Pricing risk due to mortality under the Wang Transform

by

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Abstract

The purpose of this thesis is to study the pricing of mortality risk in life annuities, when using the so-called Wang's Transform which is popular in certain quarters of actuarial science. This is a distortion operator that transforms the mortality distribution into risk-adjusted mortalities. By applying this to a given mortality table, we will price life annuities with both distributions and discuss the underlying risk of using wrong mortalities.

Words: life insurance, life annuities, mortality risk, Wang's Transform, mortality bonds, insurance securitization, hedging, discounting.

Contents

Chapter 1

Introduction

Longevity risk is a major issue for insurers and pension funds. When pricing a life insurance product it is important that the mortalities used don't deviate too much from the actual mortalities in the future, as this could lead to severe underestimation of the reserve. Mortality tables are based on historical data. Because of a continuously increase in expected lifetime since The Second World War, the historical data quickly become obsolete.

In this thesis, we will study the pricing of mortality risk in life annuities when using the Wang Transform:

$$g_\lambda(u) = \Phi[\Phi^{-1}(u) - \lambda].$$

The distortion operator transforms the mortality distribution into risk-adjusted mortalities. By applying this to a given mortality table, we will price life annuities with both distributions and discuss the underlying risk of using wrong mortalities. The risk-adjusted mortalities will also be used further to price a mortality bond.

It is assumed that the reader knows basic statistics and also a little about life insurance. In Chapter 2 will life insurance basics be introduced, and also necessary background material for further use in the thesis. The concept of mortality bonds is introduced with examples. We will look at the theory of distortion operators, and especially we introduce the Wang Transform and how it can be used on survival probabilities.

In Chapter 3 will we expain how a life annuity can be priced. We will use both the mortalities from a given table and the risk-adjusted mortalities in our calculations, and see if there actually is a difference.

In Chapter 4 will we go deeper into one of the mortality bonds from Chapter 2 and look at how it can be priced with the use of the risk-adjusted mortalities obtained from the Wang Transform in Chapter 3.

Finally, we will compare and discuss the results to see if the Wang Transform can be used as a universal framework for adjusting mortality tables when the historical data is obsolete.

Chapter 2

Life insurance basics

2.1 Annuities

2.1.1 Introduction

An annuity is defined as a sequence of payments of limited duration which we denote by n. The payments can either take place at the end of each period (*in arrears*), or at the beginning (*in advance*); see [9]. If the payments start at time 0, the present value is denoted by $\ddot{a}_{\overline{n}|}$, and with survival probabilities ${}_k p_{l_0}$ and discount rate d, given by

$$\ddot{a}_{\overline{n}|} = \sum_{k=0}^{n-1} d^k {}_k p_{l_0}. \tag{2.1}$$

Similarly, if the payments occur at the end of the periods, the present value, now denoted $a_{\overline{n}|}$, is

$$a_{\overline{n}|} = \sum_{k=1}^{n} d^k {}_k p_{l_0}. \tag{2.2}$$

In other words, taking the payment agreed on at time k (here set equal to 1) and multiplying with the probability that it is actually made, adding over all k and discounting, the present value of the annuity emerges; see [7].

2.1.2 Life tables

An important part of annuities is the survival probabilities ${}_kp_l$. Often the payment stream is broken off when the individual dies, and we have to correct for it. To do this, we have to model how long people live. It can then be transformed to a life table specified through the conditional probabilities

$$\underbrace{{}_kp_l = P(L \geq l + k | L \geq l)}_{\text{survival probabilities}} \quad \text{and} \quad \underbrace{{}_kq_l = P(k + l - 1 \leq L < l + k | L \geq l)}_{\text{mortalities}}. \tag{2.3}$$

To the left we have the probability of surviving k periods given that the initial age is l, whereas the right is the probability that the individual survives k-1 periods then dies during the next, given initial age l.

Using the one-step probabilities ${}_1p_l = p_l$ and ${}_1q_l = q_l$, we can construct a life table through recursion,

$${}_{k+1}p_l = (1 - q_{l+k}) \cdot {}_kp_l, \qquad \text{k} = 0, 1, ... \qquad \text{starting at} \qquad {}_0p_l = 1, \tag{2.4}$$

and for the mortalities we have

$${}_{k+1}q_l = q_{l+k} \cdot {}_kp_l, \qquad \text{k} = 0, 1, ... \tag{2.5}$$

2.1.3 The concept of discounting

To find the present value of an annuity we have to discount. This is because the payments are to be received in the future. Money is subject to inflation and has above all the ability to earn interest, therefore one money unit today is worth more than one money unit tomorrow. Discounting is the process of determining how tomorrow's money unit is devaluated.

Let's say that a payment F will be made k years ahead, then the present value of this payment, also called the discounted value, is $P = F/(1 + r)^k$, where r is called the discount yield.

There are several ways of determining the discount rate. We have

$$\underbrace{d_k = \frac{1}{(1 + r)^k}}_{\text{technical rate}}, \qquad \underbrace{d_k = P_0(0 : k) = \frac{1}{(1 + \bar{r}_0(k))^k}}_{\text{fair value discounting}}, \qquad \underbrace{d_k = \frac{Q_k}{(1 + r)^k}}_{\text{inflation included}}.$$

The technical rate r is determined administratively. It is the interest rate charged to banks and other depository institutions for loans received from the central bank. It is vulnerable to bias as the central bank changes it according to which direction they want to push the economy. A low interest rate makes liabilities very attractive, while high values are used to keep liabilities low.

That weakness is avoided with fair value discounting. The discounts now are market bond prices $P_0(0:k)$ closely related to the market interest rate curve $\bar{r_0}(k)$. The bias is gone, but both bond prices and interest rate curves fluctuate, and also the market-based present valuation with them. The fair value discounts in the future are not known, and this also induces uncertainty in the valuation.

It may be the liabilities depend on inflation. In traditional defined benefit schemes where pension rights and contributions are linked to some prior price or wage index Q_k, we enter inflation by $d_k \cdot Q_k$. This can be done with the fair value discount as well as the technical rate.

2.1.4 Life annuities

A life annuity is a financial contract in form of an insurance product according to which a seller - typically a life insurance company - makes a series of future payments to a buyer - an annuitant - in exchange for the immediate payment of a lump sum (single-payment annuity) or a series of payments (regular-payment annuity), prior to the onset of the annuity.

As mentioned, the payment stream has an unknown duration based principally upon the death of the annuitant. Then the contract will terminate and the remainder of the fund accumulated is forfeited unless there are other annuitants or beneficiaries in the contract. This is a form of longevity insurance: the uncertainty of an individual's lifespan is transferred from the individual to the insurer, which reduces its own uncertainty by pooling many clients.

A life annuity can be divided into two phases: the accumulation phase and the distribution phase. During the accumulation phase the annuitant deposits and accumulates money into an account. Then during the distribution phase the insurance company makes payments until the death of the annuitant. The type of contract decides how long each phase lasts.

Fixed and variable annuities
A fixed annuity consists of payments in fixed amounts or increases by a fixed percentage. A variable one is when the amounts vary according to the investment performance of a specified set of investments, typically bonds and equity mutual funds.

Guaranteed annuities
The issuer is required to make annuity payments for at least a certain number of years, called the "period certain". If the annuitant outlives the specified period, annuity payments will then continue until death. However if the annuitant dies before expiration of the period, the annuitant's estate of beneficiary is entitled to collect the remaining payments certain. This is a way of reducing the risk of loss for the annuitant, but in return the annuity payments will be smaller than with an ordinary annuity.

Joint annuities
This is a multiple annuitant product that includes joint-life and joint-survivor annuities. The payments stop upon death of one or both of the annuitants, depending on what was agreed on in the contract. A type of contract can be structured so that a married couple receives payments until the second spouse's death. In joint-survivor annuities, sometimes the payments are reduced to the second annuitant after the death of the first.

Impaired life annuities
If there is a medical diagnosis which is severe enough to reduce life expectancy, the terms offered will often be improved compared to an ordinary annuity.

The present value of life annuities
Annuities are often used to save money for retirement, e.g. pension schemes. The type of contract we will focus on is fixed annuities. The ordinary benefit type have contributions π up to some retirement age l_r, and then benefits s are recieved after that. The cash flows can be written like (2.1) and (2.2).

Assuming payments are made in advance, we get that the expected present value for the entire scheme is

$$\ddot{a}_{\overline{\infty}|} = -\pi \sum_{k=0}^{l_r - l_0 - 1} d^k {}_k p_{l_0} + s \sum_{k=l_r-l_0}^{\infty} d^k {}_k p_{l_0}, \tag{2.6}$$

the usual convention being that the contributions are counted negative (as this is something the policy holder has to pay).

The equivalence principle
An important concept in pricing life insurance is the principle of equivalence. Then the expected value of payments into and out of the scheme is equalized, i.e. (2.6) is set equal to zero. Solving for π, we get the premium a pension holder has to pay to receive the agreed on pension benefit s after retirement. Then there is no profit for the insurer, but no expenses or risk are covered. In real life the companies add a loading to cover the expenses, but we will disregard this for now.

2.1.5 Life table risk

In section 2.1.2 life tables and how they are obtained were introduced. Now we will look at the risk inherent in this. The mortalities are estimated from historical data, so it is a risk of the data being obsolete. Since The Second World War, there has been a trend of one-year increases per ten years of survival in the expected lifetime, thanks to advancements in medicine and raised awareness of personal hygiene.

Random error is inevitable, but negligible for large countries. There is a different story when it comes to small countries and pension schemes. Historical data are now more scarce and it has been discovered that life tables for pension schemes differ substantially from the country average. The target group that buys life annuities are usually the group of good health who are afraid of outliving their savings.

We also have the systematic error or bias. This is when the historical material is too old or applies to the wrong social group, also called selection bias. Let's say that a newly started life insurance company has access to mortalities for their entire country or the life annuitants in another country. What data should they choose to base their calculations on? The smaller data set applies to the right group, but to the wrong country. The larger data set applies to the wrong group, but the right country. All the choices that are made regarding the life table lead to an error of some type. Using a data set that applies to the correct population will remove the bias, but the random error will be large. Using a larger data set to reduce the random error will introduce bias.

2.2 Mortality bonds

2.2.1 Introduction

Longevity risk is a major issue for insurers and pension funds. The calculation of expected present values requires an appropriate dynamic mortality model in order to avoid underestimation of the future costs. Actuaries are increasingly using life tables that include forecasts of future trends of mortality, but there is the danger that the mortality projections turn out to be incorrect. Longevity risk occur principally when the annuitants live longer than predicted by the projected life tables. A very good hedge against mortality improvement risk is mortality bonds where the coupon payments depend on the proportion of the population surviving to particular ages; see [8].

There has since The Second World War not only been a substantial increase in expected lifetime, it was also a baby boom period in the immediate post-war decade. These so-called "baby boomers" are now reaching retirement age and are starting their distribution phases. This means that the annuity providers are in big demand of liquidity, and a mortality bond can come in handy as is dealth with next.

2.2.2 Example of a mortality bond

An insurer buys reinsurance from a special purpose company (SPC), which issues bonds to investors. The bond contract and reinsurance transfer the risk from the annuity provider to these investors. The company invests the premium and cash from the sale of the bonds in default-free securities; see Figure 2.1 for an overview. To understand

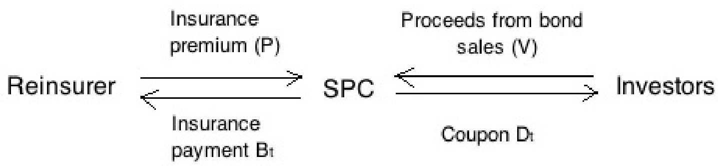


Figure 2.1: Mortality Bond Cash Flow Diagram

the concept of a mortality bond consider the following example.

Suppose an insurer must pay immediate life annuities to n_x annuitants all aged x initially. If we set the payment rate at 1000/year annuitant, and let n_{x+k} denote the number of survivors to year k, the insurer pays $1000n_{x+k}$ to its annuitants. We will define a bond contract to hedge the risk that the insurer's payments exceed an agreed upon level.

The insurer buys reinsurance from the SPC for a premium P at time 0. The contract has fixed trigger levels X_k such that the SPC pays the insurer the excess of the actual payments over this level. In year k, the insurer pays $1000n_{x+k}$ to its annuitants. If the payments exceed the trigger level for that year, the SPC pays the excess up to a maximum amount 1000C. Then in each year k=1,2,. . . ,K the insurer collects the benefit B_k from the SPC determined by formula (2.7):

$$B_k = \begin{cases} 1000C, & \text{if } n_{x+k} > X_k + C, \\ 1000(n_{x+k} - X_k), & \text{if } X_k < n_{x+k} \leq X_k + C, \\ 0, & \text{if } n_{x+k} \leq X_k. \end{cases} \tag{2.7}$$

The insurer's cash flow to annuitants at k is now offset by positive cash flow from the insurance:

$$\begin{aligned} \text{Insurer's net cash flow} &= 1000n_{x+k} - B_k \\ &= \begin{cases} 1000(n_{x+k} - C), & \text{if } n_{x+k} > X_k + C, \\ 1000X_k, & \text{if } X_k < n_{x+k} \leq X_k + C, \\ 1000n_{x+k}, & \text{if } n_{x+k} \leq X_k. \end{cases} \end{aligned} \tag{2.8}$$

Now, there are no "basis risk" in the reinsurance. That arises when the hedge is not exactly the same as the reinsurer's risk, but this mortality bond cover that.

The cash flows between the SPC, the investors, and the insurer can be described as in Figure 2.1. First, the SPC's payments to the investors:

$$D_k = \begin{cases} 0, & \text{if } n_{x+k} > X_k + C, \\ 1000C - B_k, & \text{if } X_k < n_{x+k} \leq X_k + C, \\ 1000C, & \text{if } n_{x+k} \leq X_k, \end{cases} \tag{2.9}$$

$$= \begin{cases} 0, & \text{if } n_{x+k} > X_k + C, \\ 1000(C + X_k - n_{x+k}), & \text{if } X_k < n_{x+k} \leq X_k + C, \\ 1000C, & \text{if } n_{x+k} \leq X_k, \end{cases} \tag{2.10}$$

where D_k is the total coupon paid to investors. The maximum value of n_{x+k} is n_x, attained when nobody has died yet, but from the perspective of 0, n_{x+k} is a random value between 0 and n_x. We denote the market price of the mortality bond as V. The aggregate cash flow out of the SPC is

$$B_k + D_k = 1000C$$

for each year k=1,..,K and the principal amount 1000F at k=K. The SPC will perform on its insurance and bond contract commitments provided that P+V is at least equal to the price W of a default-free fixed-coupon bond with annual coupon 1000C and principal 1000F valued with the bond market discount factors:

$$P + V \geq W = 1000Fd(0, K) + \sum_{k=1}^{K} 1000Cd(0, k). \tag{2.11}$$

In other words, the SPC can buy a "straight bond" and have exactly the required cash flow it needs to meet its obligation to the insurer and the investors, if the insurance premium and proceeds from sale of the mortality bonds are sufficient. Each year, they will receive 1000C as the straigth bond coupon and then pays D_k to the investors and B_k to the insurer. The case is always that 1000C=$D_k + B_k$ is exactly enough to meet its obligations.

2.2.3 Types of mortality bonds

There are many types of mortality bonds, but they can be divided into two main categories:

1. Principal-at-risk
2. Coupon-based

For the first type, the investor risks losing all or part of the principal if the relevant mortality event occurs. An example of this is the Swiss Re mortality bond issued in December 2003. The second type has coupon payments that are mortality dependent. This can be a smooth function of a mortality index, or it can be specified in "at-risk" terms. Then the investor loses some or all of the coupon if the mortality index crosses som threshold. An example of this is the EIB/BNP longevity bond announced in November 2004; see [4] for more details.

