}_i \longrightarrow H$. However, $\sum' \widehat{G}_i$ is dense in the direct product $\prod \widehat{G}_i$. Consequently $\sum' \widehat{G}_i \longrightarrow H$ defines a compact representation $\prod \widehat{G}_i \longrightarrow H$ giving the commutative diagram

$$\begin{array}{ccccc} \widehat{G}_j & \longrightarrow & \sum' \widehat{G}_i & \longrightarrow & \prod \widehat{G}_i \\ \uparrow & & \uparrow & & \downarrow \\ G_j & \longrightarrow & \sum' G_i & \longrightarrow & H \end{array}$$

Moreover, the canonical representation $\rho = (\rho_i)_{i \in I}$ of $\sum' G_i$ into $\sum' \widehat{G}_i$ clearly respects the maps of (6) , hence completes (6) to

$$\begin{array}{ccccc} \widehat{G}_j & \longrightarrow & \sum' \widehat{G}_i & \longrightarrow & \prod \widehat{G}_i \\ \uparrow & & \uparrow & & \downarrow \\ G_j & \longrightarrow & \sum' G_i & \longrightarrow & H \end{array}$$

Since $\sum' G_i \longrightarrow \sum' \widehat{G}_i \longrightarrow \prod \widehat{G}_i$ is a compact representation of $\sum' G_i$ and since by (7) every compact representation of $\sum' G_i$ lifts to $\prod \widehat{G}_i$, $\sum' G_i \longrightarrow \prod \widehat{G}_i$ must be the imbedding of $\sum' G_i$ into its Bohr compactification. In particular $\widehat{\sum' G_i} = \prod \widehat{G}_i$. The second part of theorem 7 now follows from the fact that $\sum' G_i$ is dense in $\prod G_i$, and therefore have the same Bohr compactification as $\prod G_i$.

2. Let $\mathbb{C}$ be a category with finite products and $\mathbb{C}_o$ a full subcategory of $\mathbb{C}$, also with finite products. By definition we have: Any object of $\mathbb{C}_o$ is an object of $\mathbb{C}$. Any object of $\mathbb{C}$ isomorphic to an object of $\mathbb{C}_o$ is an object of $\mathbb{C}_o$. If A_o, B_o are two objects of $\mathbb{C}_o$, $\mathrm{Hom}_{\mathbb{C}_o}(A_o,B_o)$ is equal to $\mathrm{Hom}_{\mathbb{C}}(A_o,B_o)$. The composition law of morphisms in $\mathbb{C}_o$ is induced from that in $\mathbb{C}$.

Let A and $L_o, M_o, \ldots, Q_o$ be objects of $\mathbb{C}$ and $\mathbb{C}_o$, respectively. We suppose that to A there is associated a finite sequence of natural numbers $a_1 = 1, a_2, \ldots, a_r$, and to each a_i a morphism $\alpha_i : A^{a_i} \longrightarrow A$. It is convenient to suppose that α_1 is the identity morphism $1_A : A \longrightarrow A$. In the same way L_o is supposed to be endowed with a structure given by a sequence $l_1 = 1, l_2, \ldots, l_s$ and morphisms $\lambda_j : L_o^{l_j} \longrightarrow L_o$, $\lambda_1 = 1_{L_o}$. Finally we are given a finite sequence of pairs of natural numbers $(l'_1,a'_1), (l'_2,a'_2), \ldots, (l'_t,a'_t)$, and to each (l'_j,a'_j) a morphism λ'_j : $L_o^{l'_j} \times A^{a'_j} \longrightarrow A$. Similar requirements are attached to $M_o, N_o, \ldots, Q_o$. The sets $A, L_o, \ldots, Q_o$ and the morphisms $\alpha_1, \ldots, \lambda_1, \ldots \lambda'_1 \ldots$ together with a set of axioms (to be defined below) constitutes an example of a $(\mathbb{C}, \mathbb{C}_o)$-algebra. A is the carrier and $L_o, M_o, \ldots, Q_o$ the operator domains for the particular algebra considered. The sequence $(n_o, a_1, \ldots, l_1, \ldots, (l'_1,a'_1), \ldots)$ where n_o is the number of operator domains, is called its similarity data. By abuse of language we speak of A rather than $(A, L_o, \ldots, a_1, \ldots)$ as the $(\mathbb{C}, \mathbb{C}_o)$-algebra. The morphisms $\alpha_1, \alpha_2, \ldots, \alpha_r$ and $\lambda'_1, \lambda'_2, \ldots, \mu'_1, \ldots$ are called the internal and external operations on A, respectively. Any meaningful composition of these gives an admissible function. Clearly, if A and B are two $(\mathbb{C}, \mathbb{C}_o)$-algebras with the same similarity data then there exists a bijective correspondence between the admissible functions of A and those of B. In fact, the collection of all such algebras form the equivalent of a similarity class or a homology class in the sense of ((5)) . A set of a x i o m s for an algebra A is an ordered set of identities

