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ALGEBRAIC LOGIC

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The purpose of this paper is to present a new and essentially simplified proof of the representation theorem for polyadic algebras (locally finite and of infinite degree). The paper is divided in two parts. In part I we have included a certain amount of background material on polyadic algebras for the double purpose of first showing the relevance of the concept and second to prepare the reader for the study of part II, where the new proof is presented. It is hoped that sufficient information on algebraic logic is contained in part I and enough details are made explicit in part II to make the reading of that part intelligible.

PART I.

Algebraic logic is the algebraic study of first order theories. Roughly, a first order language is formed from a class of individual variables and a class of predicates by means of the usual connectives of logic. In more detail: Let $\underline{V}$ be the class of variables, $\underline{V} = \{ v_i \mid i \in I \}$, where I is some non-empty set, usually the natural numbers, and let $\underline{P}$ be the class of predicates, $\underline{P} = \{ \pi \}$. With every $\pi \in \underline{P}$ there is associated a natural number $n(\pi)$ giving the rank of the predicate, i.e. the number of argument-places of π. The atomic formulas of our language are

$$\pi(v_{i_1}, v_{i_2}, \ldots, v_{i_{n(\pi)}}) .$$

The class of all formulas is defined inductively as usual by means of the logical connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\exists$ (there exists). If Γ is any class of formulas, we denote by

$$\vdash_\Gamma \alpha$$

the relation that the formula α is deducible from the formulas of Γ as assumption formulas. (We assume that Γ includes a set of axioms for pure first order logic.)

The deducibility relation makes it possible to introduce the following equivalence relation in the class of formulas:

$$\alpha \underset{\Gamma}{\sim} \beta \qquad \text{iff} \qquad \vdash_{\Gamma} (\alpha \longleftrightarrow \beta).$$

Let F_{Γ} denote the set of equivalence classes according to this relation. In an obvious way it may be considered as a Boolean algebra, defining e.g. $[\alpha]'$ as $[\neg\alpha]$, etc.

But our set F_{Γ} can be given additional structure, in fact we may introduce algebraic equivalents of the logical operations of existential quantification and substitution of variables for variables. As the analysis is a little tricky, we omit the details and sketch the results.

Basically a substitution is a Boolean endomorphism of F_{Γ} , e.g. the substitution in a conjunction $\alpha \wedge \beta$ is performed by substituting in α and β separately and taking the conjunction of the resulting formulas. If β is obtained from α by substituting variables v_j for variables v_i , we may associate a map $\tau : I \longrightarrow I$ (I the index set of $\underline{V}$) , such that $\tau(v_i) = v_j$ and $\tau =$ identity on variables not involved in the substitution. Denote the substitution derived from τ by $S(\tau)$. In general we may associate with each $\tau \in I^I$ a substitution $S(\tau)$ which, being a substitution, satisfies the endomorphism formulas:

(S_1) $\qquad S(\tau)(p \wedge q) = S(\tau)p \wedge S(\tau)q$,

(S_2) $\qquad S(\tau)p' = (S(\tau)p)'$, where p,q denotes elements of F_Γ , i.e. equivalence classes of formulas.

Roughly we may introduce a quantifier on F_Γ by defining

$$\exists\lambda\,[\alpha] = [\,\exists v_i \alpha\,].$$

(It may be necessary at some places to rename bound variables, a fact which can easily be taken care of, but which makes our exposition at places somewhat inexact.)

Existential quantifiers commute with each other and it is convenient to introduce them several at a time. Omitting the details, we assert that it is possible to introduce on F_Γ an operator $\exists(J)$ associated with each $J \subset I$ such that $\exists(J)$ has the usual properties of the logical quantifier:

$(\exists_1)$ $\quad \exists(J)0 = 0$,

$(\exists_2)$ $\quad p \leqslant \exists(J)p$,

$(\exists_3)$ $\quad \exists(J)(p \wedge \exists(J)q) = \exists(J)p \wedge \exists(J)q$.

(Here 0 denotes the equivalence class of a logically false formula, e.g. $\alpha \wedge \neg\alpha$; 1 denotes the class of formulas deducible from Γ ; $(\exists_2)$ expresses the usual logical axiom $\alpha(v_i) \longrightarrow \exists v_j \alpha(v_j)$.)

