Matematisk Seminar
Universitetet i Oslo

Nr. 2
Februar 1963

ALGEBRAIC PROPERTIES OF UNIFORMLY CONTINUOUS FUNCTIONS

By

Jens Erik Fenstad

1. INTRODUCTION

The objects we are going to study are real-valued uniformly continuous functions on some uniform space. The space will be denoted by $(X, \mathcal{U})$ and the collection of functions by $U(X, \mathcal{U})$, or shorter $U(X)$ if no misunderstanding is likely to arise about which structure on the set X we are considering.

The main purpose is to characterize the set $U(X)$ by its algebraic properties, and the present exposition is devoted to a report on a solution of this problem, which we have recently obtained. (The details will be published elsewhere.)

A commutative lattice-ordered group G is (i) a commutative group and (ii) a lattice, which satisfies

$$\text{(iii)} \quad a \leq b \Longrightarrow a + c \leq b + c \quad \text{for} \quad a, b, c \in G \ .$$

It is elementary to show that $U(X)$ in the pointwise defined operation is a commutative l-group. In $U(X)$ we may also introduce the pointwise defined multiplication, but unfortunately $U(X)$ is not in general closed under multiplication, an obvious counter-example being for $X = R$ (the reals) the function $f(x) = x$; $f \circ f$ is not uniformly continuous.

Using the maximal l-ideals of an l-group (we drop the qualification "commutative" in the sequel) it is not difficult to represent the l-group as an l-group on the maxime ideal space. Then one may give this set the coarsest uniformity, making all the functions corresponding to the elements of the given l-group uniformly continuous. The main difficulty is to obtain conditions that our functions be the set of a l l uniformly continuous functions in the structure introduced. To this end one needs an approximation theorem for uniformly continuous functions, and our first main result is a Stone-Weierstrass type theorem for $U(X)$ giving sufficient conditions on an l-subgroup G of $U(X)$ to be uniformly dense in $U(X)$. Our theorem entails a version of the ordinary Stone-Weierstrass theorem for compact spaces.

The next main result is the characterization theorem showing that every divisible (in the algebraic sense) and commutative l-group which is closed under suprema of certain families of elements, can be represented as a group $U(X)$ for some suitable uniform space $(X, \mathcal{U})$. This theorem is completed by the determination of a maximal category of uniqueness for the representation space $(X, \mathcal{U})$, this category being representable as the class of all closed subsets of products of real lines in the obvious induced uniformity.

Our theory is applied to the theory of proximity spaces, giving a solution of a problem which Yu.M. Smirnov stated in an address to the Stockholm Congress. Otherwise our results on proximity structures are mainly negative, as we give counter-examples to the "obvious fact" that the set $P(X)$ of real-valued p-continuous functions is an l-group.

2. APPROXIMATION THEORY

A family $\{f_i \mid i \in I\}$ of uniformly continuous functions is called *uniform* (or, in standard terminology, uniformly equicontinuous) if $\forall \varepsilon > 0$ there exists a $V \in \mathcal{U}$ such that

$$(x,y) \in V \Longrightarrow |f_i(x) - f_i(y)| < \varepsilon \qquad \text{for all } i \in I$$

A family $\{f_i \mid i \in I\}$ is called *locally finite* if for each $i \in I$, $|f_i| \wedge |f_j| = 0$ for all but a finite number of indices $j \in I$.

Let $\mathcal{F}$ be a collection of subsets of X and let $V \in \mathcal{U}$. The family $\{f_A \mid A \in \mathcal{F}\}$ is called *separating of order* V if it is uniform and satisfies for each $A \in \mathcal{F}$ the requirements that $0 \leq f_A \leq 1$ $f_A(x) = 1$ if $x \in A$, and $f_A(y) = 0$ if $y \in X - V(A)$. With every uniform space $(X, \mathcal{U})$ we may associate a cardinal number $m(X)$, the main property of which is that if X is compact, then $m(X)$ is the least infinite cardinal. $m(X)$ is introduced in order to obtain an approximation theorem for $U(X)$ which will yield the Stone-Weierstrass theorem for compact spaces.

Two conditions on an l-subgroup G of $U(X)$ is essential for the further development.