$$\omega_i = \pi_i \qquad i = 1, 2, \ldots, n$$

where ω_i, π_i are admissible functions for A . From now on we restrict our attention to a fixed but arbitrary class of algebras whose members are subject to the conditions that they belong to one fixed similarity class and share a common set of operator domains and a common set of axioms. In the sequel we reserve the expression $(\mathbb{C}, \mathbb{C}_o)$ -algebra (or just algebra) exclusively for the members of this class, which we write $\mathbb{A}(\mathbb{C}, \mathbb{C}_o)$.

Within $\mathbb{A}(\mathbb{C}, \mathbb{C}_o)$ the notion of a $(\mathbb{C}, \mathbb{C}_o)$ - r e p r e s e n t a t i o n $\xi : A \longrightarrow B$ is well defined. It is a morphism respecting the structure of A and B in the obvious sense. With the representations as morphisms our class of algebras is transformed into a category also written $\mathbb{A}(\mathbb{C}, \mathbb{C}_o)$. In the same way the subcollection of $\mathbb{C}_o$ -algebras (i.e. with $\mathbb{C}_o$ -carriers) and representations form a category $\mathbb{A}(\mathbb{C}_o)$, which is clearly a full subcategory of $\mathbb{A}(\mathbb{C}, \mathbb{C}_o)$.

D e f i n i t i o n . Let $\mathbb{K}$ and $\mathbb{K}_o$ be a category and a full subcategory and let K be an object of $\mathbb{K}$. By the solution for K of the universal problem posed by $(\mathbb{K}, \mathbb{K}_o)$ is meant an ordered pair $(\rho, \hat{K})$ consisting of a $\mathbb{K}_o$ -object $\hat{K}$ and a morphism $\rho : K \longrightarrow \hat{K}$ such that

U.1. For any object K_o of $\mathbb{K}_o$ and any morphism $\xi : K \longrightarrow K_o$ there exists a morphism $\hat{\xi} : \hat{K} \longrightarrow K_o$ uniquely defined by the equation $\xi = \hat{\xi} \circ \rho$.

The requirements in the definition may briefly be refrased by saying that ρ is a uniquely factorizing morphism for K . Clearly ρ (and $\hat{K}$) is determined up to functorial isomorphism by U.1 .

We now return to the categories $\mathbb{A}(\mathbb{C}, \mathbb{C}_o)$ and $\mathbb{A}(\mathbb{C}_o)$.

T h e o r e m 8 . If $\mathbb{C}$ is the category of topological spaces and $\mathbb{C}_o$ the subcategory of compact spaces, then the universal problem posed by $(\mathbb{A}(\mathbb{C},\mathbb{C}_o), \mathbb{A}(\mathbb{C}_o))$ admits a solution for any $(\mathbb{C},\mathbb{C}_o)$-algebra A . In fact if A is any such algebra then the solution can be realized as the separated completion of A with respect to the finest uniform structure $\mathcal{U}$ such that

a) $\mathcal{U}$ is precompact,

b) $\mathcal{U}$ is compatible with the algebra operations on A ,

c) $\mathcal{U}$ defines a topology coarser than the initial topology on A .

P r o o f . The verification runs along the same lines as in the case of topological groups. The structure $\mathcal{U}$ considered in the theorem is constructed as the supremum $\mathcal{U}_B$ of structures on A satisfying a), b), c). The proof that $\mathcal{U}_B$ really satisfies a), b), c) itself is straightforward, and we omit it. Let $(\rho, \hat{A})$ denote the separated completion of A with respect to $\mathcal{U}_B$ and consider any two operations $\alpha_i : A^{a_i} \longrightarrow A$ and $\lambda'_j : L_o^{l'_j} \times A^{a'_j} \longrightarrow A$. According to what has been said these two operations are uniformly continuous with respect to the relevant product structures on their domains of definitions. Consequently they may be extended to the associated completed spaces, i.e. we get operations

$$\hat{\alpha}_i : \hat{A}^{a_i} \longrightarrow \hat{A} \quad ,$$

and

$$\hat{\lambda}'_j : L_o^{l'_j} \times \hat{A}^{a'_j} \longrightarrow \hat{A}$$

(since L_o is compact, hence complete, cf. ((3)) ch. II, § 3, prop. 15 and prop.

