Matematisk Seminar
Universitetet i Oslo

Nr. 3
Februar 1963

ADDITIVE IDEAL SYSTEMS AND COMMUTATIVE ALGEBRA

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1. INTRODUCTION

The theory of r-ideals of Prüfer-Krull-Lorenzen ((1)) , ((2)) , ((3)) was developed in order to study the arithmetics of integral domains and ordered groups. It is a curious fact that until recently there has been made no serious attempt to pursue the success of this theory into the domain of general commutative algebra. In the book of Jaffard ((4)) there are a few scattered results in this direction, but we believe that ((5)) represents the first systematic step in carrying out such a project. The article ((5)) only presented a selection of very elementary and classical results of commutative algebra within the framework of the theory of x-ideals. But the results of that paper already indicated considerable further possibilities as well as a gain in the understanding of the scope of some of the most basic results of commutative algebra. On the other hand it was equally clear that new axioms had to be added if further substantial progress was going to be made. One main obstacle seemed to be the lack of an appropriate substitute for the usual congruence modulo an ideal in a ring. The congruence introduced in ((5)) is a fairly natural one, but it does not reduce to the usual one in the case of rings. It therefore came as a considerable surprise when it turned out that this congruence combined with an extra condition is nevertheless able to take care of crucial additive arguments as well as arguments involving residue class rings. The purpose of the present communication is to exhibit some of the first consequences of this extra condition, which we shall term the additivity axiom.

In the first section we recall some of the most fundamental definitions concerning x-systems. We give in particular the definition of a homomorphism and relate this notion with that of a congruence modulo an x-ideal. We next exhibit the various forms of the additivity axiom and show its intimate connection with the canonical homomorphisms, with the operations on x-ideals, with modularity, with the second isomorphism theorem and with Noetherianness. As an example of a slightly more advanced application of the additivity axiom we prove a wide generalization of Matusita's basic result on Dedekindian rings.

2. SYSTEMS OF IDEALS, HOMOMORPHISMS AND CONGRUENCES

We shall say that there is defined an (integral) <u>x-system</u> or an <u>ideal system</u> in a commutative semi-group S if to every subset A of S there corresponds a subset A_x of S such that

I $\quad A \subseteq A_x$

II $\quad A \subseteq B_x \implies A_x \subseteq B_x$

III $\quad AB_x \subseteq B_x \cap (AB)_x$

We shall sometimes use the letter x as name of the given ideal system as for instance in Theorem 2A below. If $A = A_x$ we shall say that A is an <u>x-ideal</u> and we shall refer to the passage from A to A_x as an <u>x-operation</u> and say that A_x is generated by A. All the x-systems considered here are supposed to be of <u>finite character</u> in the sense that the x-ideal generated by A equals the set-theoretic union of all the x-ideals generated by finite subsets of A. The operations of x-union (resp. x-product) denoted by $\cup_x$ (resp. $\circ_x$) are defined by $A \cup_x B = (A \cup B)_x$ (resp. $A \circ_x B = (A \circ B)_x$). The given x-system is said to be <u>principal</u> if $\{a\}_x = S \circ a$ for every $a \in S$. If S is a semi-group with cancellation we can define a <u>fractionary</u> <u>x-system</u> in the group of quotients of S.

Let S_1 and S_2 be two semi-groups each of which is equipped with an x-system denoted respectively by x_1 and x_2. A mapping φ of S_1 into

S_2 is called an (x_1, x_2)-*homomorphism* or shortly a *homomorphism* if

$$1^\circ \quad \varphi(ab) = \varphi(a)\varphi(b)$$

$$2^\circ \quad \varphi(A_{x_1}) \subseteq (\varphi(A))_{x_2}$$

2° is equivalent to saying that the inverse image of an x_2-ideal in S_2 is an x_1-ideal in S_1. Given an x-system in S and an x-ideal $A_x \subseteq S$ we say that b and c are x-*congruent* or simply *congruent* modulo A_x and write $b \equiv c (\mathrm{mod}\ A_x)$ whenever $(A_x, b)_x = (A_x, c)_x$. In the case of ordinary ideals in rings this congruence is coarser than the usual one and it is therefore somewhat surprising that it can nevertheless be used to establish generalizations of ring-theoretic results. The essential properties of this congruence and its relation to the notion of an (x_1, x_2)-homomorphism is given by the following