A(1): G contains separating functions, i.e. to each $V \in \mathcal{U}$ and each subset-collection $\mathcal{F}$ such that card $(\mathcal{F}) < m(X)$, there exists a separating family $\{f_A \mid A \in \mathcal{F}\}$ of order V.

A(2): G is closed under the formation of supremum of locally finite uniform families of cardinality strictly less than $m(X)$.

It is important to note that the supremum is relative to G. Thus if $\{f_i\}$ is an admissible family, by A(2) there exists in G a function $f = \bigvee_G f_i$ characterized by $f \geq f_i$ for all $i \in I$, and if $g \geq f_i$ for all $i \in I$, where $g \in G$, then $g \geq f$. It is not at all obvious that f is going to be the pointwise supremum of the family $\{f_i\}$, in fact, while it is true for those l-subgroups G in which we are interested, the proof is somewhat involved. Thus, if G contains the rational constants and satisfies A(1), then, whenever $\{f_i\}_{i \in I}$ has a supremum in G, this supremum is the pointwise one. This entails that

$$((\bigvee_G f_i))(x) = \bigvee_{i \in I} (f_i(x))$$

We are now in a position to state the main approximation theorem.

T h e o r e m 1. <u>Let G be an l-subgroup of $U(X)$ containing the rational constants. If G satisfies the conditions A(1) and A(2), then G is uniformly dense in $U(X)$.</u>

(G is uniformly dense in $U(X)$, if for all $f \in U(X)$ and all $\varepsilon > 0$ there exists a $g \in G$ such that $|f(x) - g(x)| < \varepsilon$, $x \in X$.) The proof, while simple in its conception, is not at all easy to carry out, and we refrain from giving any indications of the details. Theorem 1 entails the Stone-Weierstrass theorem, the deduction being a rather straightforward exercise in how to reduce open covers of compact sets to finite covers.

For the representation theory a modified version of theorem 1 will be better suited. A family $\{f_i \mid i \in I\}$ is called **bounded** if there exists a real constant M such that $|f_i| \leq M$ for all $i \in I$. $\{f_i \mid i \in I\}$ is called **admissible** if it is either locally finite or bounded. The modified conditions reads:

$A'(1)$: G contains separating functions, i.e. for each $V \in \mathcal{U}$ there exists a uniform family $\{f_x \mid x \in X\}$ which is separating of order V.

$A'(2)$: G contains the supremum of any of its admissible uniform families.

Let us recall the definition of a divisible commutative group: G is called **divisible** if the equation $a = nx$ has a solution in x for every $a \in G$ and natural number n.

Theorem 1'. <u>Let G be a divisible l-subgroup of $U(X)$ containing the identity function. If G satisfies $A'(1)$ and $A'(2)$, then G is uniformly dense in $U(X)$.</u>

3. REPRESENTATION THEORY

It will be necessary to recall some properties of l-groups. An **l-ideal** I in G is a subgroup of G such that

$$a \in I \quad \text{and} \quad |b| \leq |a| \Longrightarrow b \in I .$$

By a homomorphism we understand a homomorphism both as group and lattice. The group is simply ordered if any two elements are comparable; it is called archimedean if $na \leq b$ for all n implies $a \leq 0$. It is rather well-known that if M is a maximal l-ideal in G, then G/M is isomorphic to an l-subgroup of R. If $e \geq 0$, $e \notin M$, then the isomorphism Φ_M can be so chosen that $\Phi_M(e) = 1$.

The main definition needed for our representation theory is the following: (E, e) is called a n o r m a l p a i r in an l-group G if E is a collection of maximal l-ideals in G such that $\bigcap E = \{0\}$ and $e \in G$ satisfies $e \geq 0$ and $e \notin \bigcup E$.

Obviously a normal pair is a substitute for the wanting multiplicative structure of $U(X)$.

If (E, e) is a normal pair in G one may associate with each $a \in G$ a real-valued function $a^* : E \to R$ in the usual way:

$$a^*(M) = \Phi_M(a)$$

It is well-known that $^* : G \to G^*$ is an isomorphism, $e^* \equiv 1$.

G is supposed to be divisible, hence every equation $nq = me$ where $(n,m) = 1$ and $n > 0$ has a unique solution q, unique because G is a commutative l-group. Let Q_e be the set of all solutions. Then Q_e^* is the rational constants in G^*.