Our resulting algebra F_Γ is <u>locally finite</u> in the sense that each $p = [\alpha]$ depends upon a finite number of variables, which is equivalent to saying that for every $p \in F_\Gamma$ there exists a cofinite set $J \subset I$ such that $\exists(J)p = p$. (J consists of the indices of those variables which do not occur free in α . $I - J$ is then called the support of p and denoted supp(p) .)

To recapitulate: F_Γ is a Boolean algebra with some additional structure. S is a map from transformations τ of I to substitutions on

F_Γ and $\exists$ is a map from subsets $J \subset I$ to quantifiers on F_Γ (a quantifier being defined through $(\exists_1) - (\exists_3)$). The maps S and $\exists$ satisfy certain obvious rules:

(P_1) $\quad \exists(\emptyset)p = p$ and $S(\delta)p = p$ if $\delta =$ identity ,

(P_2) $\quad \exists(J \cup K) = \exists(J)\exists(K)$ and $S(\sigma\tau) = S(\sigma)S(\tau)$.

Here (P_1) is a triviality: if we don't quantify or substitute, we don't. And (P_2) expresses that combined substitutions and quantifications can be performed one after another.

Of the many other properties of S and we put down:

(P_3) $\quad$ If $\sigma = \tau$ on $I - J$, then $S(\sigma)\exists(J) = S(\tau)\exists(J)$.

(P_4) $\quad$ If τ is one-to-one on the subset $\tau^{-1}J$ of I , then

$$\exists(J)S(\tau) = S(\tau)\exists(\tau^{-1}J) .$$

The reason why we put down these properties is that they characterize our algebraic entity F_Γ in the following sense.

A polyadic algebra $(A, I, S, \exists)$ consists of a Boolean algebra A , a non-empty set I and two maps S and $\exists$, where S is a map from transformations $\tau \in I^I$ to Boolean endomorphisms on A (i.e. $S(\tau)$ satisfies (S_1) and (S_2)) and $\exists$ is a map from subsets $J \subset I$ to quantifiers on A (i.e. $\exists(J)$ is an operator on A satisfying $(\exists_1) - (\exists_3)$). The maps S and $\exists$ are required to satisfy conditions $(P_1) - (P_4)$.

If we suppose that our polyadic algebra has infinite degree (i.e. the set I is infinite) and is locally finite (i.e. for every $p \in A$ there is some cofinite set $J \subset I$ such that $\exists(J)p = p$), then our polyadic algebra is isomorphic to some algebra of formulas F_Γ .

In a locally finite algebra of infinite degree one has the following important identity. Let $\sigma J_{*} \tau$ denote the relation that $\sigma i = \tau i$ for all $i \in I - J$, then

$$(1) \qquad S(\tau)\exists(J)p = \vee\{S(\sigma)p \mid \sigma J_{*}\tau\} .$$

The general algebraic theory of polyadic algebras is not too difficult, giving the easy, but important result that each polyadic algebra is semi-simple.

The other chief example of a polyadic algebra is derived from the notion of interpreted language or model. Our formulas are then supposed to say something about a set X and certain relations defined on X. Our variables $\underline{V}$ now denote elements of X and a predicate π is interpreted as a $n(\pi)$-termed relation on X, i.e. as a subset of $X^{n(\pi)}$. It is convenient to consider a predicate π, or more generally, every formula α as a map from $X^n \longrightarrow \underline{\underline{O}}$, where n is the number of free variables occurring in α and $\underline{\underline{O}}$ is the two-elemented Boolean algebra of $0 = \text{false}$ and $1 = \text{true}$. Then $\alpha(x_{i_1}, \ldots, x_{i_n}) = 1$ iff $x_{i_1}, \ldots, x_{i_n}$ are elements of X which satisfy the formula α.

Instead of associating with each α a map from $X^n \longrightarrow \underline{\underline{O}}$, and thus having the inconvenience of considering different cartesian products X^n, we may introduce α as a map

$$\alpha : X^I \longrightarrow \underline{\underline{O}} ,$$

defined in the following way.

Let α be an atomic formula $\pi(v_{i_1}, \ldots, v_{i_{n(\pi)}})$. If $x \in X^I$ we define $\alpha(x) = 1$, iff $x_{i_1}, \ldots, x_{i_{n(\pi)}}$ satisfy π, i.e. $(x_{i_1}, \ldots, x_{i_{n(\pi)}}) \in \pi$, when the latter is considered as a subset of

$X^{n(\pi)}$. The maps associated with the formulas $\alpha \wedge \beta$, $\alpha \vee \beta$ and $\neg \alpha$ are defined by the obvious pointwise operations making the set of maps, F^{φ} , into a Boolean algebra.