The Swiss Re mortality bond

The Swiss Re bond was a three-year life catastrophe bond maturing on January 1, 2007. This was to reduce their exposure to catastrophic mortality deterioration (e.g. if a pandemic occur). The issue size was $400m. Investors would receive quarterly coupons set at three-month U.S. dollar LIBOR + 135 basis points.

The principal was unprotected and depended on what happened to the constructed index of mortality rates across five countries: the United States of America, United Kingdom, France, Italy and Switzerland. The principal would be repayable in full if the mortality index didn't exceed 1.3 times the 2002 base level during any of the three years. It was reduced by 5% for every 0.01 increase in the mortality index above this threshold and it was completely exhausted if the index exceeded 1.5 times the base level. The payoff schedule is shown in Table 2.1.

The bond was issued via a special purpose vehicle (SPV) called Vita Capital (VC). VC invested the $400m principal in bonds and swapped the income stream on these for a LIBOR-linked cash flow. They distributed the quarterly income to investors and any principle repayment at maturity; see Figure 2.2 for an overview. The benefits of using a SPV are that the cash flows are kept off balance sheet (which is good from Swiss Re's point of view) and the credit risk is reduced (which is good from the investor's point of view).

Payment at maturity (K)	100%-$\sum_k loss_k$	if $\sum_k loss_k < 100\%$
	0%	if $\sum_k loss_k \geq 100\%$
Loss percentage in year k = $loss_k$	0%	if $q_k < 1.3q_0$
	$[(q_k - 1.3q_0)/(0.2q_0)] \times 100\%$	if $1.3q_0 \leq q_k \leq 1.5q_0$
	100%	if $1.5q_0 \leq q_k$
where:	q_0=base index $q_k = \sum_j C_j \sum_i (G^m A_i q^m_{i,j,k} + G^f A_i q^f_{i,j,k})$	
Key:	$q^m_{i,j,k}$=mortality rate (deaths per 100,000) for males in the age group i for country j $q^f_{i,j,k}$=mortality rate (deaths per 100,000) for females in the age group i for country j C_j = weight attached to country j A_i = weight attributed to age group i (same for males and females) G^m and G^f=gender weights applied to males and females respectively The following country weights apply: U.S.A. 70%, U.K. 15%, France 7.5%, Italy 5%, Switzerland 2.5%, male 65%, female 35%	

Table 2.1: Swiss Re mortality bond payoff schedule

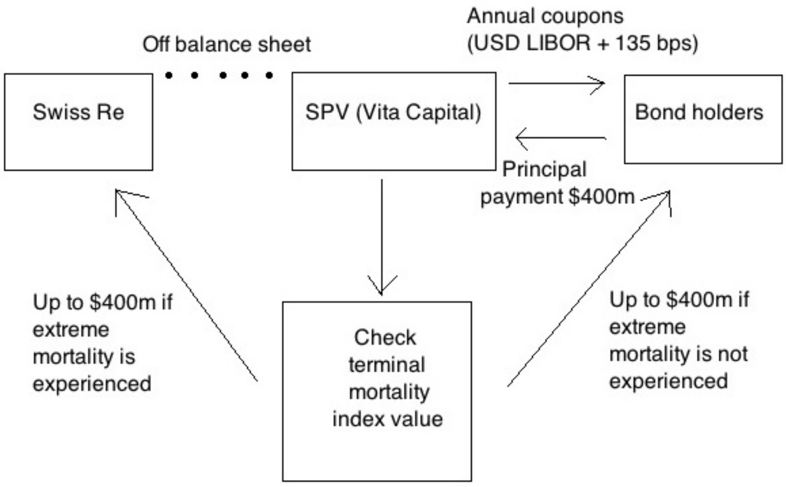


Figure 2.2: The structure of Swiss Re mortality bond

The EIB/BNP longevity bond

In 2004, BNP Paribas announced a long-term longevity bond targeted at pension plans and other annuity providers. The security was to be issued by the European Investment Bank (EIB), with BNP Paribas as the designer and originator and Partner Re as the longevity risk insurer. The 25-year maturity bond had a face value of £540m. The bond was an annuity with floating coupon payments, with the coupon payments linked to a cohort survivor index based on the realised mortality rates of English and Welsh males aged 65 in 2002. The initial coupon was set at £50m.

We will refer to December 31, 2004 as time k=0, and December 31, 2005 as time k=1 etc. Then we have that the survivor index $S(k)$ can be constructed as follows:

$$S(0) = 1$$
$$S(1) = S(0) \times (1 - m(2003, 65))$$
$$S(k) = S(0) \times (1 - m(2003, 65)) \times (1 - m(2004, 66)) \times \ldots \times (1 - m(2002 + k, 64 + k)).$$

where $m(y, x)$ is the crude central death rate for age x published in year y. At each k=1,2,...,25, the bond pays a coupon of £50m × S(k). The cash flows are illustrated in Figure 2.3.

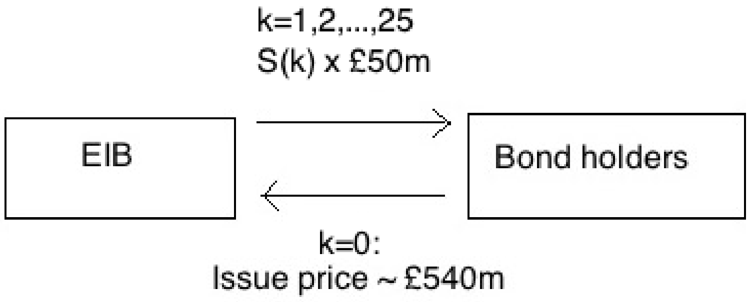


Figure 2.3: Cash flows from the EIB/BNP bond, as viewed by investors

There are also issues of credit risk to consider, which makes everything a bit more complex, see Figure 2.4 for details on the involvement of BNP Paribas and Partner Re.

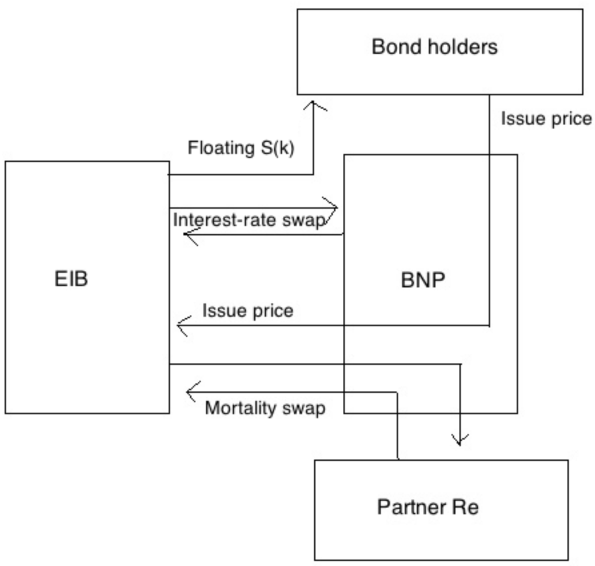


Figure 2.4: Cash flows from the EIB/BNP bond

As we can see, things are much more complicated now. The longevity bond is made up of 3 components.

- A floating rate annuity bond issued by the EIB with a commitment to pay in euros (€).
- A cross-currency interest-rate swap between EIB and BNP Paribas, in which EIB pays floating euros and receives fixed sterling, $\hat{S}(k)$, which has to be set to ensure that the swap has zero value at initiation.
- A mortality swap between the EIB and Partner Re, in which the EIB exchanges the fixed sterling $\hat{S}(k)$ for the floating sterling $S(k)$.

It's a bit more complicated than the Swiss Re bond, and it was withdrawn for redesign in late 2005.

2.3 The Wang Transform

2.3.1 Introduction

The expected utility theory has dominated the financial and insurance economics for the past half century, and it has had a big influence in actuarial risk theory; see [5], [6] or [10]. From this, a dual theory of risk has emerged in the economic literature by Yaari [20] and others.

In finance, the first major pricing theory is the capital asset pricing model (CAPM). We also have option-pricing theory, with among others the widely accepted Black-Scholes formula in [3]. Some researchers noted the resemblance between an option and a stop-loss reinsurance cover, which called for an analogous approach to pricing insurance risks. However we have to remember there are still big differences between the two pricing methods. As the option-pricing methodology defines a price as the minimal cost of setting up a hedging portfolio, the actuarial pricing is based on the actuarial present value of costs and the law of large numbers.

Wang has proposed a method of pricing risk that unifies four different approaches: (i) the traditional actuarial standard deviation loading principle, (ii) Yaari´s economic theory of risk, (iii) CAPM, and (iv) option-pricing theory; see [17]. The method named the Wang Transform is based on distorting the survival function of an insurance risk.

2.3.2 Distortion operators in insurance pricing

Let X be a non-negative loss random variable with cumulative distribution function F_X, and with $S_X = 1 - F_X$ as its survival function. The net insurance premium (excluding other expenses) is

$$E[X] = \int_0^\infty y \mathrm{d}F_X(y) = \int_0^\infty S_X(y)\mathrm{d}y. \tag{2.12}$$

An insurance layer $X_{(a,a+m]}$ of X is defined by the payoff function

$$X_{(a,a+m]} = \begin{cases} 0, & \text{when } 0 \leq X < a, \\ X - a, & \text{when } a \leq X < a+m, \\ m, & \text{when } a+m \leq X, \end{cases} \tag{2.13}$$

where a is the attachment point (also called deductible) and m is the payment limit.

The survival function of this insurance layer is given by S_X as

$$S_{X_{(a,a+m]}}(y) = \begin{cases} S_X(a+y), & \text{when } 0 \leq y < m, \\ 0, & \text{when } m \leq y. \end{cases} \tag{2.14}$$

Hence, the expected loss for the layer $X_{(a,a+m]}$ can be calculated by

$$E[X_{(a,a+m]}] = \int_0^\infty S_{X_{(a,a+m]}}(y)\mathrm{d}y = \int_a^{a+m} S_X(x)\mathrm{d}x. \tag{2.15}$$

Inspired by Venter [16], Wang [19] suggested that the premium could be calculated by transforming the survival function through

$$H_g[X] = \int_0^\infty g[S_X(x)]\mathrm{d}x, \tag{2.16}$$

where the so-called distortion operator g is an increasing function over (0,1) with g(0)=0 and g(1)=1. A distortion operator transforms a probability distribution S_X to a new distribution $g[S_X]$. The mean value

$H_g[X]$ is meant to represent the risk-adjusted premium, expenses excluded. From (2.15) and (2.16), we now get the risk-adjusted premium of a risk layer as

$$H_g[X_{(a,a+m]}] = \int_0^\infty g[S_{X_{(a,a+m]}}(y)]\mathrm{d}y = \int_a^{a+m} g[S_X(x)]\mathrm{d}x. \tag{2.17}$$

For general insurance pricing, the distortion operator g should meet the following criteria:

- $0 < g(u) < 1$, $g(0) = 0$ and $g(1) = 1$,
- $g(u)$ is increasing (where it exists, $g'(u) \geq 0$),
- $g(u)$ is concave (where it exists, $g''(u) \leq 0$),
- $g'(0) = \infty$.

Furthermore, the dual distortion function of g is given by:

$$\tilde{g}(u) = 1 - g(1-u), \qquad \text{u} \in [0,1].$$

2.3.3 The distortion operator

The price of an insurance risk is called a risk-adjusted premium, expenses excluded. Wang has proposed a new distortion operator in the general class of Wang which are transformations that can be applied on (2.16); see [19]. The proportional hazard transform; see [18], is the simplest member of the class with

$$g(x) = x^{\frac{1}{p}}, \qquad p \geq 1. \tag{2.18}$$

Unlike the PH-transform, the new distortion operator is equally applicable to assets and losses.

Let $\Phi(x)$ be the standard normal cumulative distribution function with probability density function

$$f(x) = \frac{d\Phi(x)}{dx} = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

for all x. Wang defines the distortion operator as

$$g_\alpha(u) = \Phi[\Phi^{-1}(u) + \alpha] \tag{2.19}$$

for 0 < u < 1 and a real-valued parameter α. As mentioned, the distortion operator (2.19) can be applied to both assets and liabilities, with opposite signs in the parameter α.

Note that g_α in equation (2.19) satisfies the following criteria:

- The limits are
$$g_\alpha(0) = \lim_{u \to 0+} g_\alpha(u) = 0, \quad \text{and} \quad g_\alpha(1) = \lim_{u \to 1-} g_\alpha(u) = 1.$$
- The first derivative is
$$\frac{dg_\alpha(u)}{du} = \frac{f(x+a)}{f(x)} = e^{-\alpha x - \alpha^2/2} > 0.$$
- The second derivative is
$$\frac{d^2 g_\alpha(u)}{du^2} = \frac{-\alpha f(x+a)}{f(x)^2}.$$
Thus, g_α is concave ($g''_\alpha < 0$) for positive α, and convex ($g''_\alpha > 0$) for negative α.
- For $\alpha > 0$,
$$g'_\alpha(0) = \lim_{0 \to 0+} \frac{dg_\alpha(u)}{du} = \lim_{x \to -\infty} e^{-\alpha x - \alpha^2/2} = +\infty.$$
- The dual distortion operator of g_α is
$$g^*_\alpha(u) = 1 - g_\alpha(1-u) = g_{-\alpha}(u).$$
In other words, a change in the sign of α and we obtain the dual distortion operator. This is due to the symmetry of the standard normal distibution around the origin.

Hence, for $\alpha > 0$, g_α meets all the necessary criteria listed for a desirable distortion operator.

2.3.4 The market price of risk

Lin and Cox applied this method to price mortality risk bonds; see [13]. Changing the sign of (2.19), the Wang transform can be written as
$$g_\lambda(u) = \Phi[\Phi^{-1}(u) - \lambda]. \tag{2.20}$$

Given a distribution with cumulative density function F(t), a "distorted" distribution F^*(t) is determined by λ according to the equation

$$F^*(t) = g_\lambda(F(t)), \tag{2.21}$$

where the parameter λ is called the market price of risk, reflecting the systematic risk of an insurer's liability X. Thus, the Wang transform will produce a "risk-adjusted" density function F^* for an insurer's given liability X.

2.3.5 Using the Wang Transform

Under the new probability measure, E^*(X) will define a risk-adjusted "fair-value" of X, which can be discounted to time zero using the risk-free rate. In terms of an annuity of the form (2.1) the formula for the price can be written

$$H(X, \lambda) = E^*(X) = s \sum_{k=0}^{n-1} d^k {}_k p^*_{l_0}, \tag{2.22}$$

where ${}_k p^*_{l_0}$ is the risk-adjusted survival probabilities obtained from Wang's transformation. Combining (2.20) and (2.21) we get

$$\begin{aligned} {}_k p^*_{l_0} &= g_\lambda({}_k p_{l_0}) \\ &= \Phi[\Phi^{-1}({}_k p_{l_0}) - \lambda] \\ &= \Phi[\Phi^{-1}(1 - {}_k q_{l_0}) - \lambda]. \end{aligned} \tag{2.23}$$

The Wang transformation adjusts the mortalities from the population average. The selection bias introduced in section 2.1.5 can now be reduced. For the transformation to be of good use, the mortalities have to shift downwards, meaning that under the distorted mortalities, people live longer. This is obtained for $\lambda > 0$. With the increase in longevity that are present, the historical data becomes obsolete fast. Applying the Wang Transform with a λ of own choice might conceivably be a good way to adjust the old mortalities, but what value of λ is to be chosen?

Chapter 3

Pricing life annuities

3.1 Introduction

When a life annuity is issued the issuer has to calculate a price for the future payments. This is usually done using the Actuarial Present Value (APV), which is the expected value of the present value of a random cash flow. As mentioned in section 2.1.4 it is often calculated using the principle of equivalence. The probability of a future payment is based on assumptions about a person's future mortality, estimated using a life table. The price can be found numerically.