T h e o r e m 1. 1. *The relation $b \equiv c (\mathrm{mod}\ A_x)$ is a congruence relation in S, thus giving rise to a quotient semi-group S/A_x and a canonical multiplicative homomorphism $\varphi : S \longrightarrow S/A_x$.* 2. *The semi-group $\bar{S} = S/A_x$ has a zero element and $\mathrm{Ker}\,\varphi = A_x$.* 3. *The family of all sets $\bar{B} \subseteq \bar{S}$ such that $\varphi^{-1}(\bar{B})$ is an x-ideal in S defines an ideal system in $\bar{S}$ denoted by $\bar{x}$. Relative to this ideal system, φ is an $(x, \bar{x})$-homomorphism and $\bar{x}$ is the finest ideal system y such that φ is an (x, y)-homomorphism. $\bar{x}$ is called the canonical ideal system in $\bar{S}$.* 4. *The canonical homomorphism φ establishes a bijection between the x-ideals of S containing A_x and the $\bar{x}$-ideals of $\bar{S}$ in the way that $\bar{B}$ is an $\bar{x}$-ideal in $\bar{S}$ if and only if it is the direct image by φ of an x-ideal containing A_x.*

3. ADDITIVE IDEAL SYSTEMS

If R is a commutative ring and $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals in R then any element in the ideal-theoretic union (sum) of $\mathfrak{a}$ and $\mathfrak{b}$ is congruent to a suitable element in $\mathfrak{b}$ modulo $\mathfrak{a}$. Since this is valid for the ordinary congruence modulo $\mathfrak{a}$ it is also valid with respect to the coarser d-congruence obtained by specialization from the general definition of an x-congruence in the preceding paragraph. This leads us to the following

D e f i n i t i o n : An x-system is said to be <u>additive</u> if the following condition is satisfied:

A. To any element $c \in A_x \cup_x B_x$ there corresponds an element $b \in B_x$ such that $c \equiv b (\text{mod } A_x)$.

Most of the usual ideal systems in rings, semi-groups and lattices are additive. In order to show the existence of ideal systems which are not additive one can take the m-system in certain finite multiplicative lattices.

T h e o r e m 2 . <u>With the same notations as in Theorem 1 the following properties are equivalent</u>

A. <u>x is additive.</u>

B. $\varphi(A_x \cup_x B_x) = \varphi(B_x)$ <u>for all</u> A_x <u>and corresponding canonical</u> φ.

C. <u>The direct image of an x-ideal in S by any canonical homomorphism is an $\bar{x}$-ideal in $\bar{S}$.</u>

D. <u>The x-operation commutes with all canonical homomorphisms, i.e.</u> $\varphi(B_x) = (\varphi(B))_{\bar{x}}$.

E. <u>Every canonical homomorphism is distributive with respect to the</u> x-union of x-ideals: $\varphi(B_x \cup_x C_x) = \varphi(B_x) \cup_{\bar{x}} \varphi(C_x)$.

F. Every canonical homomorphism is distributive with respect to the x-union of arbitrary sets: $\varphi(B \cup_x C) = \varphi(B) \cup_{\bar{x}} \varphi(C)$.

G. Every canonical homomorphism is distributive with respect to the x-multiplication of x-ideals: $\varphi(B_x o_x C_x) = \varphi(B_x) o_{\bar{x}} \varphi(C_x)$.

H. Every canonical homomorphism is distributive with respect to the x-multiplication of arbitrary subsets of S : $\varphi(B o_x C) = \varphi(B) o_{\bar{x}} \varphi(C)$.

