Matematisk Seminar
Universitetet i Oslo

Nr. 4
Februar 1963

EXTENSION OF POSITIVE LINEAR FUNCTIONALS DEFINED ON COFINAL SUBSPACES

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It has since long been known that a positive linear functional defined on a cofinal linear subspace of a vector space admits a positive, algebraic extension (see for instance the book by Shohat and Tamarkin ((6))). The problem to find when such an extension is continuous poses itself. This problem is a special case of a more general extension problem which we have treated in ((3)) . However, the present case where the subspace is cofinal exhibits so many properties not valid in general that a special inquiry seems to be justified.

N o t a t i o n . E denotes a locally convex topological vector space over the real numbers R , P a convex cone in E , F a linear subspace of E , and $f \neq 0$ a positive linear functional on F , that is $f(p) \geqq 0$ for any $p \in P \cap F$. F is called c o f i n a l if for any $x \in E$ there exists $y \in F$ such that $y - x \in P$, or otherwise stated, if $E = F + P$. Usually F is assumed cofinal but sometimes we shall only require that $F + P$ is a linear subspace. The subspace generated by an element x in E is denoted $[x]$.

1. SOME GENERAL RESULTS

Since we assume that $f \neq 0$ and that f is a positive linear functional, it follows that $f^{-1}(0) + P$ is a convex cone different from E . More precisely, we state

L e m m a 1 . <u>Assume that $E = F + P$. Then if $f(e) > 0$ for some $e \in P \cap F$, it follows that e is an order unit of $f^{-1}(0) + P$. On the other hand, if $f = 0$ on $P \cap F$, then $f^{-1}(0) + P$ is a hyperplane in E .</u>

P r o o f . For any $a \in F \setminus f^{-1}(0)$, we have $E = F + P = f^{-1}(0) + P + [a]$. In particular, if $e \in \{P \cap F\} \setminus f^{-1}(0)$, then for any x in E ,

$\lambda e - x \in f^{-1}(0) + P$ for some real λ. Since $e \in f^{-1}(0) + P$, the same assertion is true with $|\lambda|$ instead of λ. This proves the first statement. To prove the second one, it suffices to show that $f^{-1}(0) + P$ is a linear subspace. Let $z = y + p$ be given, where $y \in f^{-1}(0)$, $p \in P$. Since $-z \in E = F + P$, we have $-z = y_1 + p_1$, with $y_1 \in F$, $p_1 \in P$. Therefore $0 = y - z + p = y + y_1 + p_1 + p$, and so $p_1 + p \in P \cap F$. Consequently $0 = f(y) + f(y_1) + f(p_1 + p) = f(y_1)$. Hence $-z \in f^{-1}(0) + P$, and therefore $f^{-1}(0) + P$ is a linear space.

P r o p o s i t i o n 1. <u>Assume that $E = F + P$. Then f admits a positive and continuous extension $\bar{f}$ to E if and only if $f^{-1}(0) + P$ is non-dense in E.</u>

P r o o f. If $\bar{f}$ exists, then $f^{-1}(0) + P$ is contained in the closed halfspace $\{x : \bar{f}(x) \geq 0\}$, and is therefore non-dense. Assume conversely that $f^{-1}(0) + P$ is non-dense. By a basic separation theorem, there exists a continuous linear functional $g \neq 0$ such that $g \geq 0$ on $f^{-1}(0) + P$. If we can find an $e \in P \cap F$ such that $f(e) > 0$, then it follows from Lemma 1 that $g(e) > 0$. Put in this case $\bar{f} = f(e)/g(e) \cdot g$. If $f = 0$ on $P \cap F$, we conclude, again using Lemma 1, that $g^{-1}(0) = f^{-1}(0) + P$. Choose $a \in F \setminus f^{-1}(0)$. Then $g(a) \neq 0$, and we define $\bar{f} = f(a)/g(a) \cdot g$. In either cases $\bar{f}$ is an extension of f of the desired kind.

It is easy to show that if P admits an interior point, then $F + P = E$ if and only if F contains an interior point of P. Hence the following corollary includes a result of Krein ((1, p. 75)).

