An observation on unique solvability of a Cauchy problem for linear partial differential equations with constant coefficients.

by

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1. A theorem, and some comments.

The purpose of this note is to show how an old trick from the theory of partial differential equations in the complex domain (Garabedian : [2] ch. 16,1) may be combined with standard results on hyperbolic equations (Hörmander [3] sections 5,4- 5,6) to give a simple proof of the following

Theorem 1. Let $A_1, \cdots, A_n$ and B denote $N \times N$-matrices with complex entries, and I the $N \times N$ unit matrix. Let $f = (f_1, \cdots, f_N)$ be an N-tuple of vector-valued distributions, defined on the real line $\mathbb{R}$, and taking their values in the space $H' = H'(\mathbb{C}^n)$ of analytic functionals on $\mathbb{C}^n$. Suppose that f vanishes on $\mathbb{R}^- = \{t \in \mathbb{R},\ t < 0\}$.

Then there exists a unique N-tuple $u = (u_1, \cdots, u_N)$ of distributions on $\mathbb{R}$, with values in H', vanishing on $\mathbb{R}^-$, and solving the partial differential equation

$$(*) \qquad I\frac{\partial u}{\partial t} + \sum_{j=1}^{n} A_j \frac{\partial u}{\partial z_j} + Bu = f.$$

Further, there exists a proper convex cone Γ^* in $\mathbb{R} \times \mathbb{C}^n$, containing the half-line $\mathbb{R}^+ \times \{0\}$, depending only on $A_1, \cdots, A_n$, and having the following property :

If all the values $f_\phi \in H'$ of f, for $\phi \in C_0^\infty(\mathbb{R})$, can be carried by the compact $K \subset \mathbb{C}^n$, and if $I \subset \mathbb{R}$ is compact, then the values $u_\phi \in H'$ of u, for $\phi \in C_0^\infty(I)$ can be carried by any compact set $\tilde{K} \in \mathbb{C}$ such that

$$I \times \tilde{K} \supset (I \times \mathbb{C}^n) \cap ((\mathbb{R}^+ \times K) + \Gamma^*)$$

(For the notation of carrier, see section 3).

The theorem will be proved by reducing it to a well-known theorem on the existence and uniqueness of solutions to the Cauchy problem for hyperbolic equations. This reduction is found in section 4 of this note, while sections 2 and 3 contain some preliminary lemmas.

Similar results, for f and u in suitable spaces of continuous functions defined on $\mathbb{R}$ with values in H' (instead of distributions) have been proved by Persson ([4]). His method is very different from ours, and quite complicated, but has the great advantage that it works for equations with variable (analytic) coefficients. I do not know whether this may be the case for the present method.

The standard technique for reducing Cauchy problems for a higher order single equation to an equivalent problem for a first order system (Courant-Hilbert [1] pp 43-46) works as well in our situation as in the classical one, so that the case of higher order single equations is taken care of in our theorem.

If the right hand side f and the solution u are continuous functions from $\mathbb{R}$ to $(H')^N$, the Cauchy problem is often formulated as the problem of finding a solution u of (*) such that $u(0)$ has a given value. This way of posing the problem may be reduced to the one in our theorem just as in the "ordinary" distribution case ([3] section 5,0 p 114-115).

2. Preliminaries on partial differential equations.

In this section we work in $\mathbb{R}^{1+m}$, with variables denoted by (t,x), $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, and we consider a system of N linear first-order partial differential equations in the form

$$(2.1) \qquad I \frac{\partial}{\partial t} U + \sum_{j=1}^{m} \Lambda_j \frac{\partial}{\partial x_j} U + BU = F$$

where $\Lambda_1, \cdots, \Lambda_m$ and B are $N \times N$-matrices with complex entries, and F and U are N-tuples of distributions on $\mathbb{R}^{1+m}$.

To shorten notations, we define a $N \times N$-matrix $P = \{P_{ij}\}$ by

$$P(\tau,\xi) = I\tau + \sum_{j=1}^{m} \Lambda_j \xi_j + B \qquad (\tau,\xi) \in \mathbb{R}^{1+m}$$

and denote the matrix of cofactors (algebraic complements) in $\{P_{ij}\}$ by ${}^{co}P$ or $\{{}^{co}P_{ij}\}$.