Coming to the substitution operator and the quantifier, we proceed in the following manner. Let $\tau_* x$ denote the following element of X^I :

$$(\tau_* x)_i = x_{\tau i} ,$$

we then define the map $S(\tau)\alpha$ in terms of α by

$$(2) \qquad S(\tau)\alpha(x) = \alpha(\tau_* x) .$$

This squares with our intuition. If α is the formula $\alpha(v_{i_1}, \ldots, v_{i_n})$, then $S(\tau)\alpha$ should correspond to the formula $\alpha(v_{\tau i_1}, \ldots, v_{\tau i_n})$. And a sequence $x \in X^I$ ought to satisfy $S(\tau)\alpha$ if it comes from a certain sequence x^{φ} which satisfies α , i.e. such that $(x^{\varphi}_{i_1}, \ldots, x^{\varphi}_{i_n}) \in \alpha$. Hence we must have $x^{\varphi}_{i_1} = x_{\tau i_1}, \ldots, x^{\varphi}_{i_n} = x_{\tau i_n}$ which is true if $x^{\varphi} = \tau_* x$.

To define the quantifier we need the following auxiliary relation: $x J_* y$, iff $x_i = y_i$ for all $i \in I - J$. Then we define $\exists(J)\alpha : X^I \to \underline{\underline{0}}$ by

$$(3) \qquad \exists(J)\alpha(x) = \vee\{\alpha(y) \mid x J_* y\} .$$

If $J = \{i\}$ we are concerned with a formula $\exists v_i \alpha(v_i, v_{j_1}, \ldots, v_{j_n})$, and we have to show that the above definition corresponds to our conception of how the satisfiability of $\exists v_i \alpha$ ought to be defined in terms of the satisfiability of α . If a sequence x shall satisfy $\exists v_i \alpha$, there must be some sequence y such that $y_{j_1} = x_{j_1}, \ldots, y_{j_n} = x_{j_n}$ and such that y satisfies α, i.e. $\alpha(y_i, y_{j_1}, \ldots, y_{j_n}) = 1$. Hence if y is identical with x except at y_i, then $(\exists v_i \alpha)(x) = \bigvee_y \alpha(y)$, and this is just the definition of the map $\exists(J)\alpha$ given above.

It is not difficult to verify that F', a set of maps $\alpha: X^I \longrightarrow \underline{\underline{O}}$ as defined above, satisfies the axioms of polyadic algebras. This particular species will be called functional polyadic algebras or models, the last because they correspond to models of formal languages. It is easy to show that each model is simple (in the algebraic sense). The converse of this assertion is the representation theorem for polyadic algebras (locally finite and of infinite degree).

T h e o r e m . <u>Every locally finite polyadic algebra of infinite degree is isomorphic to a subdirect product of models.</u>

The representation theorem for polyadic algebras has important logical significance - it is, in fact, the algebraic counterpart of the Gödel completeness theorem.

This theorem asserts that any consistent set Δ of formulas has a model. For simplicity assume that we have one formula, $\alpha(v_{i_1}, \ldots, v_{i_n})$. The consistency assumptions signifies that $p = [\alpha(v_{i_1}, \ldots, v_{i_n})] \neq 0$ in the polyadic algebra F_Γ obtained from the provability relation in first order logic (i.e. Γ contains only the usual axioms for first order logic). As polyadic algebras are semi-simple, $p \neq 0$ implies that there exists a maximal polyadic ideal M in F_Γ such that $p \notin M$. Then p does not map onto the 0 of F_Γ/M. As every simple polyadic algebra is isomorphic to a model, we have a further map $p \longrightarrow p'$ of p into some model F' over a set X such that $p' \neq 0$ in F', i.e. there exists a sequence $x \in X^I$ such that $p'(x) = 1$. But this is the same to say that the elements $x_{i_1}, \ldots, x_{i_n} \in X$ satisfy $\alpha(v_1, \ldots, v_n)$ in the model

F' . Thus we have obtained Gödel's theorem.

PART II.

In this part we are going to sketch a new proof of the representation theorem. Whatever is needed from the elementary theory of polyadic algebras which is not contained in part I or explained in part II, will be found in Halmos, Algebraic logic II, Fund. Math. 43 (1956), p. 255-325.