Algorithm 1: Present value of life annuities

0. Input: $l_0, K, d = 1/(1+r), \{q_l\}, s$
1. $\ddot{a} \leftarrow 0$, $p \leftarrow 1$, $l \leftarrow l_0 - 1$
2. for $k = 0, 1, \ldots, K-1$ repeat
3. $\quad \ddot{a} \leftarrow \ddot{a} + p \quad$ and $\quad l \leftarrow l + 1$
4. $\quad p \leftarrow p(1 - q_l)d \qquad$ % Recall that ${}_kp_{l_0} = (1 - q_{l_0+k-1})_{k-1}p_{l_0}$
5. $a \leftarrow \ddot{a} + p - 1$
6. Return $s \cdot \ddot{a}$ and $s \cdot a$.

This is $\ddot{a}$ and a from equation (2.1) and (2.2)

The concept will be used to estimate the market price of risk λ. Using a mortality table and known prices of annuities, λ can be estimated numerically by solving equation (2.22) for λ.

$$
\begin{aligned}
H(X, \lambda) &= s \sum_{k=0}^{n-1} d^k {}_k p^*_{l_0} \\
&= s \sum_{k=0}^{n-1} d^k \Phi[\Phi^{-1}(1 - {}_k q_{l_0}) - \lambda]. \qquad (3.1)
\end{aligned}
$$

Algorithm 2: Market Price of Risk

0. Input: $d = 1/(1+r), \{q_l\}, s, l_0, l_e, gender$
1. $L = function(\lambda, input)$
2. $K = l_e - l_0$
3. If (gender=male) then $q \leftarrow q_{male}$ else $q \leftarrow q_{female}$
4. $H(X, \lambda) \leftarrow s \sum_{k=0}^{K} d^k \Phi[\Phi^{-1}(1 - {}_k q_l) - \lambda]$ $\quad \% Equation\,(3.1)$
5. list $H(X, \lambda)$
6. Solve $L(\lambda, input)$ for λ given $H(X, \lambda)$
%This can be done using *uniroot* in R

s, l_0 and gender are variables, others kept fixed.

We will then apply the Wang Transform with the obtained λ's on the mortality table as in equation (2.23), and plot the two distributions to compare the actual distribution to the transformed distribution.

The objebtive is to look at the stability of λ. As mentioned earlier, the market price of risk is reflecting the systematic risk of an insurer's liability X. For the Wang Transform to be a universal framework, λ has to be stable.

It is reasonable to think that $\lambda = \lambda_{l_0,g}$ such that it depends on age, but also on gender. If a 25 year old female and a 45 year old male want the same contract, it is reasonable to think that the young female is a bigger risk to the company. There is larger uncertainty about her future, in addition females have a tendency to live longer than males.

3.2 Detailed procedure

To obtain a life table we use the 1996 IAM 2000 Mortality Table; see A.1 or [11]. We will assume a technical rate of interest r of 3% and 6% to get the discount rate $d = 1/(1 + r)$. Best's Review gives us the prices for Single Premium Immediate Annuities (SPIA's) for 99 different companies; see [12]. With prices from Canada Life (CL), Franklin Life (FL), Hartford Life (HL) and Nationwide Insurance (NI); see Table 3.3, we will use Algorithm 2 to get the market price of risk by solving the following equation numerically:

$$\pi = s * 12 \sum_{k=0}^{n-1} d^k \Phi[\Phi^{-1}(1 - {}_k q_{l_0}) - \lambda]. \tag{3.2}$$

The prices in Best's review are monthly payouts on a single premium immediate annuity with a one-time premia of \$100,000. This means that the annuitant pays a lump sum, and then the benefit payouts start immediately after. Since the prices are monthly, but the mortalities are one-year mortalities, s is multiplied with 12.

The prices are different between the companies, but also inside each company the prices vary for the different ages and type of gender. We will get one λ for each price, but as we only have prices for six different age groups we will have to use interpolation and extrapolation for the remaining ages when we plot the distorted survival functions. In Figure 3.1, the black circles represent the price one would get from Canada Life when signing a contract at the age $x = 55, 60, 65, 70, 75$ and 80.

3.2.1 Interpolation

In the mathematical field of numerical analysis, interpolation is a method of constructing new data points within the range of already known data points. In Figure 3.1 we want to find the values for the red dots. There are several ways of doing so, the one more complex than the other, but we will stick to the very simplest.

Piecewise constant interpolation
This is also called nearest-neighbor interpolation. The method is to

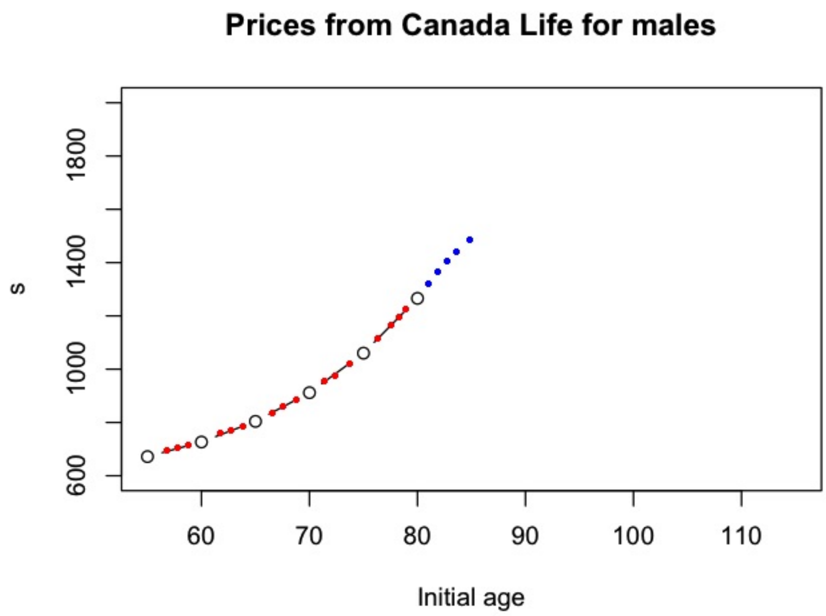


Figure 3.1: Prices from Canada Life for males

locate the nearest data value, and assign the same value. In simple problems, this method is unlikely to be used as linear interpolation is almost as easy, but in higher dimensions, this could be a good choice for its speed and simplicity.

Linear interpolation

This is one of the simplest interpolation methods. It takes two data points and find the weighted average between them. Say that we have (x_1, y_1) and (x_3, y_3) and wants to find y_2. Then we use the following formula:

$$y_2 = \frac{(x_2 - x_1)(y_3 - y_1)}{(x_3 - x_1)} + y_1. \tag{3.3}$$

The slope between x_1 and x_2 will now be the same as the slope between x_1 and x_3. Linear interpolation is quick and easy, but not very precise. We could use polynomial interpolation or spline interpolation instead, but it depends on how important the error is, see [14] for more on this.

We will use the linear interpolation method on the prices from Best's review to estimate λ's for each age $x \in$ (55,80), and then plot the distorted survival probabilities.

Algorithm 3: Interpolation

0. Input: x= age vector, y= price vector, n=length(x)
1. $P = function(x, y)$
2. for $i = 1, \ldots, n$ repeat
3. $y_i = \frac{(x_i - x_1)(y_n - y_1)}{(x_n - x_1)} + x_1$
4. list y

age and *price* are divided into 5 groups, each group containing two known prices as its end points. Run the algorithm separately for the 5 groups and merge the price vectors into one.

3.2.2 Extrapolation

In mathematics, extrapolation is the process of estimating beyond the original observation range. In Figure 3.1 we want to estimate values for the blue dots. It is similar to interpolation, but subject to greater uncertainty and a higher risk of producing meaningless results. Extrapolation may also apply to human experience, granting that one expand known experience into an area not known, e.g. a driver extrapolates the road outside their sight when driving.

Linear extrapolation
It is almost the same as linear interpolation, but now we create a tangent line at the end of the known data and extend it beyond the limit. A good result will only be provided when used on a fairly linear function or not too far beyond the known data.

If the two data points nearest the point x_3 to be extrapolated are (x_1, y_1) and (x_2, y_2), linear extrapolation gives the formula:

$$y_3 = \frac{(x_3 - x_2)(y_2 - y_1)}{(x_2 - x_1)} + y_1. \tag{3.4}$$

We will use extrapolation on the ages $x \in$ (80,115), but as this group is unlikely to invest their savings in a SPIA, we will instead use nearest-point extrapolation and assign all the ages the same price as age 80. This will lead to a little lower benefit than they probably would get if signing a contract, but that means the company issuing the SPIA will gain on average. When inserted in the Wang transform, the prices are used on different lengths of annuities (the mortalities used will differ from the different ages), so we will still get different values of λ.

3.3 Results and discussion

Before we analyse the results, some assumptions will be made. It is expected that the market price of risk goes down as the age goes up. This is because the older you are, the fewer expected payouts will there be in the future. When we get to the older age groups, the "risky" people have usually already died. The selection bias will then be small, as the mortalities for the group of annuitants don't deviate too much from the country average anymore. It might also be a higher market price of risk for females than for males, as females have a longer life expectancy, and hence more expected payouts in the future.

The market price of risk for males and females are shown in Table 3.1 and Table 3.2 for the two different interest rates. Figures (3.2)-(3.5) are plots of the same values. As mentioned in section 2.3.5 for the transformed mortalities to be of good use we will have to have $\lambda > 0$. Then the mortalities will go down, implying a longer expected lifetime.

Canada Life
Starting with Canada Life consider Figure 3.2. When $r = 3\%$, females have a higher price of risk than males, as expected. The ratio of the risks decreases with age, probably coming from the fact that the uncertainties inside the gender groups become smaller as the age goes up. We also note that the market price of risk is decreasing as the age is increasing. Currently, our assumptions are fulfilled, but when the discount $r = 6\%$, things change.

Now males are more risky, which seems odd, as the risk shouldn't change between groups just because of a change in the discount. The

Different values of the market price of risk, r=3%

	Males				Females			
	CL	FL	HL	NI	CL	FL	HL	NI
55	1.117	0.934	1.052	0.917	1.261	1.080	1.202	1.095
60	0.981	0.782	0.914	0.788	1.098	0.892	1.025	0.945
65	0.842	0.633	0.780	0.658	0.938	0.712	0.862	0.796
70	0.712	0.505	0.654	0.546	0.781	0.541	0.711	0.652
75	0.604	0.403	0.564	0.480	0.632	0.393	0.575	0.520
80	0.517	0.331	0.509	0.457	0.504	0.273	0.477	0.426

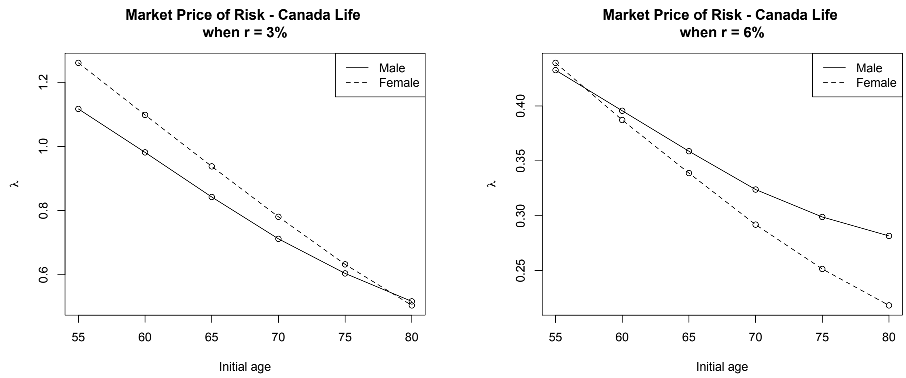
Table 3.1: Examples of λ evaluations obtained using the Wang Transform with r=3%

Different values of the market price of risk, r=6%

	Males				Females			
	CL	FL	HL	NI	CL	FL	HL	NI
55	0.433	0.036	0.301	-0.007	0.439	-0.041	0.299	0.006
60	0.396	0.019	0.276	0.032	0.387	-0.081	0.235	0.053
65	0.359	0.012	0.260	0.055	0.339	-0.098	0.202	0.076
70	0.324	0.018	0.241	0.081	0.292	-0.109	0.182	0.083
75	0.299	0.029	0.247	0.134	0.251	-0.099	0.171	0.092
80	0.282	0.050	0.271	0.208	0.218	-0.086	0.183	0.118

Table 3.2: Examples of λ evaluations obtained using the Wang Transform with r=6%

ratio of the risks are also increasing with age, something that isn't expected. Other than that, the market price of risk still decreases with age, so that assumption still holds true. Also, we notice that $\lambda > 0$ for both discount rates and genders, so the transformed mortalities will be of good use.



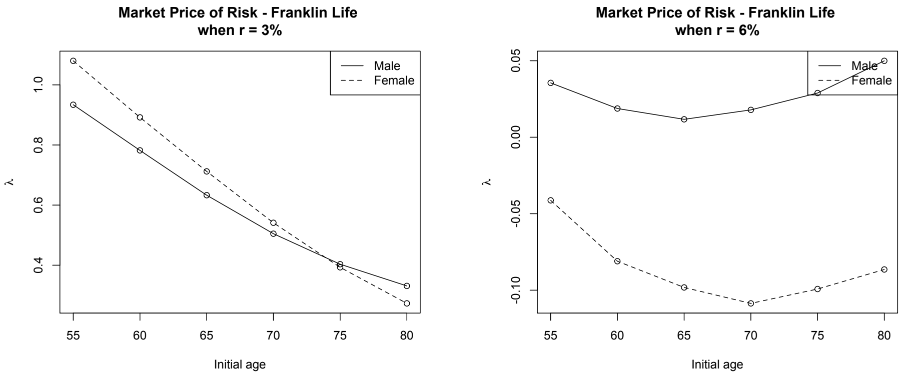
(a) Canada Life 3% (b) Canada Life 6%

Figure 3.2: Prices from Canada Life

Franklin Life

The next example is Franklin Life in Figure 3.3. The 3% discount produces the expected. Females have higher risk than males, and the ratio decreases with age. At the age of 75 males become of more risk, but this is just because it is a small age group with little data to base our calculations on. Also, the market price of risk decreases with age, and all λ's > 0.

For the 6% discount, we get that all λ's ≈ 0, and for females we also get $\lambda < 0$, which shouldn't be. Then we will get an upward shift in the mortality curve, meaning that the group of females we look at have shorter expected lifetime. In Figure A.2 we have plotted the transformed mortalities against the actual distribution. As we can see, the transformed mortalities have become higher, which will lead to severe underestimation of the need of liquidity. Also note that the risk for both gender starts with a decrease, before it ends with an increase.

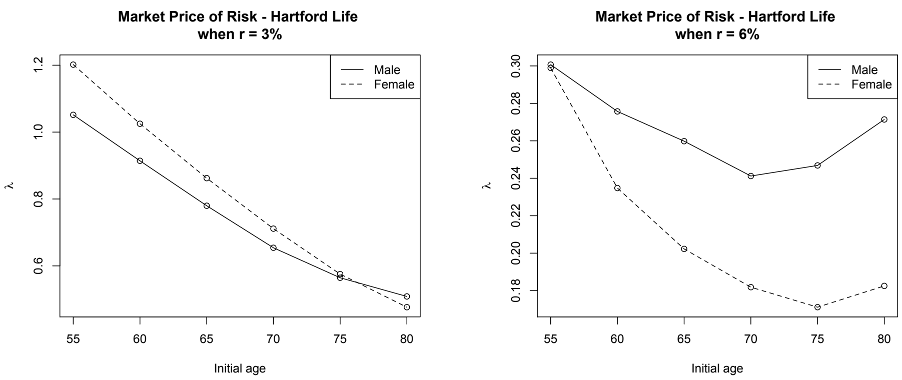


(a) Franklin Life 3% (b) Franklin Life 6%

Figure 3.3: Prices from Franklin Life

Hartford Life

Hartford Life in Figure 3.4 has expected values for $r = 3\%$. Just as the other two, the risk decreases with age, females are of higher risk than males and all $\lambda's > 0$. For the 6% discount, we get that the risk increases after the age of 75, also males are of much higher risk, again something implausible.