The notion of admissible family carries over to general l-groups. It will be convenient to restate the definition of a bounded family: $\{a_i \mid i \in I\}$ is called bounded if there exists an $a \geq 0$ in G such that $|a_i| \leq a$ for all $i \in I$. If G is supposed to contain the supremum of the admissible families in Q_e, then it is easily verified that the set R_e of such suprema is isomorphic to R by the *-map.

The last preparation needed for our representation theorem is a definition, within the context of general l-groups, of uniform families.

There exists for each $M \in E$ a uniquely defined map $a \to a^M$ satisfying (i) $a^M = a - r$ for some $r \in R_e$, and (ii) $a^M \in M$. Indeed, let r be the unique element in $R_e \cap \Phi^{-1}(\Phi_M(a))$. For each $N \in E$ an order relation $\underset{N}{<}$ is defined by

$$a \underset{N}{<} b \iff \phi_N(a) < \phi_N(b) \quad .$$

Define the subset $V(q,a)$, $q > 0$, $q \in Q_e$ and $a \in G$:

$$V(q,a) = \left\{ (M,N) \mid \quad |a^M| \underset{N}{\leq} q \right\}.$$

A family $\{ a_i \mid i \in I \}$ in G is called u n i f o r m if for every $q > 0$ in Q_e there exists a finite set of elements $b_1, \ \ldots \ , b_n \in G$ and a $q' > 0$ in Q_e such that

$$\bigcap_{j=1}^{n} V(q', b_j) \subseteq \bigcap_{i \in I} V(q, a_i) \quad .$$

Now we have our main theorem on the algebraic structure of uniformly continuous functions.

T h e o r e m 2 . <u>A commutative l-group G is isomorphic to an l-group $U(X)$ for some uniform space $(X, \mathcal{U})$, if and only if G is divisible and there exists a normal pair (E, e) in G relative to which G contains the supremum of any of its admissible uniform families.</u>

There is one subtlety to note. The definition of uniform families presupposes the definition of R_e and that R_e is isomorphic to R for each ϕ_M . Hence we need a modification stating the theorem as we have done. R_e must initially be defined as those suprema q_n in Q_e which exists in G , and the map $a \longrightarrow a^M$ cannot be taken to be defined on all of G but only for those a such that $R_e \cap \phi_M^{-1}(\phi_M(a)) \neq \emptyset$. It is then fairly simple to show that we, in fact, obtain what we need for our theorem.

Concerning the proof, it is very easy to show by means of the approximation theorem that G^* is uniformly dense in $U(E)$ when E is given the coarsest uniformity making the functions in G^* uniformly continuous. To obtain that G^* is actually all of $U(E)$ requires a somewhat more involved argument.

Concerning the problem of uniqueness of the representing space of theorem 2, we have obtained the following result. A uniform space $(X, \mathcal{U})$ is called functionally complete if it is complete and has the coarsest uniformity making all the functions in $U(X)$ uniformly continuous.

Theorem 3. Let $\mathcal{W}$ be the category of functionally complete spaces. Then $\mathcal{W}$ is a category of uniqueness in the sense that given any two structures $(X_i, \mathcal{U}_i) \in \mathcal{W}$, $i = 1,2$, there exists a uniform isomorphism $\psi: (X_1, \mathcal{U}_1) \simeq (X_2, \mathcal{U}_2)$, if and only if there exists an l-isomorphism $\phi: U(X_1) \cong U(X_2)$ such that $\phi(1) = 1$. $\mathcal{W}$ is a maximal category of uniqueness and may be represented as the category of closed subsets of products of real lines in the induced uniformity.

The third part of our work is concerned with the relationship between uniformity and proximity. The positive result is a characterization of those subsets of $P(X)$, the class of all real-valued p-continuous functions, which are the subsets $U(X)$ for some compatible uniformity, - theorem 2 is an immediate answer. Otherwise, as we have remarked in the introduction, our results are counter-examples.