The proof is divided in four parts. First we give a construction of free polyadic algebras using well-known techniques of general algebra. In the next part we study local representations, and in the third section we transcribe a part of the theory of ultraproducts as a method of building a general representation from local ones. In the last part we obtain the full representation theorem.

1. Free algebras.

Let X and I be non-empty sets and let j be a map from X to finite subsets of I. Then a locally finite polyadic algebra $(F,I,S,\exists)$ is called *free* on (X,j) if for every $\varphi : X \longrightarrow B$ where B is a locally finite algebra with index set I and $\operatorname{supp}(\varphi(x)) \subset j(x)$, there exists a polyadic homomorphism $f : F \longrightarrow B$ such that $f \circ i = \varphi$, where $i : X \longrightarrow F$ is a fixed injection.

Free polyadic algebra exist. The proof is almost standard. Let W be a set of cardinality equal to the polyadic "words" on X , consider all pairs $\lambda = (A_\lambda, \varphi_\lambda)$ where $A_\lambda \subset W$ and $\operatorname{supp}(\varphi_\lambda(x)) \subset j(x)$ for all $x \in X$. Take the cartesian product of all A_λ and consider the subset F_0 of all families (a_λ) where $\operatorname{supp}(a_\lambda)$ is contained in some fixed finite subset J of I for all λ . Under the pointwise operations F_0 is easily shown to be a polyadic algebra which is locally finite. Take for F the subalgebra generated by the elements $i(x) = (\varphi_\lambda(x))$.

It is now immediate that F with the injection i is a free algebra on (X,j) and that every locally finite polyadic algebra with index set I is a homomorphic image of some free algebra F .

2. Local representation.

Let A be any I-algebra (i.e. A is an algebra $(A,I,S,\exists)$). Then A is the homomorphic image of a free algebra F . The set $M = \{p \in F;\ f(p) = 1\}$ where $f : F \longrightarrow A$, is a polyadic filter in F . Let $p_o \in M$, we want to construct a model A_o and a homomorphism $f_o : F \longrightarrow A_o$ such that $f_o(p_o) \neq 0$.

A preliminary reduction is necessary. Denote by J the support of p_o and let I_o be an infinite countable subset of I such that $J \subset I_o \subset I$. Define $F_o = \{p;\ \exists(I - I_o)p = p\}$, then F_o can be considered as an I_o-algebra. Further considering F_o as a subset of F , it is easily seen that F_o generates F as an I-algebra. This is so because for any $x \in X$ there is a one-to-one τ such that $S(\tau)x \in F_o$. Further one may verify that the construction of a homomorphism f'_o of F_o into a model A'_o will give a homomorphism f_o of F into a model A_o , where A_o is "generated" by the A'_o in the same way as F_o generates F .

A'_o will be a subset of the set of all functions from $I_o^{I_o}$ into $\underline{\underline{O}}$. As I_o is denumerable there exists in F_o an ultra-filter M_o (in the Boolean sense) such that $p_o \in M_o$ and such that if $S(\tau)\exists(J)p \in M_o$, there exists a transformation σ such that $\sigma\, J_* \tau$ and $S(\sigma)p \in M_o$. (M_o is an ultra-filter preserving countable unions.) Denote by $\Phi : F_o \longrightarrow \underline{\underline{O}}$ the map defined by $\Phi(p) = 1$ if and only if $p \in M_o$. Then A'_o consists of all functions $f'_o(p)$ from $I_o^{I_o} \longrightarrow \underline{\underline{O}}$ given by

$$(4) \qquad f'_o(p)(\tau) = \Phi(S(\tau)p) \ .$$

We do not prove in detail that f'_o is a homomorphism, but show as an example that f_o' commutes with $\exists$:

$$f_o'(\exists(J)p)(\tau) = \Phi(S(\tau)\exists(J)p) = \Phi(V\{S(\sigma)p \; ; \; \sigma J_* \tau\}) =$$

$$= V\{\Phi(S(\sigma)p) \; ; \; \sigma J_* \tau\} = \bigvee\{f_o'(p)(\sigma) \; ; \; \sigma J_* \tau\} = \exists(J)f_o'(p)(\tau) .$$

Here the first equality follows from definition (4) , the second from (1) , the middle equality follows from the special property of M_o , the next one from (4) , and the last one from definition (3) . The calculation $f_o'(p_o)(\delta) = \Phi(S(\delta)p_o) = \Phi(p_o) = 1$ (because $p_o \in M_o$), shows that $f_o'(p_o) \neq 0$.