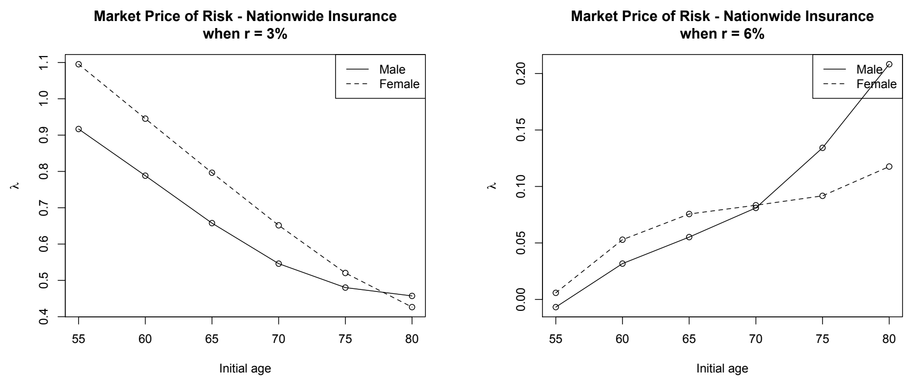


(a) Hartford Life 3% (b) Hartford Life 6%

Figure 3.4: Prices from Hartford Life

Nationwide Insurance

At last we come to Nationwide Insurance in Figure 3.5. The 3% values are reasonable, with all assumptions looking OK, but the 6% values are the opposite of what we expect. Except from the fact that females are of higher risk than males until the age of 70, we get that the risk increases with age, and we even get $\lambda < 0$ for a male aged 55. Hence, the 6% discount doesn't seem to give good values.



(a) Nationwide Insurance 3%　　(b) Nationwide Insurance 6%

Figure 3.5: Prices from Nationwide Insurance

Comparison

Now, we want to compare the values between the companies. When using the Wang Transform to distort the mortalities we need a value of λ for each age and gender, but what values to choose?

Looking at the 3% discount values for males we get big discrepancies inside each age group. Of course, this comes from the fact that the different companies have different prices, also with big discrepancies there. Canada Life and Hartford Life have chosen to give their annuitants higher benefit payouts than Franklin Life and Nationwide Insurance. Hence, they get a higher market price of risk as well. This can come from several facts, but Franklin Life and Nationwide Insurance have probably used a higher loading in their calculations, and by that assigning a higher risk to their customers than the other two. We see that the same tendency fits for the females as well.

The 6% discount values for males have the same tendency. Canada Life and Hartford Life have a much higher market price of risk than the other two. Because of the increase in λ, Nationwide Insurance gets a lot closer to the other two in this scenario. Unlike the other, Franklin Life has values close to 0 for all age groups, implying that their customers are of little risk, in other words, they take a much higher loading than the other three.

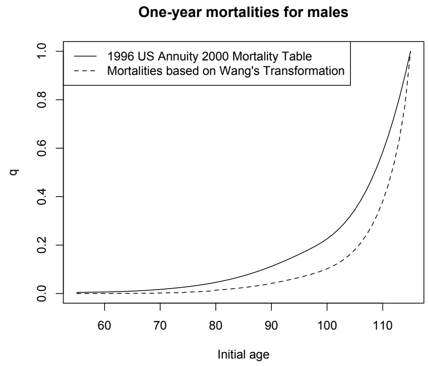
When we look at the 6% values for females we still have that Canada Life and Hartford Life have the highest market price of risk, but now all the values for Franklin Life < 0. This is not good, using transformed mortalities based on this will lead to big underestimation. Again we have that Nationwide Insurance starts around zero, a lot less than the other two, but the increase in risk decreases the ratio.

It seems as though the Wang Transform works for the discount rate of 3%, but neither of the results for 6% is as expected. Therefore, when we use the market price of risk further to compare the two mortality distributions, we will only use $r = 3\%$.

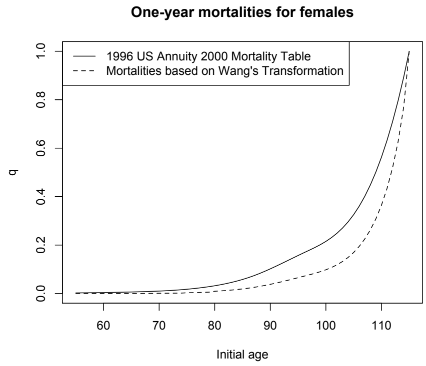
Prices from Best's review [12]

	Males				Females			
	CL	FL	HL	NI	CL	FL	HL	NI
55	671.70	612	649	607	627.13	575	609	579
60	726.44	656	701	658	669.96	607	646	622
65	804.02	720	777	729	729.13	654	702	680
70	911.69	813	882	831	812.49	722	784	761
75	1060.03	943	1035	985	936.41	827	908	882
80	1265.68	1129	1259	1219	1118.95	984	1101	1070

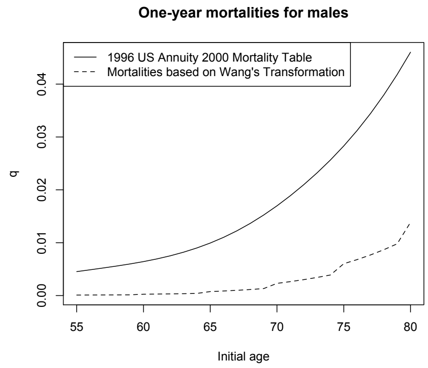
Table 3.3: Single Premium Immediate Annuities as of May 1, 1996
Lifetime Only Option - $100,000 Single Premium



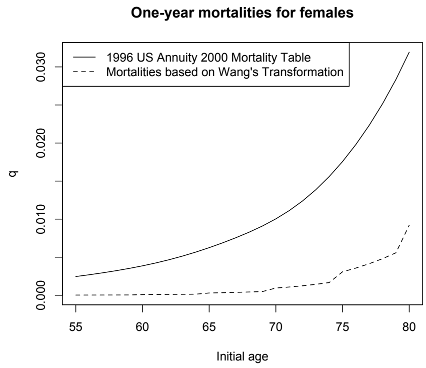
(a) Canada Life Males



(b) Canada Life Females



(c) Canada Life Males

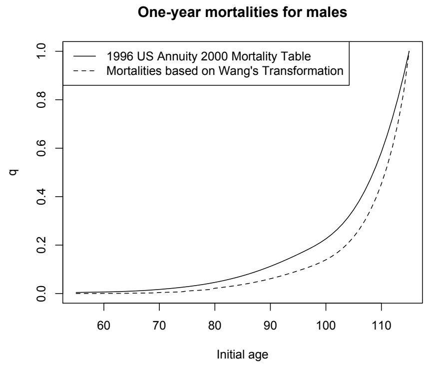


(d) Canada Life Females

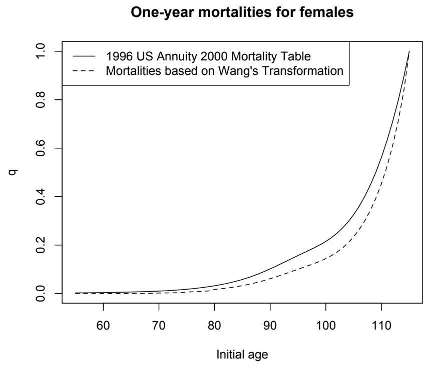
Figure 3.6: Wang transform used on Canada Life

The transformed mortalities

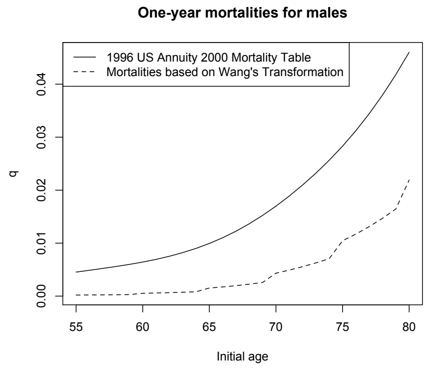
Figures 3.6 - 3.9 shows us the 1996 IAM 2000 Mortality Table plotted against the new distorted distribution for the four different companies. As we can see, all the transformed distributions have reduced mortalities. This is what we want as life annuity customers usually have a better expected survival than the country average. We can think of the mortality table as the actual distribution, which requires a distortion to obtain market prices. That is, a risk premium is required for pricing annuities.



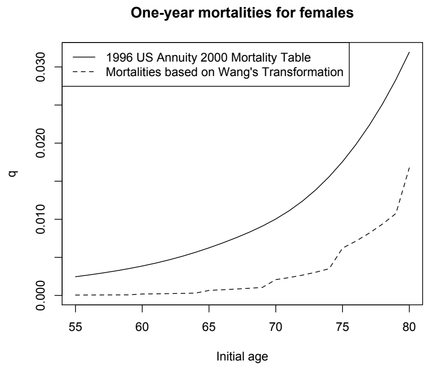
(a) Franklin Life Males



(b) Franklin Life Females



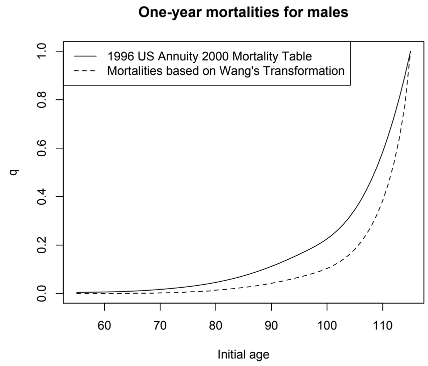
(c) Franklin Life Males



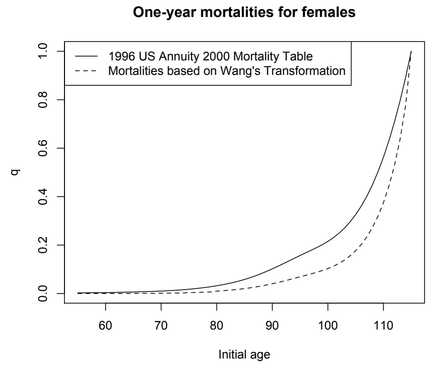
(d) Franklin Life Females

Figure 3.7: Wang transform used on Franklin Life

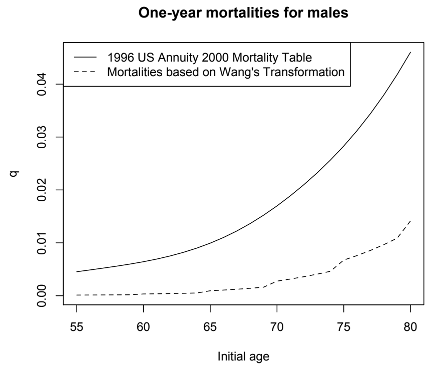
The male and female mortalities are plotted in different plots for an easier view. Remembering back to section 3.2.2, we chose to assign the same price to all ages x > 80 when we extrapolated. Because of this, and also because annuitants at this age usually don't make annuity contracts at this time, we have chosen to look at the cropped plots for the mortality distributions as well. The mortality plots with $x \in (55, 115)$ aren't easy to interpret for the ages under 80. For the ages $x \in (55, 70)$ it looks as though the distorted mortalities are approximately the same as the original. Cropping the plot and looking at $x \in (55, 80)$ we see that this really isn't the case. The discrepancies are now easier to see.



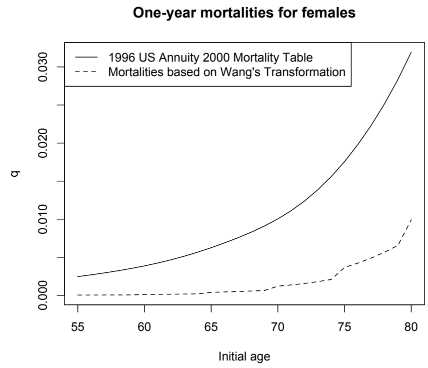
(a) Hartford Life Males



(b) Hartford Life Females



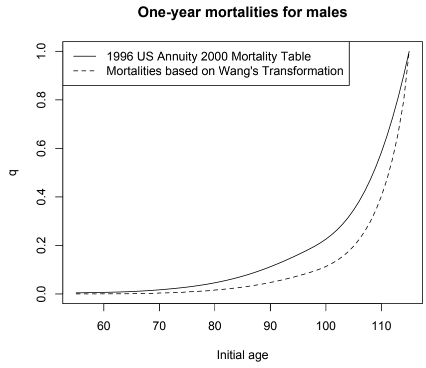
(c) Hartford Life Males



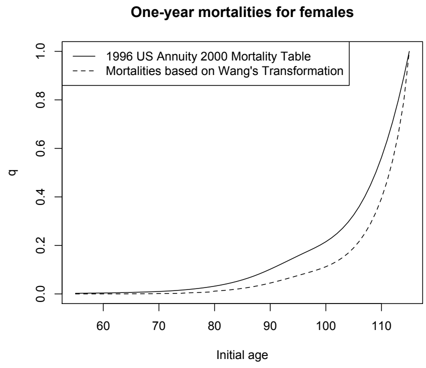
(d) Hartford Life Females

Figure 3.8: Wang transform used on Hartford Life

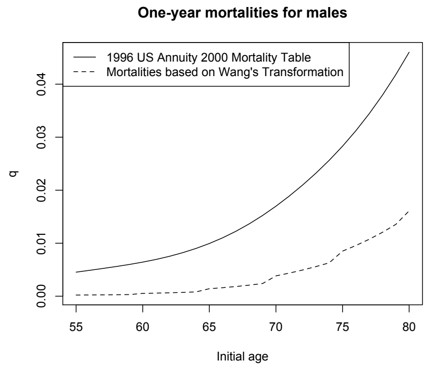
Remembering that Canada Life and Hartford Life had higher values of λ, we notice that their distorted distributions have lower mortalities than Franklin Life and Nationwide Insurance. Comparing the distributions between males and females, we also notice that the female mortality distribution have lower values than the males. This comes from the fact that the original mortalities was smaller to begin with, and also that the market price of risk was higher, so we subtract a higher value in the transformation.



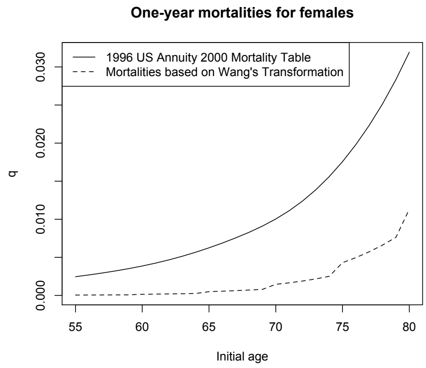
(a) Nationwide Insurance Males



(b) Nationwide Insurance Females



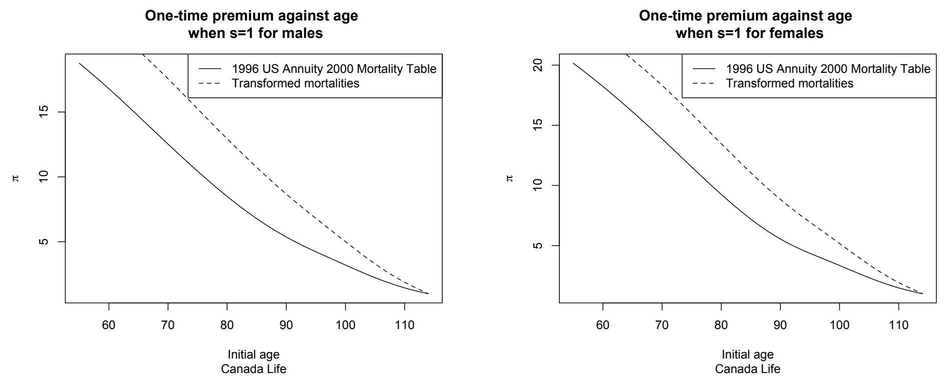
(c) Nationwide Insurance Males



(d) Nationwide Insurance Females

Figure 3.9: Wang transform used on Nationwide Insurance

Let's say that the 1996 US Annuity 2000 Mortality Table is the data a company has access to, and that these data are obsolete. By using the Wang transform (2.23) on them we get transformed mortalities. The risk-adjusted mortalities are fulfilling what we need to price annuities, and we will now use Algorithm 1 to calculate the one-time premium of a life annuity that pays s=1 money unit/year, when using both distributions.