In view of the formula

$$({}^{co}P)\cdot P = P\cdot({}^{co}P) = (\det P)\cdot I$$

the N-tuple of distributions $E = (E_1, \cdots, E_N)$ is a fundamental solution of (2.1) if and only if

$$(2.2) \qquad (\det P)E_j = \left(\sum_{k=1}^{N} {}^{co}P_{jk}\right)\delta, \qquad 1 \leq j \leq N.$$

According to [3] section 5,4 the differential polynomial $(\det P)(\frac{\partial}{\partial t}, \frac{\partial}{\partial x})$ is hyperbolic with respect to the t-axis if the hyperplane $t = 0$ is non-characteristic, and if there exists a number $\tau_0 > 0$ such that

$$(2.3) \qquad (\det P)(i\tau + i\gamma, i\xi) = (i)^N \det\{I(\tau+\gamma) + \sum_{j=1}^{m} \Lambda_j \xi_j - iB\}$$

considered as a polynomial in $\gamma \in \mathbb{C}$, has all its zeros in the strip $|\mathrm{Im}\gamma| < \tau_o$, for every $(\tau,\xi) \in \mathbb{R}^{1+m}$. (The factors i in (2.3) occur because we have passed from differential to ordinary polynomials by replacing $(\frac{\partial}{\partial t}, \frac{\partial}{\partial x})$ with (τ,ξ) instead of with $(i\tau, \xi)$ as in [3]).

A sufficient, but, as simple examples show, not necessary condition for (det P) to be hyperbolic, is given in the following

<u>Lemma 2.1.</u> If the matrices $\Lambda_1, \dots, \Lambda_m$ are Hermitian, then the differential polynomial

$$(\det P)(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}) = \det\{I \frac{\partial}{\partial t} + \sum_{j=1}^{m} \Lambda_j \frac{\partial}{\partial x_j} + B\}$$

is hyperbolic with respect to the t-axis.

<u>Proof.</u> According to (2.3) we have to prove that for some $\tau_o > 0$, independent of $(\tau,\xi) \in \mathbb{R}^{1+m}$, all the eigenvalues γ of the matrix

$$I\tau + \sum_{j=1}^{m} \Lambda_j \xi_j - iB$$

satisfy the inequality $|\mathrm{Im}\gamma| < \tau_o$.

We denote the Hilbert space norm and inner product in $\mathbb{C}^N$ by $\|\cdot\|$ and $<,>$ respectively, and let γ be an eigenvalue, $v \in \mathbb{C}^N$ a corresponding eigenvector, with $\|v\| = 1$. Then we have

$$\gamma = <\gamma v, v> = <(iB - I\tau - \Sigma \Lambda_j \xi_j)v, v> = i<Bv,v> - \tau - \sum_{j=1}^{m} \xi_j <\Lambda_j v, v>$$

When the Λ_j are Hermitian, $<\Lambda_j v, v>$ are real, and we get

$$|\mathrm{Im}\gamma| = |\mathrm{Re}\langle Bv,v\rangle| \leq \|B\|$$

for every $\langle\tau,\xi\rangle \in \mathbb{R}^{1+m}$. This proves the lemma.

We denote by Δ the principal part of the differential polynomial (det P); by homogeneity, we have

$$\Delta(\tau,\xi) = i^N \det\{I\tau + \sum_{j=1}^{m} \Lambda_j \xi_j\}$$

If (det P) is hyperbolic with respect to the t-axis, then so is Δ ([3] th.5,5,2) and in that case the cones $\Gamma(\Delta)$ and $\Gamma^*(\Delta)$ are defined by

$$\Gamma(\Delta) = \{(\tau,\xi) \in \mathbb{R}^{1+m};\ \Delta(\tau+\gamma,\xi) = 0 \Rightarrow \gamma < 0\}$$

and

$$\Gamma^*(\Delta) = \{(t,x) \in \mathbb{R}^{1+m};\quad t\tau + \sum_{j=1}^{m} x_j\xi_j \geq 0 \text{ when } (\tau,\xi) \in \Gamma(\Delta)\}$$

They are both convex cones containing the vector (1,0), Γ is open and Γ^* closed, both Γ and, except for the origin, Γ^* are contained in the half-space $\{(t,x);\ t > 0\}$. ([3] section 5.5).

The existence theorem we need is obtained by combining lemma 1 above with theorems 5,6,1 and 5,6,3 in [3]. We state it as

<u>Lemma 2,2.</u> If the matrices $\Lambda_1,\cdots,\Lambda_m$ are Hermitian, the system (2.1) has a unique fundamental solution E with supp $E \subset \{(t,x);\ t \geq 0\}$, and in fact supp E is contained in the proper convex cone $\Gamma^*(\Delta)$.