C o r o l l a r y 1. <u>If $E = F + P$, and $f^{-1}(0) + P$ has a non-empty interior, then f admits a positive and continuous extension.</u>

P r o o f. Let e be an interior point of $f^{-1}(0) + P$. Then $-e \notin f^{-1}(0) + P$, because $f^{-1}(0) + P \neq E$. But neither can $-e$ belong to the closure of $f^{-1}(0) + P$, because if so, it would follow ((1, p. 54)) that $-\frac{1}{2}e$ was an interior point of P and therefore $-e \in P$.

The next corollary is just the statement referred to in the introduction.

C o r o l l a r y 2. If $E = F + P$, then f admits a positive, algebraic extension to E.

P r o o f. It follows from Lemma 1 that in the finest locally convex topology on E, $f^{-1}(0) + P$ is either a closed hyperplane or admits an interior point.

2. USE OF AN OPEN MAPPING THEOREM

From now on E is assumed to be metrizable and complete.

L e m m a 2. Let S be a closed convex cone in E, L a topological vectorspace, and let $u : S \to L$ be additive and continuous. Assume further that $\overline{u(S \cap U)}$ is a zero-neighbourhood in L for each member U of a fundamental system for the zero-neighbourhoods in E. Then $u(S \cap U)$ is a zero-neighbourhood for each U.

P r o o f. The argument given in Grothendieck's book ((2, p. 69)) applies with only minor modifications.

C o r o l l a r y. Assume that A and B are two closed convex cones in E such that $E = A - B$. Then $A \cap V - B \cap V$ is a zero-neighbourhood whenever V is a zero-neighbourhood.

P r o o f. $A \times B$ is a closed convex cone in $E \times E$. Define $u : A \times B \to E$ by $(a,b) \to a - b$. Then

$$u([A \times B] \cap [V \times V]) = A \cap V - B \cap V .$$

We call this set K . Then $K \cap -K$ is convex and symmetric. Since $E = A - B = B - A$, it is also easily seen that $K \cap -K$ is absorbing. Hence $\overline{K \cap -K}$ is a barrel, and therefore a zero-neighbourhood.

R e m a r k : In case $A = B$, the above corollary is due to Klee ((4)) . He applied in the proof another kind of an open mapping theorem than our Lemma 2.

P r o p o s i t i o n 2 . Assume that F and P are closed, that $E = F + P$, and that f is continuous. Then f admits a positive and continuous extension to E .

P r o o f . Let $\overline{f}$ be a positive, algebraic extension of f . Let $\varepsilon > 0$ be given, and choose a neighbourhood V such that $|f(y)| \leq \varepsilon$ whenever $y \in F \cap V$. Using the positivity of $\overline{f}$ we find that if $x, -x \in F \cap V - P \cap V$, then $|\overline{f}(x)| \leq \varepsilon$. In virtue of the corollary of Lemma 2, we can conclude that $\overline{f}$ is continuous.

Our next aim is to show that the proposition above can be extended to the case where we only assume that $F + P$ is a subspace of finite codimension. We need the following result, which has an interest on its own.

P r o p o s i t i o n 3 . If F and P are closed and $F + P$ is a linear subspace of finite codimension in E , then $F + P$ is closed.

P r o o f . First we assume that $F + P$ is a hyperplane in E , say $E = F + P + [\, a \,]$. Let $M = F + [\, a \,]$. Then M is a closed subspace of E . Define g on M by $g(a) = 1$, $g = 0$ on F . g is continuous, since F is closed. We have that $g(p) = 0$ whenever $p \in P \cap M$. Because let $p = \lambda a + y$, with $y \in F$. Then $\lambda a = p - y \in (P + F) \cap [\, a \,]$, and therefore $\lambda = 0$, and this means that $g(p) = 0$. According to Proposition 2, g admits a positive and continuous extension to E . Hence, by Proposition 1, $g^{-1}(0) + P = F + P$ is non-dense in E , and being a hyperplane $F + P$ has to be closed.