(a) Canada Life Males (b) Canada Life Females

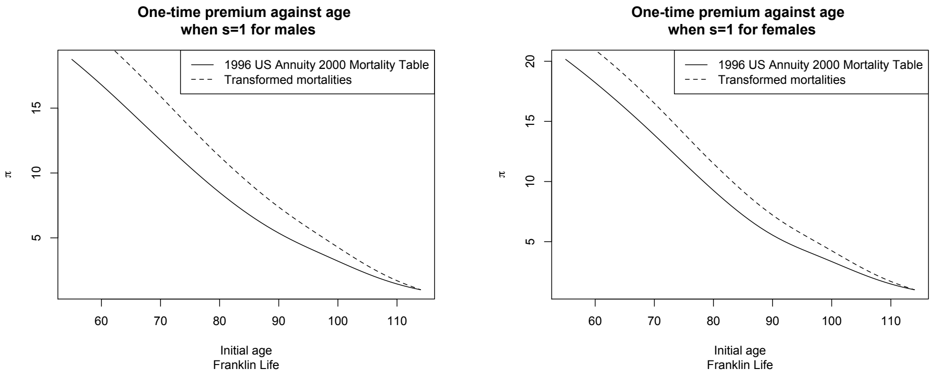
Figure 3.10: One-time premium for an annuity where $s = 1$, based on Canada Life's transformed mortalities

3.3.1 Using the transformed mortalities in annuities

Figures 3.10 - 3.13 shows us the result when we apply the market price of risk in Table 3.1. As we can see, if the company had used the obsolete data set they would have underestimated the premium, which again would lead to their reserve being to small. Hence, using an obsolete data set could cause a company to go bankrupt.

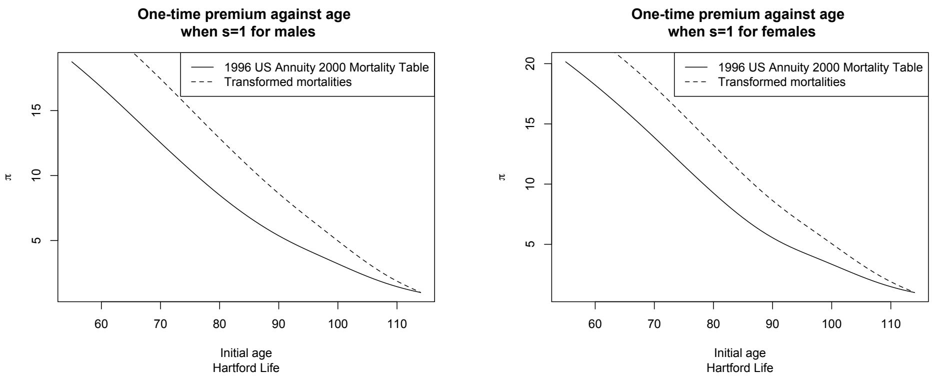
We also note that the one-time premium is higher for females than for males. This is because the distribution phase in this contract lasts until death. Not separating between gender when using a mortality table would lead to severe underestimation for the female clients, and overestimation for the male clients. If one is lucky, the over- and underestimation can hedge each other, but it is unlikely that this hedge is perfect. Hence, it is important to separate between male and female mortalities during calculations.

We also notice that the one-time premium obtained when using the risk-adjusted mortalities are a bit higher for Canada Life and Hartford Life, than for Franklin Life and Nationwide Insurance. As mentioned earlier this comes from the fact that the latter two takes a higher loading in their contracts, which probably reduces their risk.



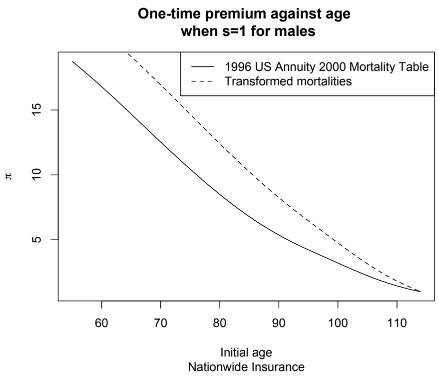
(a) Franklin Life Males (b) Franklin Life Females

Figure 3.11: One-time premium for an annuity where $s = 1$, based on Franklin Life's transformed mortalities

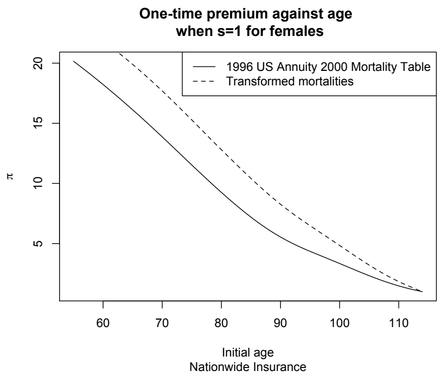


(a) Hartford Life Males (b) Hartford Life Females

Figure 3.12: One-time premium for an annuity where $s = 1$, based on Hartford Life's transformed mortalities



(a) Nationwide Insurance Males



(b) Nationwide Insurance Females

Figure 3.13: One-time premium for an annuity where $s = 1$, based on Nationwide Insurance's transformed mortalities

Chapter 4

Pricing mortality bonds

4.1 Introduction

As mentioned in Section 2.3.4, Lin and Cox applied the transformed mortality distribution obtained from the Wang Transform to price mortality risk bonds. We will look back at the example given in Section 2.2.2. The mortality risk bond here can also be called a longevity bond because the hedge is against too high payments in annuities, which arise when the mortaliy rate have been overestimated - or in other words when the increase in longevity is higher than expected.

4.2 Mathematics

4.2.1 The bond price

In the bond market, we have cash flows $\{D_k\}$ given by equation (2.10). This gives us that the bond price of a mortality bond with face value F can be written

$$V = Fd(0,K) + \sum_{k=1}^{K} E^*[D_k]d(0,k). \tag{4.1}$$

The face amount F is not at risk, it will be paid at time K regardless of the number of surviving annuitants. We will use the same discount factor as in Chapter 3, i.e. $r = 3\%$. The survival distribution will be the one we derived with the Wang Transform in Chapter 3. We will only use the survival distribution obtained with the prices from Canada Life.

4.2.2 The mortality bond strike levels X_k

The contract are set at different strike levels X_k. We will use the same strike levels as Lin and Cox, which they derived using the Renshaw, Haberman and Hatzopoulos method to predict the force of mortality; see [15]. The improvement levels were determined by the average of 30-year force of mortality improvement forcecast for the age groups 65-74, 75-84 and 85-94. That gave the improvement levels in Table 4.1.

Age group	Change of force of mortality
65-74	-0.0070
75-84	-0.0093
85-94	-0.0103

Table 4.1: The improvement levels to determine the strike levels

Now we can determine the strike levels X_k:

$$X_k = \begin{cases} n_x \cdot {}_kp_x \cdot e^{0.0070t}, & \text{for } k = 1, \dots, 10, \\ n_x \cdot {}_kp_x \cdot e^{0.07} e^{0.0093(t-10)}, & \text{for } k = 11, \dots, 20, \\ n_x \cdot {}_kp_x \cdot e^{0.163} \cdot e^{0.0103(t-20)}, & \text{for } k = 21, \dots, 30, \end{cases} \tag{4.2}$$

where ${}_kp_x$ is the survival probabilities from the 1996 IAM 2000 Annuity table.

4.2.3 The coupon payments D_k

Now we need to calculate $E^*[D_k]$. From (2.10), the coupon payment can be written as

$$\frac{1}{1000}D_k = \begin{cases} 0, & \text{if } n_{x+k} > X_k + X, \\ C + X_k - n_{x+k} & \text{if } X_k < n_{x+k} \leq X_k + C, \\ C, & \text{if } n_{x+k} \leq X_k, \end{cases} \tag{4.3}$$

$$= C - (n_{x+k} - X_k)_+ + (n_{x+k} - X_k - C)_+. \tag{4.4}$$

Hence,

$$\frac{1}{1000}E^*[D_k] = C - E^*[(n_{x+k} - X_k)_+] + E^*[(n_{x+k} - X_k - C)_+]. \tag{4.5}$$

4.2.4 Calculation

We have that the distribution of n_{x+k} is the distribution of the number of survivors from n_x who survive to age $x+k$, which occurs with probability ${}_kp_x^*$. Therefore n_{x+k} has a binomial distribution with parameters n_x and ${}_kp_x^*$. Since n_x is a large value, we have that n_{x+k} is approximately normally distributed with mean $E^*[n_{x+k}] = \mu_k^* = n_x \cdot {}_kp_x^*$ and variance $V^*[n_{x+k}] = \sigma_k^{*2} = n_x \cdot {}_kp_x^* \cdot (1 - {}_kp_x^*)$.

Integrating by parts, we get that for a random variable X with $E[X] < \infty$:

$$\begin{aligned} E[(X-a)_+] &= \int_a^\infty [1 - F(t)]\mathrm{d}t \\ &= \int_a^\infty [1 - \Phi(t)]\mathrm{d}t. \end{aligned}$$

We can write this as

$$\begin{aligned} \Psi(a) &= \int_a^\infty [1 - \Phi(t)]\mathrm{d}t \\ &= \phi(a) - a[1 - \Phi(a)]; \end{aligned}$$

see [13] for more details. As the functions $\phi(a)$ and $\Phi(a)$ are easy to calculate, we now express $E^*[D_k]$ in terms of them:

$$E^*[D_k] = 1000 \cdot \{C - \sigma_k^*[\Psi(a_k) - \Psi(a_k + C/\sigma_k^*)]\}, \tag{4.6}$$

where $a_k = (X_k - \mu_k^*)/\sigma_k^*$.

Inserting (4.6) in (4.1), the bond price V can be calculated. Letting $\lambda_{65,m}$=0.842 and $\lambda_{65,f}$=0.938, we find that the mortality bond price when we assume that n_{65}=10,000 for each gender, F=10,000,000 and C=0.07, is V_{male}=4,119,868 and V_{female}=4,120,117.

Using that the face value of the straight bond W=10,000,000, we can calculate the premium P that the insurer pays the SPC. In Section 2.2.2 we mentioned that SPC would perform on its insurance and commitments given that P+V was at least equal to W. Lin and Cox sets W=10,000,000 so we will do the same. This gives that $P_{male} = 5,880,132$ and $P_{female} = 5,879,883$. They also state that the total premium from annuitants is $\pi_{male} = 99,650,768$ and $\pi_{female} = 107,232,089$. Comparing the total immediate annuity premium the insurer collects from its annuitants, the reinsurance premium the insurer pays the SPC is only a proportion of the total annuity premium: 5.9% for males and 5.5% for females.

Lin and Cox of course get other values as their values for λ differs a lot from ours. They have used other values for the discount, and may also have used different calculations. This indicates that λ is not so stable.

Chapter 5

Discussion with possible extensions

We have looked at the stability of the market price of risk λ obtained from Wang's Transform. It seems as a good idea to transform the mortalities so they have a shift downwards compared to the country average. As the group buying annuities often have a longer life expectancy than the country average, it can be a large underestimation in the reserve when using the mortalities of a country. To find a value of $\lambda_{l_0,g}$ for $l_0 \in (55, 80)$ and $g \in (male, female)$, we used prices of annuities to solve (2.23) numerically.

There were big differences between the gender groups and age groups just by a little change in the discount, implying that there would be difficult to find universal values of λ. When we used $r = 3\%$, all our assumptions were OK, so we used the market price of risk obtained with that discount in our further calculations.

To plot the two mortality distributions against each other, we had to interpolate and extrapolate the prices to find values of λ for all ages $x \in (55, 115)$. The transformed mortality distributions had a shift downwards from the actual distribution, just as we wanted for pricing life annuities. Even thought the value of λ may not be "the right one", the transformed mortalities are better to use than the historical ones as they come from a data set that may be obsolete.

We calculated the one-time premium of a life annuity with benefit payments $s = 1$ money unit/year until death. We got that the transformed mortalities gave a much higher premium than the historical mortalities. An insurance company is obliged to have a reserve for future payments, and if the distorted mortalities are closer to the real ones than the historical ones, a company only using the historical data could risk bankruptcy. Hence, risk due to mortality is important to take serious.

One way for campanies to cover some of their risk is to use a loading that covers more than just the expences, which we saw that Franklin Life and Nationwide Insurance probably was doing. This lead to their market price of risk being smaller than for Canada Life and Hartford Life. If they in addition had used the market price of risk from one of these companies instead of their own, they would get a much higher one-time premium. If the pension holders are willing to pay this price for the annuity, they will have a good cover of future risk.

So the stability of λ was not present between companies. To use the Wang Transform with a decided value of λ isn't difficult, but the universal market price of risk is not present. One have to be careful not to think that just by using the Wang Transform with a random λ, future risk is covered.

Lin and Cox suggested to use the risk-adjusted mortalities to price a mortality bond. We did so using $\lambda_{65,g}$ from Canada Life, and got that mortality bonds could be a good way of hedging ones mortality risk. In our calculations, we got that just a little proportion of the total annuity premium would go to pay the reinsurance premium. If the annuitants lived longer than expected, the issuer would get parts of the excess covered, up to a maximum amount.

Again, as we got values different than Lin and Cox, the stability of λ is not very good. It can be a smart tool to handle mortality risk, but the uncertainties are too big to use it alone, without an extra loading and one should also probably adjust the value a little higher just to be safe.

We chose to calculate the market price of risk λ by using $r = 3\%$. Possible extensions to this thesis could be to do the calculations with other values and methods of discounting. As mentioned in Section 2.1.3 one could also use the fair value discounting with the market yield curve. One possibility is also to use stochastic interest rates obtained by using e.g. Vašiček or Black-Karasinski.

Also, one can use different mortality tables in the calculations. Using Algortihm 2, one can calculate the market price of risk for different mortality tables and different prices of annuities, and see whether there are a trend in the values or if they are all over the place. If there is a trend, one can look at this and use the average value as the universal value for each age and gender.