3. Prime extensions.

Let F be a free polyadic algebra and let $\{A_N \; ; \; N \in \underline{\underline{N}}\}$ be a family of $\underline{\underline{O}}$-valued algebras. Assume that we have given for each $N \in \underline{\underline{N}}$ a homomorphism $f_N : F \longrightarrow A_N$. Let X be the product set $\prod X_N$. On $\underline{\underline{N}}$ choose an ultrafilter D and define an equivalence relation in X by $x \sim y$ iff $\{N \; ; \; x_N = y_N\} \in D$. Denote the equivalence class corresponding to x by x/D and let X_D be the set of equivalence classes. If $x \in X^I$ we define $x_D = (x_i/D) \in X_D$ and $x(N) = (pr_N x_i) \in X_N^I$.

These preliminary definitions are used to construct a map $f : F \longrightarrow A_{\underline{\underline{N}}}$, where $A_{\underline{\underline{N}}}$ is a set of functions from X_D^I to $\underline{\underline{O}}$. Let Φ denote the map from subsets of $\underline{\underline{N}}$ to $\underline{\underline{O}}$ taking the value 1 on the sets belonging to the ultrafilter D , then the definition of f reads

$$f(p)(x_D) = \Phi(\{N \in \underline{\underline{N}} \; ; \; f_N(p)(x(N)) = 1\}) . \tag{5}$$

Let us as an example show that $f(S(\tau)p) = S(\tau)f(p)$, which in particular gives that $A_{\underline{\underline{N}}}$ is closed under $S(\tau)$:

$$f(S(\tau)p)(x_D) = \Phi(\{N ; f_N(S(\tau)p)(x(N)) = 1\}) =$$

$$= \Phi(\{N ; f_N(p)(\tau_* x(N)) = 1\}) = f(p)(\tau_* x_D) =$$

$$= S(\tau)f(p)(x_D) .$$

Using the fact that each f_N is a polyadic homomorphism and that D is an ultrafilter, the rest of the proof is very similar to a proof of Kochen.

4. General representation.

Let A be a non-trivial simple I -algebra, and let F be a free algebra and f_A a homomorphism of F onto A . Denote by M the set of elements p such that $f_A(p) = 1$, and let $\underline{\underline{N}}$ be the class of finite subsets of M . By local representation we then have maps $f_N : F \longrightarrow A_N$, $N \in \underline{\underline{N}}$, such that if $p \in N$ then $f_N(p) \neq 0$. (This is not exactly what was proved in section 2, but follows immediately.)

Define the following ultrafilter on $\underline{\underline{N}}$. Let $F_p = \{ N \in \underline{\underline{N}} ; p \in N \}$, it is then easy to show that $\{ F_p \}$ has the finite intersection property, hence there is some ultrafilter D such that $F_p \in D$. Construct the $\underline{\underline{O}}$ - valued algebra $A_{\underline{\underline{N}}}$ as in section 3 and let f_o be the map there constructed of F onto $A_{\underline{\underline{N}}}$. We are going to show that $A \cong A_{\underline{\underline{N}}}$, the isomorphism map f will be defined as follows: For any p A there exists a $p_1 \in F$ such that $f_A(p_1) = p$, set

$$f(p) = f_{\underline{\underline{N}}}(p_1) .$$

The first thing to verify is that f is uniquely defined.

Thus let p_1, p_2 be elements of F such that $f_A(p_1) = f_A(p_2)$. One has to show that $f(p_1) = f(p_2)$, which is the same to show that $f(p_o) = 1$ where $p_o = (p_1' \vee p_2) \wedge (p_1 \vee p_2')$. Now $f_A(p_o) = 1$, hence $p_o \in M$. It is then easily seen that there exists a $q \in M$ such that $\mathrm{supp}(q) = \emptyset$ and $q \leq p_o$. Hence it is sufficient to show that $f(q) = 1$.

But $q \in M$, thus there exists a $N \in \underline{\underline{N}}$ and a $x(N)$ such that $f_N(q)(x(N)) = 1$, Using the fact that $\mathrm{supp}(q) = \emptyset$, we obtain $F_q \subseteq \{ N ; f_N(q)(x(N)) = 1 \}$. But $F_q \in D$ and thus $f_{\underline{\underline{N}}}(q)(x_D) = 1$, which entails, using the fact once more that $\mathrm{supp}(q) = \emptyset$, that $f_{\underline{\underline{N}}}(q) = 1$.

Using the fact that A is simple, one easily shows that f is injective.