I. The canonical map $^{A}x/A_x \cap B_x \longrightarrow A_x \cup_x {}^{B}x/B_x$ is a surjection.

J. The canonical map $^{A}x/A_x \cap B_x \longrightarrow A_x \cup_x {}^{B}x/B_x$ is a bijection.

R e m a r k . We can of course also formulate the additivity axiom without using congruences or canonical homomorphisms in the following way. If $c \in A_x \cup_x B_x$ there exists a $b \in B_x$ such that $(A_x, b)_x = (A_x, c)_x$. In this form the additivity condition was also discovered independently by Azriel Rosenfeld and Erling Hansén.

4. MODULARITY AND NOETHERIANNESS

An x-system is said to be modular if its family of x-ideals forms a modular lattice under inclusion. The property of modularity is of importance in various connections, but is not satisfied for all x-systems. It is therefore of interest that we have the following

T h e o r e m 3 . Every additive x-system is modular.

P r o o f : If $B_x \subseteq A_x$ we have to show that $A_x \cap (B_x \cup_x C_x) \subseteq B_x \cup_x (A_x \cap C_x)$. Suppose that $a \in A_x \cap (B_x \cup_x C_x)$. From $a \in B_x \cup_x C_x$ and additivity we conclude that there exists an element $c \in C_x$ such that $a \equiv c (\text{mod } B_x)$. This implies $c \in (B_x, a)_x$ and $c \in A_x \cap C_x$. Using this together with $a \equiv c (\mathrm{mod}\ B_x)$ we obtain $a \in (B_x, c)_x \subseteq B_x \cup_x (A_x \cap C_x)$.

In ((5)) we defined S as x-Noetherian (we shall in this case also say that x is Noetherian) if x is of finite character and the following two conditions are valid.

I. S satisfies the ascending chain condition for x-ideals.

II. Every irreducible x-ideal in S is primary.

It is a fundamental fact in ordinary ideal theory of rings that I implies II . This is not valid for all x-systems of finite character as was shown by an example in ((5)) . But we have the following

T h e o r e m 4 . The implication I $\Longrightarrow$ II is valid for any principal and additive x-system of finite character.

In view of the results of ((5)) , the decomposition theorems of E. Noether therefore hold for all principal and additive x-systems of finite character.

5. MATUSITA SYSTEMS

Let S be a semigroup with cancellation. We define an x-system as a Matusita system if every x-ideal in S can be written as an x-product of prime x-ideals. We then have the following

T h e o r e m 5 . An additive and fractional Matusita system is Dedekindian, i.e. the given product representation is unique.

P r o o f : The present proof follows closely that of Zariski-Samuel in the case x = d . The five simple preparatory lemmas given in Zariski-Samuel are all of a multiplicative nature and their proofs apply verbatim to the general case of x-ideals (prop. 27 in ((5)) generalizes Lemma 3).

We refer below to these lemmas by using the same numbering as in Zariski-Samuel. We first show that every invertible proper prime-ideal in S is maximal. Denoting the quotient semi-group S/P_x by $\bar{S}$ and the residue class of a by $\bar{a}$ we claim that

$$(1) \qquad (P_x,a)_x/P_x = \bar{S}\,\bar{a}$$

whenever $a \notin P_x$. Let $c \in (P_x,a)_x$. From the fact that x is supposed to be principal and additive it follows that there exists an $s \in S$ such that $c \equiv sa (\mathrm{mod}\ P_x)$, i.e. $\bar{c} = \bar{s}\,\bar{a}$. This shows that the left-hand side of (1) is contained in the right-hand side. The reverse inclusion is a consequence of $Sa = (P_x,a)_x$. Since x is supposed to be a Matusita system we have the decompositions

$$(2) \qquad (P_x,a)_x = \prod_{i=1}^{n} P_x^{(i)} \quad \text{and} \quad (P_x,a^2)_x = \prod_{j=1}^{m} Q_x^{(j)}$$

where the $P_x^{(i)}$'s and $Q_x^{(j)}$'s are prime-x-ideals and the product sign denotes the x-product.