To state the result on $P(X)$ we recall some definitions. An admissible family $\{f_i \mid i \in I\}$ in some subset G of $P(X)$ is a family which is either locally finite or bounded. Bounded is here, as above, used in the sense dominated, i.e. $\{f_i \mid i \in I\}$ is bounded in G if there exists an $f \in G$ such that $|f_i| \leq f$ for all $i \in I$. A uniform family is defined by the property that for any $\varepsilon > 0$ there shall exist a $\delta > 0$ and functions $g_1, g_2, \ldots, g_n \in G$ such that

$$|g_j(x) - g_j(y)| < \delta \quad \text{for} \quad j = 1, \ldots, n$$

$$\Longrightarrow \quad |f_i(x) - f_i(y)| < \varepsilon \qquad \text{for all} \quad i \in I .$$

The answer to Smirnov's problem is now

T h e o r e m 4 . Let $(X, \mathcal{P})$ be a proximity space and let $P(X, \mathcal{P})$ be the set of all real-valued p-continuous functions on the space. A subset G of $P(X)$ is an l-group $U(X)$ for some compatible uniformity $(X, \mathcal{U})$, if and only if G is a divisible l-group which contains the set $P_B(X)$ of bounded functions in $P(X)$ and is closed under suprema of admissible uniform families.

4. CONCLUDING REMARKS

Given a $U(X)$ we have a natural exact sequence of l-groups and l-homomorphisms:

$$0 \longrightarrow U_B(X) \longrightarrow U(X) \longrightarrow A \longrightarrow 0 \quad ,$$

where $U_B(X)$ is the l-ideal of bounded functions in $U(X)$ and A is the quotient group. The group $U_B(X)$ is a group $C(X')$ for some compact space, viz. $X' = \hat{X}$ where we complete with respect to the totally bounded uniformity deduced from $\mathcal{U}$; hence U_B has some very natural algebraic characterizations.

A is a divisible l-group without any torsion elements, thus by general theorems A is a weak direct sum of rationals

$$U(X)/U_B(X) \cong \coprod Q \quad .$$

An open problem is to "compute" the group $\coprod Q$, i.e. to compute the cardinality of the index set of the sum.

Thus combining informations on U_B and the quotient $\coprod Q$ we could obtain algebraic characterizations of the groups $U(X)$.

This requires that we discuss the problem of l-group extensions. Let A and B be two l-groups, then X and an l-homomorphism ϕ is an extension of A by B if we have the exact sequence

$$0 \longrightarrow B \longrightarrow X \xrightarrow{\Phi} A \longrightarrow 0 \quad .$$

Our groups U_B are divisible, hence in the category of abelian groups X would be uniquely determined to within ordinary group isomorphisms. The situation in our case is rather more complicated as the following example shows.

Let $R^+ = [0, \rightarrow \rangle$, then there exists a map $\Phi : U(R^+) \rightarrow U(R^+)$ such that Φ is a group isomorphism and a lattice isomorphism on $U_B(R^+)$, but Φ is n o t a lattice isomorphism of $U(R^+)$.

Define $f_0(x) = x$ and $f_1(x) = x$ if $0 \leq x \leq 1$, $f_1(x) = x^{\frac{1}{2}}$ if $x > 1$. Let Φ_B be the identity on $U_B(R^+)$, then we can extend it to a map of the groups (not l-groups):

$$\Phi' : U_B + (rf_i) \longrightarrow U_B + (rf_i) \;, \qquad i = 0,1,$$

where $r \in R$ and $\Phi'(rf_0) = r \cdot f_1$, $\Phi'(r \cdot f_1) = r \cdot f_0$. Owing to the fact that if B is a divisible l-subgroup of A and $x \in A - B$, then the sum $B + (x)$ is direct, we may extend the isomorphism Φ' to a group isomorphism $\Phi : U(R^+) \longrightarrow U(R^+)$. The map Φ is n o t a lattice isomorphism.

Next, if $A = \coprod Q$ and we have an extension of A by some group U_B

$$0 \longrightarrow U_B \longrightarrow X \longrightarrow A \longrightarrow 0 \;,$$

is then X an l-group U(X) which admits U_B as its group of bounded functions?

We have not yet had time to study the problems outlined in these concluding remarks, but we suspect that our theorem 2, giving an algebraic characterization of U(X) as a commutative, divisible l-group, may be of use in obtaining some of the answers.