18, cor. 2). In the same way the other operations are extended. Obviously $(\hat{A}, L_o, \ldots, Q_o, \hat{\alpha}_1, \ldots, \hat{\lambda}_1, \ldots, \hat{\lambda}'_1, \ldots)$ has the similarity data of the members of $\mathbb{A}(\mathbb{C}, \mathbb{C}_o)$, and its admissible functions are extentions of admissible functions for A . Moreover, if $\omega = \pi$ is any axiom of A then by the principle of extension of identities (cf. ((3, ch. I, p. 87))) $\hat{\omega} = \hat{\pi}$. It follows that $\hat{A}$ is an $\mathbb{A}(\mathbb{C}_o)$ -algebra. Finally, suppose $\xi : A \longrightarrow B$ is a continuous representation (i.e. a $(\mathbb{C}, \mathbb{C}_o)$ -representation) of A into a compact algebra B . Then the inverse image of the uniform structure on B is a uniform structure on A which is obviously precompact and defines a topology on A coarser than the given one. Moreover, direct verification shows that the algebra operations on A are uniformly continuous with respect to the new structure. Hence the inverse image of the uniform structure on B satisfies all three requirements a), b), c). Consequently it is coarser than $\mathcal{U}_B$. It follows that ξ is $\mathcal{U}_B$ -uniformly continuous, and since B is complete ξ may be extended to $\hat{\xi} : \hat{A} \longrightarrow B$ such that $\xi = \hat{\xi} \circ \rho$. Clearly, $(\rho, \hat{A})$ is the solution for A of the universal problem posed by $(\mathbb{A}(\mathbb{C}, \mathbb{C}_o), \mathbb{A}(\mathbb{C}_o))$.

In the rest of this paper we write $\Pi\mathbb{A}$ for $\mathbb{A}(\mathbb{C}, \mathbb{C}_o)$ and $\Pi_c\mathbb{A}$ for $\mathbb{A}(\mathbb{C}_o)$ when $\mathbb{C}$ and $\mathbb{C}_o$ is the category of topological spaces and of compact spaces, respectively. The objects of $\Pi\mathbb{A}$ and $\Pi_c\mathbb{A}$ are termed topological algebras and compact algebras. Besides we shall follow previous practice and speak about compact (but not necessarily dense) representations and Bohr compactifications.

T h e o r e m 9 . Suppose that $\Pi\mathbb{A}$ and $\Pi_c\mathbb{A}$ both admit infinite direct products and $\Pi\mathbb{A}$ infinite direct sums (in the categorial sense), and that every compact representation of a sum can be lifted to a uniquely determined compact representation of the corresponding product. Then the functor $A \rightsquigarrow \hat{A}$ retracting $\Pi\mathbb{A}$ onto $\Pi_c\mathbb{A}$ commute with the operation of forming products.

In fact, with the conditions imposed on $\Pi\mathbb{A}$ and $\Pi_c\mathbb{A}$ the second part of the proof of theorem 7 with nominal changes works also in the general case. In particular theorem 9 applies to topological rings and topological semigroups. Under proper restrictions on $\Pi/\mathbb{A}$ and $\Pi_c/\mathbb{A}$ one should also be able to generalize the first part of theorem 7. We do not attempt to carry out this.

Before we leave the general aspects of theory, we return for a moment to our categories $\mathbb{K}$ and $\mathbb{K}_o$. Consider the functor $K_o \rightsquigarrow \mathrm{Hom}(K,K_o)$ of $\mathbb{K}_o$ into $\mathbb{E}\mathbb{M}\mathbb{S}$, the category of sets and maps, for a given object K of $\mathbb{K}$. This functor is representable in the sense of Grotendieck ((5, p. 8)) if it is isomorphic to any functor $K_o \rightsquigarrow \mathrm{Hom}(K_o',K_o)$, where K_o' is an object of $\mathbb{K}_o$. The representative K_o' is then determined up to a unique isomorphism. It follows that if $(\rho,\hat{K})$ is a solution for K of the universal problem posed by $(\mathbb{K},\mathbb{K}_o)$, then $K_o \rightsquigarrow \mathrm{Hom}(K,K_o)$ is in fact representable with $\hat{K}$ as representative. The concept of representable functor, however, does not seem comprehensive enough to express the morphism $\rho : K \longrightarrow K$, the part of the solution which is of prime interest, nor to express the functor $\wedge$.