$$(3) \qquad \bar{S}\cdot\bar{a} = \prod_{i=1}^{n} P_x^{(i)}/P_x \quad \text{and} \quad \bar{S}\,\bar{a}^2 = \prod_{j=1}^{m} Q_x^{(j)}/P_x$$

where the ideals $P_x^{(i)}/P_x = \varphi_{P_x}(P_x^{(i)})$ and $Q_x^{(j)}/P_x = \varphi_{P_x}(Q_x^{(j)})$ are prime $\bar{x}$-ideals in $\bar{S}$ (by Theorem 3 C). Since

$$(4) \qquad \bar{S}\,\bar{a}^2 = (\bar{S}\,\bar{a})^2 = \prod_{i=1}^{n} (P_x^{(i)}/P_x)^2$$

it follows from Lemma 5 on comparing (3) and (4) that the $\bar{x}$-ideals $Q_x^{(j)}/P_x$ are the $\bar{x}$-ideals $P_x^{(i)}/P_x$ each repeated twice. Thus $m = 2n$ and we can renumber the $Q_x^{(j)}$ such that $Q_x^{(2i)}/P_x = Q_x^{(2i-1)}/P_x = P_x^{(i)}/P_x$ and thus also $Q_x^{(2i)} = Q_x^{(2i-1)} = P_x^{(i)}$ since any x-ideal containing P_x is a union of equivalence classes modulo P_x (Theorem 1). Applying this to (2) we obtain $(P_x, a^2)_x = (P_x, a)_x^2$ which implies $P_x = (P_x, a)_x^2 = P_x^2 \cup_x (P_x \circ \{a\}) \cup_x \{a^2\} = (P_x^2, a)_x$. If therefore $c \in P_x$ the assumption in the theorem implies the existence of an $s \in S$ such that $c \equiv sa (\mathrm{mod}\ P_x^2)$. Hence $c \equiv sa (\mathrm{mod}\ P_x)$ and $sa \in P_x$. Since P_x is prime and $a \notin P_x$ we have $s \in P_x$. We obtain $(P_x^2, c)_x = (P_x^2, sa)_x \subseteq (P_x^2, aP_x)_x$ for all $c \in P_x$, i.e. $P_x \subseteq (P_x^2, aP_x)_x$. Since the reverse inclusion is obvious we have $P_x = (P_x^2, aP_x)_x = P_x \circ (P_x \cup_x \{a\})$. If P_x is invertible we can multiply this equality by P_x^{-1} and we obtain $S = (P_x, a)_x$ which shows that P_x is maximal. According to Lemma 5 we only need to prove that every proper prime x-ideal P_x is invertible. Let $b \in P_x$ and write $(b) = Sb = \prod_{i=1}^{n} P_x^{(i)} \subseteq P_x$. This implies $P_x^{(i)} \subseteq P_x$ for a suitable i. By Lemma 4 every $P_x^{(i)}$ is invertible and hence maximal according to the first part of the proof. Thus $P_x^{(i)} = P_x$ for some $P_x^{(i)}$ and P_x is invertible. The theorem now follows from Lemma 5.

C o r o l l a r y 1 . (Matusita) <u>If every proper ideal in an integral domain R can be written as a product of prime ideals then this decomposition is unique.</u>

C o r o l l a r y 2 . <u>If every proper s-ideal in an integral domain R can be written as a product of prime s-ideals then R is a discrete valuation ring of rank one.</u>

Using the results on x-Dedekindian semi-groups which was established in ((5)) the following result is now immediate.

T h e o r e m 6 . Let S be a semi-group with cancellation law in which there is given an additive fractionary x-system. Then the following properties are equivalent

1. x is a Matusita system.
2. Every x-ideal in S can be written uniquely as an x-product of prime x-ideals.
3. The fractionary x-ideals form a group under x-multiplication.
4. S is x-Dedekindian (i.e. satisfies the three Noetherian axioms).

P r o o f : $1 \Longrightarrow 2$ has been established above. $2 \Longrightarrow 3$: The first part of the proof of Theorem 11 p. 274 in Zariski-Samuel carries over verbatim to the present case. $3 \Longrightarrow 4$ follows from Theorem 25 in ((5)) and $4 \Longrightarrow 1$ is obvious.

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