Appendix A

Appendix

A.1 1996 IAM 2000 Mortality Table

Age	Male	Female
1	0	0
2	0	0
3	0	0
4	0	0
5	0.291	0.171
6	0.27	0.141
7	0.257	0.118
8	0.294	0.118
9	0.325	0.121
10	0.35	0.126
11	0.371	0.133
12	0.388	0.142
13	0.402	0.152
14	0.414	0.164
15	0.425	0.177
16	0.437	0.19
17	0.449	0.204
18	0.463	0.219
19	0.48	0.234
20	0.499	0.25
21	0.519	0.265

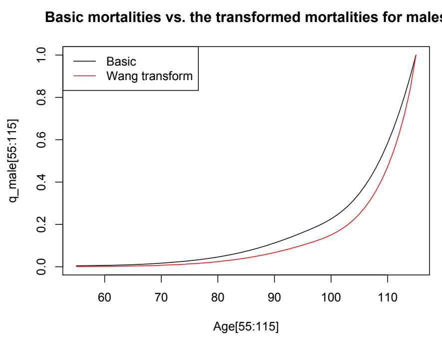
22	0.542	0.281
23	0.566	0.298
24	0.592	0.314
25	0.616	0.331
26	0.639	0.347
27	0.659	0.362
28	0.675	0.376
29	0.687	0.389
30	0.694	0.402
31	0.699	0.414
32	0.7	0.425
33	0.701	0.436
34	0.702	0.449
35	0.704	0.463
36	0.719	0.481
37	0.749	0.504
38	0.796	0.532
39	0.864	0.567
40	0.953	0.609
41	1.065	0.658
42	1.201	0.715
43	1.362	0.781
44	1.547	0.855
45	1.752	0.939
46	1.974	1.035
47	2.211	1.141
48	2.46	1.261
49	2.721	1.393
50	2.994	1.538
51	3.279	1.695
52	3.576	1.864
53	3.884	2.047
54	4.203	2.244
55	4.534	2.457
56	4.876	2.689
57	5.228	2.942
58	5.593	3.218
59	5.988	3.523
60	6.428	3.863
61	6.933	4.242
62	7.52	4.668

63	8.207	5.144
64	9.008	5.671
65	9.94	6.25
66	11.016	6.878
67	12.251	7.555
68	13.657	8.287
69	15.233	9.102
70	16.979	10.034
71	18.891	11.117
72	20.967	12.386
73	23.209	13.871
74	25.644	15.592
75	28.304	17.564
76	31.22	19.805
77	34.425	22.328
78	37.948	25.158
79	41.812	28.341
80	46.037	31.933
81	50.643	35.985
82	55.651	40.552
83	61.08	45.69
84	66.948	51.456
85	73.275	57.913
86	80.076	65.119
87	87.37	73.136
88	95.169	81.991
89	103.455	91.577
90	112.208	101.758
91	121.402	112.395
92	131.017	123.349
93	141.03	134.486
94	151.422	145.689
95	162.179	156.846
96	173.279	167.841
97	184.706	178.563
98	196.946	189.604
99	210.484	201.557
100	225.806	215.013
101	243.398	230.565
102	263.745	248.805
103	287.334	270.326

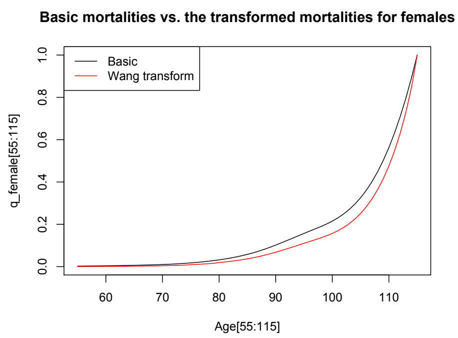
104	314.649	295.719
105	346.177	325.576
106	382.403	360.491
107	423.813	401.054
108	470.893	447.86
109	524.128	501.498
110	584.004	562.563
111	651.007	631.645
112	725.622	709.338
113	808.336	796.233
114	899.633	892.923
115	1000	1000

Table A.1: 1996 IAM US Annuity 2000 Table, $1000 \cdot q_x$

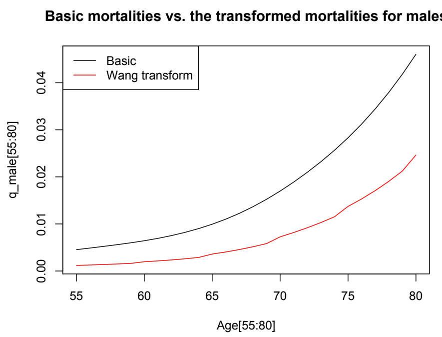
A.2 Plots



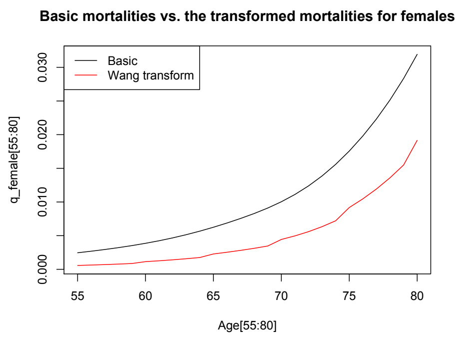
(a) Canada Life Males



(b) Canada Life Females

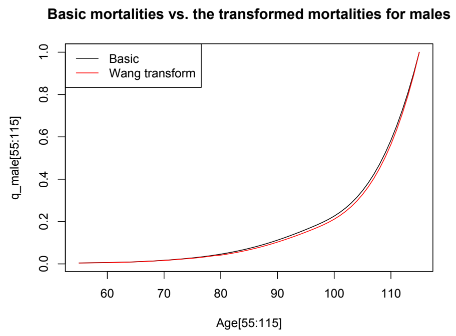


(c) Canada Life Males

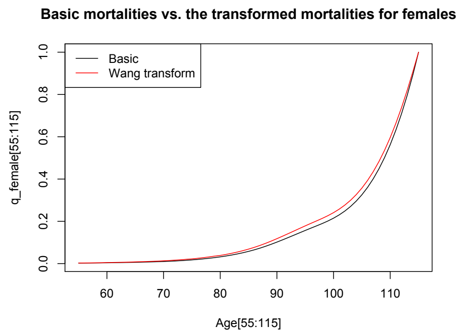


(d) Canada Life Females

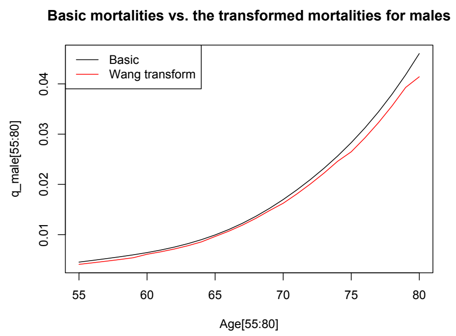
Figure A.1: Wang transform used on Canada Life, when $r = 6\%$



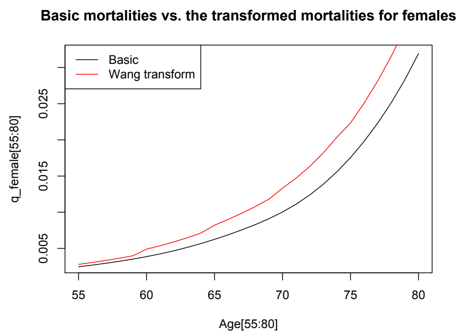
(a) Franklin Life Males



(b) Franklin Life Females

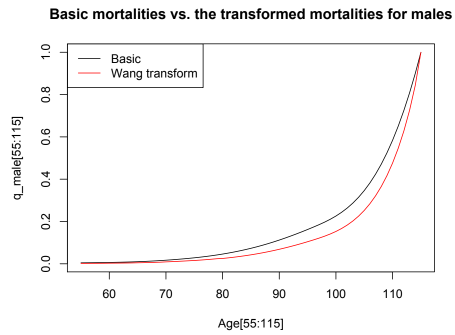


(c) Franklin Life Males

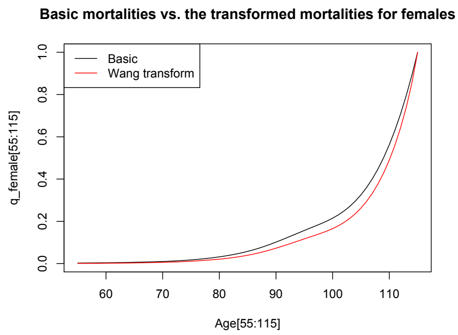


(d) Franklin Life Females

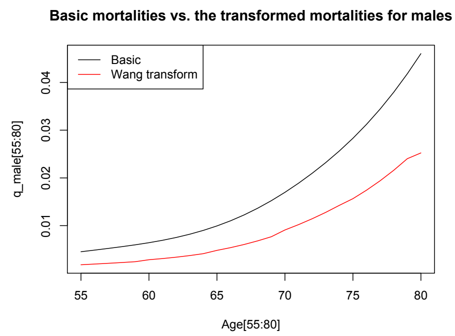
Figure A.2: Wang transform used on Franklin Life, when $r = 6\%$



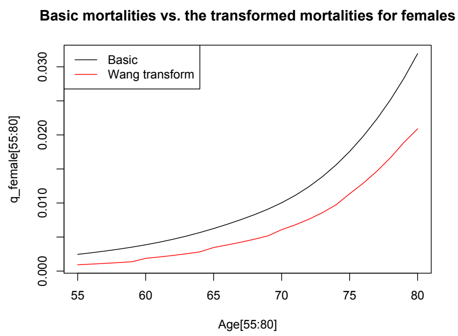
(a) Hartford Life Males



(b) Hartford Life Females

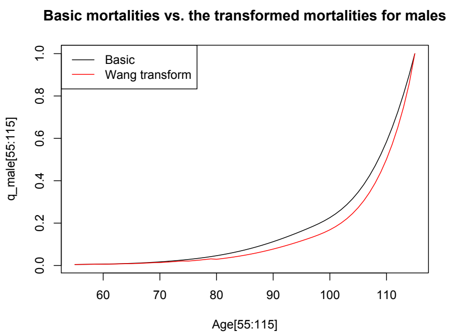


(c) Hartford Life Males

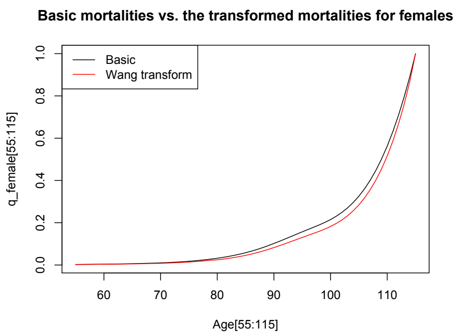


(d) Hartford Life Females

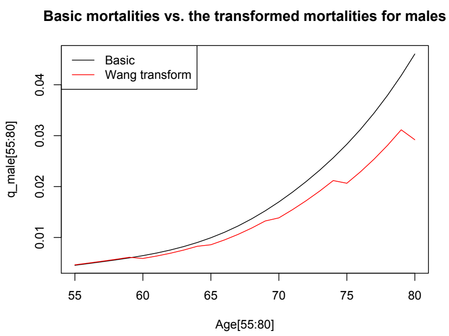
Figure A.3: Wang transform used on Hartford Life, when $r = 6\%$



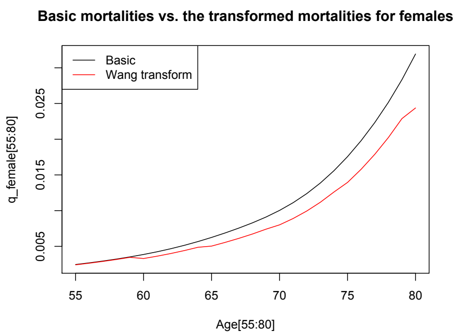
(a) Nationwide Insurance Males



(b) Nationwide Insurance Females



(c) Nationwide Insurance Males



(d) Nationwide Insurance Females

Figure A.4: Wang transform used on Nationwide Insurance, when $r = 6\%$

A.3 R-code

A.3.1 Market Price of Risk

Listing A.1: Market price of risk

```
# reading the table, 1000q_x
q_x=read.table("/Users/Solveig/Dropbox/Masteroppgave/Data/basictable.txt",header=T)

L=function(lambda,r,q,s,x0,le,male)
{
# to be optimalized wrt lambda
# r = fixed interest rate
# q = mortality table
# s = monthly payout from SPIA
# x0 = initial age
# le = maximum age set to 115
# male: TRUE/FALSE
# dividing with 1000 to get the mortalities
q_male=q$Male/1000
q_female=q$Female/1000
# K = number of time periods
K=le-x0
# discount
d=1/(1+r)
# s is monthly, q is in years
s=s*12
# calculating k_q_x0 and inserting them in a matrix
if(male) q=q_male else
         q=q_female
q_=c(q,rep(1,le))
kq=matrix(0,K+1,le)
for (l in 0:le)
{
kq[1:K+1,l]=1-cumprod(1-q_[l:(l+K-1)])
}
# the Wang transform
A=s*sum(d**(0:K)*(pnorm(qnorm(1-kq[1:(K+1),x0])-lambda)))
list(A=A)
}

##############################################################################
### s, x0 and male are variables, others kept fixed
```

```
f=function(lambda,r=0.03,q=q_x,s=680,x0=65,le=115,male=FALSE) L(lambda,r,q,s,x0,le,male)$A
fzero=function(lambda,pi_x0) f(lambda)-pi_x0
uni=uniroot(fzero,c(-10,10),pi_x0=100000)

lambda=uni$root
lambda
```

Listing A.2: Canada Life

```
##### Canada Life #####

source("MPOR2.R")

# initial age
age=c(55,60,65,70,75,80)

# row 1=male, row 2=female
# SPIA payouts
sCL=matrix(c(671.7,726.44,804.02,911.69,1060.03,1265.68,627.13,669.96,729.13,812.49,936.41,1118.95),byrow=T,ncol=6)

# estimating the Wang transform
l_male=1:6*0
l_female=1:6*0
gender=c(TRUE,FALSE)

for (i in 1:length(age))
{
for (j in 1:2)
{
f=function(lambda,r=0.03,q=q_x,s=sCL[j,i],x0=age[i],le=115,male=gender[j]) L(lambda,r,q,s,x0,le,male)$A
fzero=function(lambda,pi_x0) f(lambda)-pi_x0
uni=uniroot(fzero,c(-10,10),pi_x0=100000)
if(gender[j]) l_male[i]=uni$root else
                l_female[i]=uni$root
}
}

# plotting the Wang transform
plot(age,l_male,"o",lty=1,main="Market Price of Risk - Canada Life \nwhen r = 3%",xlab="Initial age",ylab=expression(lambda),ylim=c(min(l_male,l_female),max(l_male,l_female)))
lines(age,l_female,"o",lty=2)
```

```
legend("topright",c("Male","Female"),lty=c(1,2),col=1)



### Basic mortalities versus the transformed mortalities
q_male=q_x$Male/1000
q_female=q_x$Female/1000
Age=q_x$Age

###### Wang transform on Males (55) ######
q_starm=55:115*0

l_male2=c(rep(l_male[1],5),rep(l_male[2],5),rep(l_male[3],5),rep(l_male[4],5),rep(l_male[5],5),rep(l_male[6],36))

for (i in 1:length(q_starm))
{
q_starm[i]=pnorm(qnorm(q_male[Age[54+i]])-l_male2[i])
}

# "vanlig" plot
plot(Age[55:115],q_male[55:115],"l",ylim=c(min(q_starm),max(q_male)),main="One-year mortalities for males",xlab="Initial age",ylab="q")
lines(Age[55:115],q_starm,"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_male[55:80],"l",ylim=c(min(q_starm),max(q_male[55:80])),main="One-year mortalities for males",xlab="Initial age",ylab="q")
lines(Age[55:80],q_starm[1:(80-55+1)],"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))


###### Wang transform on Females (55) #######
q_starf=55:115*0

l_female2=c(rep(l_female[1],5),rep(l_female[2],5),rep(l_female[3],5),rep(l_female[4],5),rep(l_female[5],5),rep(l_female[6],36))

for (i in 1:length(q_starf))
{
q_starf[i]=pnorm(qnorm(q_female[Age[54+i]])-l_female2[i])
}

# "vanlig" plot
```

```
plot(Age[55:115],q_female[55:115],"l",ylim=c(min(q_starf),max(q_female)),main="One-year mortalities for females",xlab="Initial age",ylab="q")
lines(Age[55:115],q_starf,"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_female[55:80],"l",ylim=c(min(q_starf),max(q_female[55:80])),main="One-year mortalities for females",xlab="Initial age",ylab="q")
lines(Age[55:80],q_starf[1:(80-55+1)],"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))
```