Let R be any topological ring and $\mathcal{T}$ the topology of R. According to theorem 9 R admits a Bohr compactification $\rho : R \longrightarrow \hat{R}$. We want to express the Bohr topology $\mathcal{T}_B$ in terms of $\mathcal{T}$.

T h e o r e m 10. Let R be any topological ring and $\mathcal{T}$ the topology of R. The system of neighbourhoods of 0 for the Bohr topology of R consists of those subsets V_o of R which admit a sequence $(V_n)_{n=1,2,\ldots}$ of sets such that

d) $2V_{n+1} \subset V_n$, $RV_{n+1} \subset V_n$ and $V_{n+1}R \subset V_n$ for $n = 1, 2, \ldots$

e) Each V_n, $n = 1, 2, \ldots$, is a symmetric relatively dense neighbourhood of 0.

P r o o f. We first show that the collection $\mathcal{V}$ of subsets V_o satisfying d) and e) is a filter. It is enough to show that $\mathcal{V}$ is closed with respect to finite intersections. Let U_o, V_o be members of $\mathcal{V}$ with corresponding sequences $(U_n)_{n=1,2,\dots}$ and $(V_n)_{n=1,2,\dots}$. Then $U_n \cap V_n$ is relatively dense (with respect to the additive structure of R, of course), cf. 1, prop. 1 and proof of theorem 1. Moreover, $2(U_{n+1} \cap V_{n+1}) \subset 2U_{n+1} \cap 2V_{n+1} \subset U_n \cap V_n$, $R(U_{n+1} \cap V_{n+1}) \subset RU_{n+1} \cap RV_{n+1} \subset U_n \cap V_n$ and $(U_{n+1} \cap V_{n+1})R \subset U_{n+1}R \cap V_{n+1}R \subset U_n \cap V_n$. Hence the sequence $(U_n \cap V_n)_{n=1,2,\dots}$ have the required properties d), e) relatively to $U \cap V$, i.e. $U \cap V \in \mathcal{V}$.

To see that $\mathcal{V}$ defines a ring topology whose associated uniform structure is compatible with the ring operations we notice that for given $V_o \in \mathcal{V}$ there exists, according to d), e), $V_1 \in \mathcal{V}$ such that

$$V_1 - V_1 = 2V_1 \subset V_o$$

Consequently $\mathcal{V}$ is compatible with the underlying additive group structure of R. In a commutative topological group, however, the group operations are in fact uniformly continuous with respect to the associated uniform structure. Hence it remains to prove that multiplication is a uniformly continuous operation in R. Let V_o be an arbitrary member of $\mathcal{V}$. Then there exists a $V_1 \in \mathcal{V}$ such that $V_1R \subset V_o$ and $RV_1 \subset V_o$. Suppose x, x' and y, y' are elements of R such that $x' \in x + V_1$, $y' \in y + V_1$. Then $x'y' \in xy + xV_1 + V_1y + V_1V_1 \subset xy + RV_1 + V_1R + V_1V_1 \subset xy + 3V_o$, which shows that multiplication is uniformly continuous.

We next observe that since the members of $\mathcal{V}$ are relatively dense, the uniform structure $\mathcal{U}$ defined by $\mathcal{V}$ is precompact. It follows that this structure satisfies the requirements a), b), c) of theorem 8.

Finally, let $\mathcal{U}'$ be some uniform structure defined on R satisfying the requirements a), b), c) and let $\mathcal{V}'$ be the corresponding filter of neighbourhoods of 0. Then for any $V_o' \in \mathcal{V}'$ there exists a sequence $(V_n')_{n=1,2,\dots}$ of members of $\mathcal{V}'$ such that d') $$2V'_{n+1} \subset V'_n \qquad n = 0, 1, \ldots$$

e') Each V'_n, $n = 1,2, \ldots,$ is a symmetric relatively dense neighbourhood of 0.