Assume now that codim. $(F + P) > 1$. If $F + P$ is not closed, then we can find an $a_1 \in \overline{F + P} \setminus F + P$. We shall show that this entails a contradiction. Put $F_o = F + P$, and let $F_1 = [a_1] + F_o$. Then F_1 cannot be closed, because otherwise the hyperplane $F_o = F + P$ in F_1 had to be closed by the first part of the proof. Now $[a_1] \cap P = \{0\}$, and therefore $P_1 = [a_1] + P$ is closed ((1, p. 78)) . We also notice that $F_1 = F + P_1$. Assume that we have succeeded in the construction of elements $a_1, \ldots, a_k$, closed cones $P_1, \ldots, P_k$ and non-closed subspaces $F_o, F_1, \ldots, F_k$ such that

$$a_i \in \overline{F}_{i-1} \setminus F_{i-1} \quad ; \quad F_i = [a_i] + F_{i-1} \quad ; \quad F_i = F + P_i \quad ; \quad i = 1, \ldots, k .$$

Choose $a_{k+1} \in \overline{F}_k \setminus F_k$, and put $F_{k+1} = [a_{k+1}] + F_k$. Hence $F_k = F + P_k$ is a hyperplane in F_{k+1} . Since P_k is closed, we conclude as above that F_{k+1} cannot be closed. The cone $P_{k+1} = [a_{k+1}] + P_k$ is, however, closed, since $[a_{k+1}] \cap P_k = \{0\}$. Furthermore $F_{k+1} = [a_{k+1}] + F + P_k = F + P_{k+1}$. Thus the induction step is possible, and we have constructed a strictly increasing sequence $\{F_k\}$ of linear subspaces F_k of E such that $F \subset F_1$. This contradicts the assumption that F has finite codimension in E .

C o r o l l a r y . <u>If $F + P$ is a subspace of finite codimension in E and f is continuous, then f admits a positive and continuous extension to E .</u>

P r o o f . Since $F + P$ is closed, it follows from Proposition 2 that f admits a positive and continuous extension to $F + P$. Any continuous extension from $F + P$ to E will then suffice.

3. TWO COUNTEREXAMPLES

In this section we show that even if F is two-dimensional and P is locally compact, the conclusion of Proposition 3 and its corollary is not valid if we drop the assumption that $F + P$ has finite codimension. As in section 2 we require E to be complete and metrizable.

P r o p o s i t i o n 4 . <u>Assume that E is infinite dimensional. Then E contains a closed, locally compact convex cone P with an $e \in P$ such that $[e] + P$ is a non-closed linear subspace.</u>

P r o o f . E contains an infinite dimensional convex and compact subset K with $0 \notin K$. We can for instance construct K in the following way: Choose a sequence $\{x_n\}$ such that $x_n \to 0$ and with $\{x_n : n = 1, \ldots\}$ infinite dimensional. Then $\{0\} \cup \{x_n : n = 1, \ldots\}$ is compact, and hence ((1, p. 81)) the closed convex hull K_o of this set is compact. In particular we can find an x in E such that $-x \notin K_o$. Then $K = K_o + x$ has the stated properties. Let Q be the cone generated by K, that is $Q = \{\lambda k : \lambda \geq 0 , k \in K\}$, and put $L = Q - Q$. Since K is a compact subset of the linear space L, it is well known ((5, p. 341)) that Q is closed and locally compact in L. Choose V as a symmetric and convex zero-neighbourhood in L such that $V \cap Q$ is compact. Put $T = V \cap Q - V \cap Q$. Then T is a compact barrel in L. Therefore L cannot be closed in E, because if otherwise, L had to be finite dimensional. Choose $e \in L$ such that $0 \notin T + e$, and let P be the cone generated by $T + e$. Then P is closed and locally compact in E. Furthermore, $[e] + P = L$ since e is an order unit of P in L. This proves our assertion, since L is not closed.

C o r o l l a r y . <u>Assume that E is infinite dimensional. Then E contains a closed locally compact convex cone P and a two dimensional subspace F such that $L = F + P$ is a non-closed linear subspace. Furthermore,</u>

there exists a positive (and continuous) linear functional f on F, such that f admits no positive and continuous extension to L.

P r o o f . Let e and F be as in Proposition 4, and choose $a \in \overline{[e] + P} \setminus ([e] + P)$. Let F be the space spanned by e and a, and define f on F by $f(a) = 1$, $f(e) = 0$. Then $f = 0$ on $P \cap F$. Put $L = [a] + [e] + P$. L is a linear space and $L = F + P$. Since $\overline{[e] + P} \cap L = L$ and $[e] + P = f^{-1}(0) + P$, the desired conclusion follows from Proposition 1.

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