Listing A.3: Franklin Life

```
##### Franklin Life #####

source("MPOR2.R")

# initial age
age=c(55,60,65,70,75,80)

# row 1=male, row 2=female
# SPIA payouts
sFL=matrix(c(612,656,720,813,943,1129,575,607,654,722,827,984),byrow=T,ncol=6)

# estimating the Wang transform
l_male=1:6*0
l_female=1:6*0
gender=c(TRUE,FALSE)

for (i in 1:length(age))
{
for (j in 1:2)
{
f=function(lambda,r=0.03,q=q_x,s=sFL[j,i],x0=age[i],le=115,male=gender[j]) L(lambda,r,q,s,x0,le,male)$A
fzero=function(lambda,pi_x0) f(lambda)-pi_x0
uni=uniroot(fzero,c(-10,10),pi_x0=100000)
if(gender[j]) l_male[i]=uni$root else
                l_female[i]=uni$root
}
}

# plotting the Wang transform
```

```
plot(age,l_male,"o",lty=1,main="Market Price of Risk - Franklin Life \nwhen r = 3%",xlab="Initial age",ylab=expression(lambda),ylim=c(min(l_male,l_female),max(l_male,l_female)))
lines(age,l_female,"o",lty=2)
legend("topright",c("Male","Female"),lty=c(1,2),col=1)


### Basic mortalities versus the transformed mortalities
q_male=q_x$Male/1000
q_female=q_x$Female/1000
Age=q_x$Age

###### Wang transform on Males (55) ######
q_starm=55:115*0

l_male2=c(rep(l_male[1],5),rep(l_male[2],5),rep(l_male[3],5),rep(l_male[4],5),rep(l_male[5],5),rep(l_male[6],36))

for (i in 1:length(q_starm))
{
q_starm[i]=pnorm(qnorm(q_male[Age[54+i]])-l_male2[i])
}

# "vanlig" plot
plot(Age[55:115],q_male[55:115],"l",ylim=c(min(q_starm),max(q_male)),main="One-year mortalities for males",xlab="Initial age",ylab="q")
lines(Age[55:115],q_starm,"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_male[55:80],"l",ylim=c(min(q_starm),max(q_male[55:80])),main="One-year mortalities for males",xlab="Initial age",ylab="q")
lines(Age[55:80],q_starm[1:(80-55+1)],"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))


###### Wang transform on Females (55) #######
q_starf=55:115*0

l_female2=c(rep(l_female[1],5),rep(l_female[2],5),rep(l_female[3],5),rep(l_female[4],5),rep(l_female[5],5),rep(l_female[6],36))

for (i in 1:length(q_starf))
{
q_starf[i]=pnorm(qnorm(q_female[Age[54+i]])-l_female2[i])
```

```
}

# "vanlig" plot
plot(Age[55:115],q_female[55:115],"l",ylim=c(min(q_starf),max(q_female)),main="One-year mortalities for females",xlab="Initial age",ylab="q")
lines(Age[55:115],q_starf,"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_female[55:80],"l",ylim=c(min(q_starf),max(q_female[55:80])),main="One-year mortalities for females",xlab="Initial age",ylab="q")
lines(Age[55:80],q_starf[1:(80-55+1)],"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))
```

Listing A.4: Hartford Life

```
##### Hartford Life #####

source("MPOR2.R")

# initial age
age=c(55,60,65,70,75,80)

# row 1=male, row 2=female
# SPIA payouts
sHL=matrix(c(649,701,777,882,1035,1259,609,646,702,784,908,1101),byrow=T,ncol=6)

# estimating the Wang transform
l_male=1:6*0
l_female=1:6*0
gender=c(TRUE,FALSE)

for (i in 1:length(age))
{
for (j in 1:2)
{
f=function(lambda,r=0.03,q=q_x,s=sHL[j,i],x0=age[i],le=115,male=gender[j]) L(lambda,r,q,s,x0,le,male)$A
fzero=function(lambda,pi_x0) f(lambda)-pi_x0
uni=uniroot(fzero,c(-10,10),pi_x0=100000)
if(gender[j]) l_male[i]=uni$root else
                l_female[i]=uni$root
}
```

```r
}

# plotting the Wang transform
plot(age,l_male,"o",lty=1,main="Market Price of Risk - Hartford Life \nwhen r = 3%",xlab="Initial age",ylab=expression(lambda),ylim=c(min(l_male,l_female),max(l_male,l_female)))
lines(age,l_female,"o",lty=2)
legend("topright",c("Male","Female"),lty=c(1,2),col=1)


### Basic mortalities versus the transformed mortalities
q_male=q_x$Male/1000
q_female=q_x$Female/1000
Age=q_x$Age

###### Wang transform on Males (55) ######
q_starm=55:115*0

l_male2=c(rep(l_male[1],5),rep(l_male[2],5),rep(l_male[3],5),rep(l_male[4],5),rep(l_male[5],5),rep(l_male[6],36))

for (i in 1:length(q_starm))
{
q_starm[i]=pnorm(qnorm(q_male[Age[54+i]])-l_male2[i])
}

# "vanlig" plot
plot(Age[55:115],q_male[55:115],"l",ylim=c(min(q_starm),max(q_male)),main="One-year mortalities for males",xlab="Initial age",ylab="q")
lines(Age[55:115],q_starm,"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_male[55:80],"l",ylim=c(min(q_starm),max(q_male[55:80])),main="One-year mortalities for males",xlab="Initial age",ylab="q")
lines(Age[55:80],q_starm[1:(80-55+1)],"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))

###### Wang transform on Females (55) #######
q_starf=55:115*0

l_female2=c(rep(l_female[1],5),rep(l_female[2],5),rep(l_female[3],5),rep(l_female[4],5),rep(l_female[5],5),rep(l_female[6],36))

for (i in 1:length(q_starf))
```

```
{
q_starf[i]=pnorm(qnorm(q_female[Age[54+i]])-l_female2[i])
}

# "vanlig" plot
plot(Age[55:115],q_female[55:115],"l",ylim=c(min(q_starf),max(q_female)),main="One-year mortalities for females",xlab="Initial age",ylab="q")
lines(Age[55:115],q_starf,"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_female[55:80],"l",ylim=c(min(q_starf),max(q_female[55:80])),main="One-year mortalities for females",xlab="Initial age",ylab="q")
lines(Age[55:80],q_starf[1:(80-55+1)],"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))
```

Listing A.5: Nationwide Insurance

```
##### Nationwide Insurance #####

source("MPOR2.R")

# initial age
age=c(55,60,65,70,75,80)

# row 1=male, row 2=female
# SPIA payouts
sNI=matrix(c(607,658,729,831,985,1219,579,622,680,761,882,1070),byrow=T,ncol=6)

# estimating the Wang transform
l_male=1:6*0
l_female=1:6*0
gender=c(TRUE,FALSE)

for (i in 1:length(age))
{
for (j in 1:2)
{
f=function(lambda,r=0.03,q=q_x,s=sNI[j,i],x0=age[i],le=115,male=gender[j]) L(lambda,r,q,s,x0,le,male)$A
fzero=function(lambda,pi_x0) f(lambda)-pi_x0
uni=uniroot(fzero,c(-10,10),pi_x0=100000)
if(gender[j]) l_male[i]=uni$root else
                l_female[i]=uni$root
```

```r
}
}

# plotting the Wang transform
plot(age,l_male,"o",lty=1,main="Market Price of Risk - Nationwide Insurance\nwhen r = 3%",xlab="Initial age",ylab=expression(lambda),ylim=c(min(l_male,l_female),max(l_male,l_female)))
lines(age,l_female,"o",lty=2)
legend("topright",c("Male","Female"),lty=c(1,2),col=1)


### Basic mortalities versus the transformed mortalities
q_male=q_x$Male/1000
q_female=q_x$Female/1000
Age=q_x$Age

###### Wang transform on Males (55) ######
q_starm=55:115*0

l_male2=c(rep(l_male[1],5),rep(l_male[2],5),rep(l_male[3],5),rep(l_male[4],5),rep(l_male[5],5),rep(l_male[6],36))

for (i in 1:length(q_starm))
{
q_starm[i]=pnorm(qnorm(q_male[Age[54+i]])-l_male2[i])
}

# "vanlig" plot
plot(Age[55:115],q_male[55:115],"l",ylim=c(min(q_starm),max(q_male)),main="One-year mortalities for males",xlab="Initial age",ylab="q")
lines(Age[55:115],q_starm,"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_male[55:80],"l",ylim=c(min(q_starm),max(q_male[55:80])),main="One-year mortalities for males",xlab="Initial age",ylab="q")
lines(Age[55:80],q_starm[1:(80-55+1)],"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))

###### Wang transform on Females (55) #######
q_starf=55:115*0

l_female2=c(rep(l_female[1],5),rep(l_female[2],5),rep(l_female[3],5),rep(l_female[4],5),rep(l_female[5],5),rep(l_female[6],36))

```

```
for (i in 1:length(q_starf))
{
q_starf[i]=pnorm(qnorm(q_female[Age[54+i]])-l_female2[i])
}

# "vanlig" plot
plot(Age[55:115],q_female[55:115],"l",ylim=c(min(q_starf),max(q_female)),main="One-year mortalities for females",xlab="Initial age",ylab="q")
lines(Age[55:115],q_starf,"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_female[55:80],"l",ylim=c(min(q_starf),max(q_female[55:80])),main="One-year mortalities for females",xlab="Initial age",ylab="q")
lines(Age[55:80],q_starf[1:(80-55+1)],"l",lty=2)
legend("topleft",c("1996 US Annuity 2000 Mortality Table","Mortalities based on Wang's Transformation"),col=1,lty=c(1,2))
```

A.3.2 Interpolation

Listing A.6: Interpolation

```
### Linear interpolation ###

P=function(ageg,priceg)
{
for (i in 1:length(ageg))
{
priceg[i]=((ageg[i]-ageg[1])*(priceg[6]-priceg[1]))/(ageg[6]-ageg[1])+priceg[1]
}
list(priceg=priceg)
}


ageg1=c(55:60)
ageg2=c(60:65)
ageg3=c(65:70)
ageg4=c(70:75)
ageg5=c(75:80)

### Canada Life ###
priceg1_CLm=c(671.7,1:4*0,726.44)
```

```
priceg2_CLm=c(726.44,1:4*0,804.02)
priceg3_CLm=c(804.02,1:4*0,911.69)
priceg4_CLm=c(911.69,1:4*0,1060.03)
priceg5_CLm=c(1060.03,1:4*0,1265.68)
priceg1_CLf=c(627.13,1:4*0,669.96)
priceg2_CLf=c(669.96,1:4*0,729.13)
priceg3_CLf=c(729.13,1:4*0,812.49)
priceg4_CLf=c(812.49,1:4*0,936.41)
priceg5_CLf=c(936.41,1:4*0,1118.95)

priceg1_CLm=P(ageg1,priceg1_CLm)$priceg
priceg2_CLm=P(ageg2,priceg2_CLm)$priceg
priceg3_CLm=P(ageg3,priceg3_CLm)$priceg
priceg4_CLm=P(ageg4,priceg4_CLm)$priceg
priceg5_CLm=P(ageg5,priceg5_CLm)$priceg
priceg1_CLf=P(ageg1,priceg1_CLf)$priceg
priceg2_CLf=P(ageg2,priceg2_CLf)$priceg
priceg3_CLf=P(ageg3,priceg3_CLf)$priceg
priceg4_CLf=P(ageg4,priceg4_CLf)$priceg
priceg5_CLf=P(ageg5,priceg5_CLf)$priceg

price_CL=matrix(c(priceg1_CLm,priceg2_CLm[-1],priceg3_CLm[-1],priceg4_CLm[-1],↩
    priceg5_CLm[-1],priceg1_CLf,priceg2_CLf[-1],priceg3_CLf[-1],priceg4_CLf↩
    [-1],priceg5_CLf[-1]),byrow=T,ncol=26)


### Franklin Life ###
priceg1_FLm=c(612,1:4*0,656)
priceg2_FLm=c(656,1:4*0,720)
priceg3_FLm=c(720,1:4*0,813)
priceg4_FLm=c(813,1:4*0,943)
priceg5_FLm=c(943,1:4*0,1129)
priceg1_FLf=c(575,1:4*0,607)
priceg2_FLf=c(607,1:4*0,654)
priceg3_FLf=c(654,1:4*0,722)
priceg4_FLf=c(722,1:4*0,827)
priceg5_FLf=c(827,1:4*0,984)

priceg1_FLm=P(ageg1,priceg1_FLm)$priceg
priceg2_FLm=P(ageg2,priceg2_FLm)$priceg
priceg3_FLm=P(ageg3,priceg3_FLm)$priceg
priceg4_FLm=P(ageg4,priceg4_FLm)$priceg
priceg5_FLm=P(ageg5,priceg5_FLm)$priceg
priceg1_FLf=P(ageg1,priceg1_FLf)$priceg
priceg2_FLf=P(ageg2,priceg2_FLf)$priceg
priceg3_FLf=P(ageg3,priceg3_FLf)$priceg
priceg4_FLf=P(ageg4,priceg4_FLf)$priceg
```

```
priceg5_FLf=P(ageg5,priceg5_FLf)$priceg

price_FL=matrix(c(priceg1_FLm,priceg2_FLm[-1],priceg3_FLm[-1],priceg4_FLm[-1],priceg5_FLm[-1],priceg1_FLf,priceg2_FLf[-1],priceg3_FLf[-1],priceg4_FLf[-1],priceg5_FLf[-1]),byrow=T,ncol=26)


### Hartford Life ###
priceg1_HLm=c(649,1:4*0,701)
priceg2_HLm=c(701,1:4*0,777)
priceg3_HLm=c(777,1:4*0,882)
priceg4_HLm=c(882,1:4*0,1035)
priceg5_HLm=c(1035,1:4*0,1259)
priceg1_HLf=c(609,1:4*0,646)
priceg2_HLf=c(646,1:4*0,702)
priceg3_HLf=c(702,1:4*0,784)
priceg4_HLf=c(784,1:4*0,908)
priceg5_HLf=c(908,1:4*0,1101)

priceg1_HLm=P(ageg1,priceg1_HLm)$priceg
priceg2_HLm=P(ageg2,priceg2_HLm)$priceg
priceg3_HLm=P(ageg3,priceg3_HLm)$priceg
priceg4_HLm=P(ageg4,priceg4_HLm)$priceg
priceg5_HLm=P(ageg5,priceg5_HLm)$priceg
priceg1_HLf=P(ageg1,priceg1_HLf)$priceg
priceg2_HLf=P(ageg2,priceg2_HLf)$priceg
priceg3_HLf=P(ageg3,priceg3_HLf)$priceg
priceg4_HLf=P(ageg4,priceg4_HLf)$priceg
priceg5_HLf=P(ageg5,priceg5_HLf)$priceg

price_HL=matrix(c(priceg1_HLm,priceg2_HLm[-1],priceg3_HLm[-1],priceg4_HLm[-1],priceg5_HLm[-1],priceg1_HLf,priceg2_HLf[-1],priceg3_HLf[-1],priceg4_HLf[-1],priceg5_HLf[-1]),byrow=T,ncol=26)


### Nationwide Insurance ###
priceg1_NIm=c(607,1:4*0,658)
priceg2_NIm=c(658,1:4*0,729)
priceg3_NIm=c(729,1:4*0,831)
priceg4_NIm=c(831,1:4*0,985)
priceg5_NIm=c(985,1:4*0,1219)
priceg1_NIf=c(579,1:4*0,622)
priceg2_NIf=c(622,1:4*0,680)
priceg3_NIf=c(680,1:4*0,761)
priceg4_NIf=c(761,1:4*0,882)
priceg5_NIf=c(882,1:4*0,1070)
```

```
priceg1_NIm=P(ageg1,priceg1_NIm)$priceg
priceg2_NIm=P(ageg2,priceg2_NIm)$priceg
priceg3_NIm=P(ageg3,priceg3_NIm)$priceg
priceg4_NIm=P(ageg4,priceg4_NIm)$priceg
priceg5_NIm=P(ageg5,priceg5_NIm)$priceg
priceg1_NIf=P(ageg1,priceg1_NIf)$priceg
priceg2_NIf=P(ageg2,priceg2_NIf)$priceg
priceg3_NIf=P(ageg3,priceg3_NIf)$priceg
priceg4_NIf=P(ageg4,priceg4_NIf)$priceg
priceg5_NIf=P(ageg5,priceg5_NIf)$priceg

price_NI=matrix(c(priceg1_NIm,priceg2_NIm[-1],priceg3_NIm[-1],priceg4_NIm[-1],↩
    priceg5_NIm[-1],priceg1_NIf,priceg2_NIf[-1],priceg3_NIf[-1],priceg4_NIf↩
    [-1],priceg5_NIf[-1]),byrow=T,ncol=26)
```