Moreover, since $\mathcal{V}'$ is supposed to define a ring topology we can find $U' \in \mathcal{V}'$ such that $U'^2 \subset V'_1$. The collection $\mathcal{C}$ of $\mathcal{V}'$-members contained in U' forms a base for $\mathcal{V}'$. By the continuity of the multiplication in R with respect to the topology of $\mathcal{V}'$ we can find to each $x \in R$ a $U'_x \in \mathcal{C}$ such that $xU'_x \subset V_1$ and $U'_x x \subset V'_1$ (this is just the continuity at 0 of the functions $y \to xy$ and $y \to yx$ for given x). Because U' as a member of $\mathcal{V}'$ is relatively dense in R there exists a finite number of elements $a_1, a_2, \ldots, a_n$ such that $R = \bigcup_{i=1}^{n} (a_i + U')$. We form $U'_o = \bigcap_{i=1}^{n} U'_{a_i}$. Then $U'_o \in \mathcal{C}$, hence $U'_o R \subset \bigcup_{i=1}^{n} U'_o a_i + U'_o U' \subset 2V'_1 \subset V'_o$. Similarly $RU'_c \subset V'_o$. It follows that R is bounded and hence that the sequence $(V'_n)_{n=1,2,\ldots}$ could have been chosen so as to satisfy the remaining property $RV'_{n+1} \subset V'_n$ and $V'_{n+1}R \subset V'_n$. But then we have $\mathcal{V}' \subset \mathcal{V}$ and therefore $\mathcal{U}' \subset \mathcal{U}$. Consequently $\mathcal{U}$ is the Bohr structure of R. This completes the proof of theorem 10.

We shall say that a topological ring with identity contains <u>arbitrarily small regular elements</u> if each neighbourhood of 0 differing from 0 contains an element which has a (multiplicative) inverse. Trivial examples are provided by function rings containing the constant . As an application of theorem 10 we give the following result:

T h e o r e m 11 . Let R be a topological ring with identity containing arbitrarily small regular elements. Then the Bohr topology of R is trivial (i.e. equal to the coarsest or the finest topology on R.

We shall say that R is <u>minimally almost periodic</u> if the Bohr topology of R is the coarsest possible topology on R. This is clearly equivalent to saying that the $\mathcal{U}_B$ -uniformly continuous functions on R reduce to the constants only, or to saying that $\hat{R}$ reduce to one point. Then we have

C o r o l l a r y 1 . Let R be a topological ring with identity containing arbitrarily small regular elements. Then R is either a finite field with discrete topology or minimally almost periodic.

In particular any topological ring with identity having arbitrarily small regular elements has a finite Bohr compactification.

Since a compact ring is its own Bohr compactification, we also get

C o r o l l a r y 2 . Any compact ring with identity containing arbitrarily small regular elements is a finite field.

In particular any compact ring with identity which algebraically is a division ring, is finite. This is a well known result due to Kaplansky.

The proof of theorem 11 with corollaries runs as follows: If the Bohr topology of R does not separate the identity 1 from 0 , i.e. if every $\mathcal{T}_B$ -neighbourhood of 0 contains the identity 1 , then 1 is mapped onto 0 by $\rho : R \longrightarrow \hat{R}$, hence Im ρ reduces to the 0 element of $\hat{R}$. It follows that $\hat{R} = \{0\}$. Since $\mathcal{T}_B$ is the inverse image by ρ of the topology on $\hat{R}$, $\mathcal{T}_B$ must be the coarsest topology on R , i.e. R is minimally almost periodic.

On the other hand, suppose that there exists a $\mathcal{T}_B$ -neighbourhood V_o of 0 in R such that $1 \notin V_o$. According to theorem 11 we can find a $\mathcal{T}_B$ -neighbourhood V_1 such that $RV_1 \subset V_o$. Suppose $V_1 \neq 0$. Then there would exist regular elements x in V_1 , which would imply $1 = x^{-1}x \in RV_1 \subset V_o$. The contradiction shows that $V_1 = \{0\}$, hence that $\mathcal{T}_B$ is discrete. But then $\mathcal{T}_B = \mathcal{T}$ (the original topology on R). Moreover $\mathcal{U}_B$ is discrete, hence complete. But a complete precompact Hausdorff structure is certainly compact, i.e. defines a compact topology. Hence $\mathcal{T} = \mathcal{T}_B$ is compact and discrete. But then R must be finite.

Moreover, since any discrete ring containing arbitrarily small regular elements is a division ring and since it is known that finite division rings are fields R must be a finite field. This completes the proof.

Finally we remark that for theorem 11 with corollaries to hold it is really enough for the ring R to contain regular elements which are arbitrarily small with respect to the Bohr topology of R .

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