A.3.3 Risk-adjusted mortalities

Listing A.7: Using the interpolated prices to calculate the market price of risk

```
source("MPOR2.R")

# initial age
age=c(55:80)

# row 1=male, row 2=female
# SPIA payouts, interpolated in R-file interpolation
price_CL
price_FL
price_HL
price_NI

##### Canada Life #####
# estimating the Wang transform
l_male=1:length(age)*0
l_female=1:length(age)*0
gender=c(TRUE,FALSE)

for (i in 1:length(age))
{
for (j in 1:2)
{
```

```
f=function(lambda,r=0.03,q=q_x,s=price_CL[j,i],x0=age[i],le=115,male=gender[j]) L(lambda,r,q,s,x0,le,male)$A
fzero=function(lambda,pi_x0) f(lambda)-pi_x0
uni=uniroot(fzero,c(-10,10),pi_x0=100000)
if(gender[j]) l_male[i]=uni$root else
                l_female[i]=uni$root
}
}

# plotting the Wang transform
plot(age,l_male,"l",main="MPOR Canada Life",xlab="Initial age",ylab="MPOR",ylim=c(min(l_male,l_female),max(l_male,l_female)))
lines(age,l_female,"l",col=2)
legend("topright",c("Male","Female"),col=c(1,2),lty=1)


### Basic mortalities versus the transformed mortalities
q_male=q_x$Male/1000
q_female=q_x$Female/1000
Age=q_x$Age

###### Wang transform on Males (55) ######
q_starm=55:115*0

l_male2=c(l_male,rep(l_male[26],36))

for (i in 1:length(q_starm))
{
q_starm[i]=pnorm(qnorm(q_male[Age[54+i]])-l_male2[i])
}

# "vanlig" plot
plot(Age[55:115],q_male[55:115],"l",ylim=c(min(q_starm),max(q_male)),main="Basic mortalities vs. the transformed mortalities for males")
lines(Age[55:115],q_starm,"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_male[55:80],"l",ylim=c(min(q_starm),max(q_male[55:80])),main="Basic mortalities vs. the transformed mortalities for males")
lines(Age[55:80],q_starm[1:(80-55+1)],"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))

###### Wang transform on Females (55) #######
q_starf=55:115*0

l_female2=c(l_female,rep(l_female[26],36))
```

```

for (i in 1:length(q_starf))
{
q_starf[i]=pnorm(qnorm(q_female[Age[54+i]])-l_female2[i])
}

# "vanlig" plot
plot(Age[55:115],q_female[55:115],"l",ylim=c(min(q_starf),max(q_female)),main="Basic mortalities vs. the transformed mortalities for females")
lines(Age[55:115],q_starf,"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_female[55:80],"l",ylim=c(min(q_starf),max(q_female[55:80])),main="Basic mortalities vs. the transformed mortalities for females")
lines(Age[55:80],q_starf[1:(80-55+1)],"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))


##### Franklin Life #####
# estimating the Wang transform
l_male=1:length(age)*0
l_female=1:length(age)*0
gender=c(TRUE,FALSE)

for (i in 1:length(age))
{
for (j in 1:2)
{
f=function(lambda,r=0.03,q=q_x,s=price_FL[j,i],x0=age[i],le=115,male=gender[j]) L(lambda,r,q,s,x0,le,male)$A
fzero=function(lambda,pi_x0) f(lambda)-pi_x0
uni=uniroot(fzero,c(-10,10),pi_x0=100000)
if(gender[j]) l_male[i]=uni$root else
                l_female[i]=uni$root
}
}

# plotting the Wang transform
plot(age,l_male,"l",main="MPOR Franklin Life",xlab="Initial age",ylab="MPOR",ylim=c(min(l_male,l_female),max(l_male,l_female)))
lines(age,l_female,"l",col=2)
legend("topright",c("Male","Female"),col=c(1,2),lty=1)


### Basic mortalities versus the transformed mortalities
q_male=q_x$Male/1000
```

```
q_female=q_x$Female/1000
Age=q_x$Age

###### Wang transform on Males (55) ######
q_starm=55:115*0

l_male2=c(l_male,rep(l_male[26],36))

for (i in 1:length(q_starm))
{
q_starm[i]=pnorm(qnorm(q_male[Age[54+i]])-l_male2[i])
}

# "vanlig" plot
plot(Age[55:115],q_male[55:115],"l",ylim=c(min(q_starm),max(q_male)),main="Basic mortalities vs. the transformed mortalities for males")
lines(Age[55:115],q_starm,"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_male[55:80],"l",ylim=c(min(q_starm),max(q_male[55:80])),main="Basic mortalities vs. the transformed mortalities for males")
lines(Age[55:80],q_starm[1:(80-55+1)],"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))

###### Wang transform on Females (55) #######
q_starf=55:115*0

l_female2=c(l_female,rep(l_female[26],36))

for (i in 1:length(q_starf))
{
q_starf[i]=pnorm(qnorm(q_female[Age[54+i]])-l_female2[i])
}

# "vanlig" plot
plot(Age[55:115],q_female[55:115],"l",ylim=c(min(q_starf),max(q_female)),main="Basic mortalities vs. the transformed mortalities for females")
lines(Age[55:115],q_starf,"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_female[55:80],"l",ylim=c(min(q_starf),max(q_female[55:80])),main="Basic mortalities vs. the transformed mortalities for females")
lines(Age[55:80],q_starf[1:(80-55+1)],"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))
```

```
##### Hartford Life #####
# estimating the Wang transform
l_male=1:length(age)*0
l_female=1:length(age)*0
gender=c(TRUE,FALSE)

for (i in 1:length(age))
{
for (j in 1:2)
{
f=function(lambda,r=0.03,q=q_x,s=price_HL[j,i],x0=age[i],le=115,male=gender[j])↩
     L(lambda,r,q,s,x0,le,male)$A
fzero=function(lambda,pi_x0) f(lambda)-pi_x0
uni=uniroot(fzero,c(-10,10),pi_x0=100000)
if(gender[j]) l_male[i]=uni$root else
                l_female[i]=uni$root
}
}

# plotting the Wang transform
plot(age,l_male,"l",main="MPOR Hartford Life",xlab="Initial age",ylab="MPOR",↩
    ylim=c(min(l_male,l_female),max(l_male,l_female)))
lines(age,l_female,"l",col=2)
legend("topright",c("Male","Female"),col=c(1,2),lty=1)


### Basic mortalities versus the transformed mortalities
q_male=q_x$Male/1000
q_female=q_x$Female/1000
Age=q_x$Age

###### Wang transform on Males (55) ######
q_starm=55:115*0

l_male2=c(l_male,rep(l_male[26],36))

for (i in 1:length(q_starm))
{
q_starm[i]=pnorm(qnorm(q_male[Age[54+i]])-l_male2[i])
}

# "vanlig" plot
plot(Age[55:115],q_male[55:115],"l",ylim=c(min(q_starm),max(q_male)),main="↩
    Basic mortalities vs. the transformed mortalities for males")
lines(Age[55:115],q_starm,"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))
```

```
# "zoomet" inn plot
plot(Age[55:80],q_male[55:80],"l",ylim=c(min(q_starm),max(q_male[55:80])),main="Basic mortalities vs. the transformed mortalities for males")
lines(Age[55:80],q_starm[1:(80-55+1)],"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))

###### Wang transform on Females (55) #######
q_starf=55:115*0

l_female2=c(l_female,rep(l_female[26],36))

for (i in 1:length(q_starf))
{
q_starf[i]=pnorm(qnorm(q_female[Age[54+i]])-l_female2[i])
}

# "vanlig" plot
plot(Age[55:115],q_female[55:115],"l",ylim=c(min(q_starf),max(q_female)),main="Basic mortalities vs. the transformed mortalities for females")
lines(Age[55:115],q_starf,"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_female[55:80],"l",ylim=c(min(q_starf),max(q_female[55:80])),main="Basic mortalities vs. the transformed mortalities for females")
lines(Age[55:80],q_starf[1:(80-55+1)],"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))


##### Nationwide Insurance #####
# estimating the Wang transform
l_male=1:length(age)*0
l_female=1:length(age)*0
gender=c(TRUE,FALSE)

for (i in 1:length(age))
{
for (j in 1:2)
{
f=function(lambda,r=0.03,q=q_x,s=price_NI[j,i],x0=age[i],le=115,male=gender[j]) L(lambda,r,q,s,x0,le,male)$A
fzero=function(lambda,pi_x0) f(lambda)-pi_x0
uni=uniroot(fzero,c(-10,10),pi_x0=100000)
if(gender[j]) l_male[i]=uni$root else
                l_female[i]=uni$root
}
```

```
}

# plotting the Wang transform
plot(age,l_male,"l",main="MPOR Nationwide Insurance",xlab="Initial age",ylab="MPOR",ylim=c(min(l_male,l_female),max(l_male,l_female)))
lines(age,l_female,"l",col=2)
legend("topright",c("Male","Female"),col=c(1,2),lty=1)


### Basic mortalities versus the transformed mortalities
q_male=q_x$Male/1000
q_female=q_x$Female/1000
Age=q_x$Age

###### Wang transform on Males (55) ######
q_starm=55:115*0

l_male2=c(l_male,rep(l_male[26],36))

for (i in 1:length(q_starm))
{
q_starm[i]=pnorm(qnorm(q_male[Age[54+i]])-l_male2[i])
}

# "vanlig" plot
plot(Age[55:115],q_male[55:115],"l",ylim=c(min(q_starm),max(q_male)),main="Basic mortalities vs. the transformed mortalities for males")
lines(Age[55:115],q_starm,"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_male[55:80],"l",ylim=c(min(q_starm),max(q_male[55:80])),main="Basic mortalities vs. the transformed mortalities for males")
lines(Age[55:80],q_starm[1:(80-55+1)],"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))

###### Wang transform on Females (55) #######
q_starf=55:115*0

l_female2=c(l_female,rep(l_female[26],36))

for (i in 1:length(q_starf))
{
q_starf[i]=pnorm(qnorm(q_female[Age[54+i]])-l_female2[i])
}

# "vanlig" plot
```

```
plot(Age[55:115],q_female[55:115],"l",ylim=c(min(q_starf),max(q_female)),main="Basic mortalities vs. the transformed mortalities for females")
lines(Age[55:115],q_starf,"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))

# "zoomet" inn plot
plot(Age[55:80],q_female[55:80],"l",ylim=c(min(q_starf),max(q_female[55:80])),main="Basic mortalities vs. the transformed mortalities for females")
lines(Age[55:80],q_starf[1:(80-55+1)],"l",col=2)
legend("topleft",c("Basic","Wang transform"),lty=1,col=c(1,2))
```

A.3.4 Using the market price of risk

Listing A.8: Present value of a life annuity/Calculating the one-time premium

```
# One-time premium against age

le=115
K=115
r=0.03
d=1/(1+r)
lr=55
s=1

q_x=read.table("/Users/Solveig/Dropbox/Masteroppgave/Data/basictable.txt",header=T)

###### for males ######

# "normal" mortalities
q_male=q_x$Male/1000

p_=c(1-q_male,rep(0,le+1))
kp=matrix(1,K+1,le+1)
for (l in 0:le+1)
{
kp[1:K+1,l]=cumprod(p_[l:(l+K-1)])
}

I=matrix(0,le+1,le+1)
I[row(I)+col(I)>lr+1]=1

```

```
l1=0:le
M=s*d**l1*kp*I
pi_l0=apply(M,2,sum)

# transformed mortalities q_starm
q_star=c(q_male[1:54],q_starm)
p_star=c(1-q_star,rep(0,le+1))
kp_star=matrix(1,K+1,le+1)
for (l in 0:le+1)
{
kp_star[1:K+1,l]=cumprod(p_star[l:(l+K-1)])
}

I_star=matrix(0,le+1,le+1)
I_star[row(I_star)+col(I_star)>lr+1]=1

l1=0:le
M_star=s*d**l1*kp_star*I_star
pi_l0_star=apply(M_star,2,sum)

# plot
l1=55; l2=114
matplot(l1:l2,pi_l0[l1:l2+1],"l",main="One-time premium against age \n when s=1 for males",xlab="Initial age", ylab=expression(pi),sub="Nationwide Insurance")
lines(l1:l2,pi_l0_star[l1:l2+1],"l",lty=2)
legend("topright", c("1996 US Annuity 2000 Mortality Table","Transformed mortalities"),lty=c(1,2),col=c(1,1))


# for females

# "normal" mortalities
q_female=q_x$Female/1000

p_=c(1-q_female,rep(0,le+1))
kp=matrix(1,K+1,le+1)
for (l in 0:le+1)
{
kp[1:K+1,l]=cumprod(p_[l:(l+K-1)])
}

I=matrix(0,le+1,le+1)
I[row(I)+col(I)>lr+1]=1

l1=0:le
M=s*d**l1*kp*I
```

```
pi_l0=apply(M,2,sum)

# transformed mortalities q_starf
q_star=c(q_female[1:54],q_starf)
p_star=c(1-q_star,rep(0,le+1))
kp_star=matrix(1,K+1,le+1)
for (l in 0:le+1)
{
kp_star[1:K+1,l]=cumprod(p_star[l:(l+K-1)])
}

I_star=matrix(0,le+1,le+1)
I_star[row(I_star)+col(I_star)>lr+1]=1

ll=0:le
M_star=s*d**ll*kp_star*I_star
pi_l0_star=apply(M_star,2,sum)

# plot
l1=55; l2=114
matplot(l1:l2,pi_l0[l1:l2+1],"l",main="One-time premium against age \n when s=1 for females",xlab="Initial age", ylab=expression(pi),sub="Nationwide Insurance")
lines(l1:l2,pi_l0_star[l1:l2+1],"l",lty=2)
legend("topright", c("1996 US Annuity 2000 Mortality Table","Transformed mortalities"),lty=c(1,2),col=c(1,1))
```

Listing A.9: Pricing the mortality bond

```
q_x=read.table("/Users/Solveig/Dropbox/Masteroppgave/Data/basictable.txt",header=T)
q_male=q_x$Male/1000
q_female=q_x$Female/1000

le=115;K1=115
p_=c(1-q_female,rep(0,le+1))
kp=matrix(1,K1+1,le+1)
for (l in 0:le+1)
{
kp[1:K1+1,l]=cumprod(p_[l:(l+K1-1)])
}

l_male=0.842
l_female=0.938

q_starm=1:length(q_male)*0
```

```
q_starf=1:length(q_female)*0
for (i in 1:length(q_starm))
{
q_starm[i]=pnorm(qnorm(q_male[i])-l_male)
q_starf[i]=pnorm(qnorm(q_female[i])-l_female)
}

p_star=c(1-q_starf,rep(0,le+1))
kp_star=matrix(1,K1+1,le+1)
for (l in 0:le+1)
{
kp_star[1:K1+1,l]=cumprod(p_star[l:(l+K1-1)])
}

K=30
x0=65; n_x=10000
X=1:K*0
for (k in 1:10)
{
X[k]=n_x*kp[k+1,x0]*exp(0.0070*k)
}
for (k in 11:20)
{
X[k]=n_x*kp[k+1,x0]*exp(0.07)*exp(0.0093*(k-10))
}
for (k in 21:30)
{
X[k]=n_x*kp[k+1,x0]*exp(0.163)*exp(0.0103*(k-20))
}

mu=1:K*0
sigma=1:K*0
for (k in 1:K)
{
mu[k]=n_x*kp_star[k+1,x0]
sigma[k]=sqrt(n_x*kp_star[k+1,x0]*(1-kp_star[k+1,x0]))
}

psi=function(a)
{
dnorm(a)-a*(1-pnorm(a))
}

C=0.07
E_D=1:K*0
for (k in 1:K)
{
```

```
a=(X[k]-mu[k])/sigma[k]
E_D[k]=1000*(C-sigma[k]*(psi(a)-psi(a+C/sigma[k])))
}

F=10000000
V=F*d**K+sum(d**(1:K)*E_D)
```

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