3. Distributions and bilinear functionals.

The usual identification of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ by

$$(3.1) \qquad (x,y) \to x + iy \;, \qquad x \in \mathbb{R}^n, \; y \in \mathbb{R}^n$$

makes the space $H = H(\mathbb{C}^n)$ of entire analytic functions on $\mathbb{C}^n$ a subspace of $C^\infty(\mathbb{R}^{2n})$. It follows from Cauchy's formulae for the derivatives of analytic functions that the topology induced on H from $C^\infty(\mathbb{R}^{2n})$ coincides with the usual topology of uniform convergence on compact sets. (Tréves [5] p.90).

An element of the dual H' of H is called an analytic functional on $\mathbb{C}^n$. By definition of the topology on H, a linear functional f on H is in H' if and only if there exists a constant C and a compact set $K \subset \mathbb{C}^n$ such that

$$|\langle f,\psi\rangle| \leq C \; \sup_{z\in k}|\psi(z)| \;, \qquad \psi \in H.$$

Such a set K is said to carry f. (Definitions of carriers vary somewhat in the literature. This one is good enough for our purpose).

In the proof of theorem 1, it will be convenient to work with the space $B = B(C_0^\infty(\mathbb{R}),H)$ of separately continuous bilinear forms on $C_0^\infty(\mathbb{R}) \times H(\mathbb{C}^n)$ instead of the space of distributions on $\mathbb{R}$ with values in H' (that is, the space of continuous linear maps from $C_0^\infty(\mathbb{R})$ to H'). It follows from Tréves [5] Prop. 42,2 (2) that these two spaces are canonically isomorphic.

We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by (3,1) and introduce the notations

$$\frac{\partial}{\partial z_j} = \tfrac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \;, \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2i}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

The obvious definition of derivations in B by duality :

$$<\frac{\partial f}{\partial t},\phi,\psi> = -<f,\phi',\psi>, \qquad <\frac{\partial f}{\partial z_j},\phi,\psi> = -<f,\phi,\frac{\partial\psi}{\partial z_j}>$$

etc., then gives the Cauchy-Riemann equations :

$$(3.2) \qquad \frac{\partial f}{\partial \bar{z}_j} = 0 \qquad 1 \leq j \leq n. \qquad f \in B$$

Lemma 3.1. Let I be a compact subset of $\mathbb{R}$, and let f be a separately continuous bilinear functional on $C_0^\infty(I) \times H(\mathbb{C}^n)$.

Then there exists a compact set $K \subset \mathbb{C}^n$ which carries the analytic functionals

$$f_\phi : \psi \to <f,\phi,\psi> \quad , \qquad \psi \in H$$

for every $\phi \in C_0^\infty(I)$, and there exists a distribution $F \in \mathcal{E}'(\mathbb{R}^{1+2n})$ with supp $F \subset I \times K$, such that

$$<f,\phi,\psi> = <F,\phi\cdot\psi>$$

for $\phi \in C_0^\infty(I)$ and $\psi \in H(\mathbb{C}^n) \subset C^\infty(\mathbb{R}^{2n})$.

Proof. Since $C_0^\infty(I)$ and H are both Fréchet spaces, a separately continuous bilinear form on their product is simultaneously continuous ([5] corollary to theorem 34,1). The existence of K then follows directly from the continuity.

Further $B(C_0^\infty(I),H)$ is canonically isomorphic to the dual of $C_0^\infty(I) \otimes_\pi H$, which is a subspace of the Fréchet space $C^\infty(\mathbb{R}^{1+2n})$ ([5] prop. 43,4 and th. 51,6). The lemma then follows from the Hahn-Banach theorem.

Lemma 3.2. Let $f \in B(C_0^\infty(\mathbb{R}),H)$ be given, and let $\{I_\nu\}_1^\infty$ be a locally finite sequence of compact sets in $\mathbb{R}$, the interiors of which cover $\mathbb{R}$.

Then for every index ν there exists a compact set $K_\nu \subset \mathbb{C}^n$ and constants k_ν and c_ν such that for all $\phi \in C_0^\infty(I_\nu)$, $\psi \in H$,

$$|\langle f,\phi,\psi\rangle| \leq c_\nu\left(\sum_{j=0}^{k_\nu} \sup_{t\in I_\nu} |\phi^{(j)}(t)|\right)\sup_{z\in K_\nu} |\psi(z)|$$

Further, there exists a distribution F on $\mathbb{R}^{1+2n}$, with

(3.3) $\quad \operatorname{supp} F \subset \bigcup_\nu I_\nu \times K_\nu,$

and such that for $\phi \in C_0^\infty(\mathbb{R})$, $\psi \in H \subset C^\infty(\mathbb{R}^{2n})$ we have

$$\langle f,\phi,\psi\rangle = \langle F,\phi\cdot\psi\rangle$$

Proof. The first statement follows from lemma 3.1. To prove the second one, let $\{\theta_\nu\}_1^\infty$ be a C_0^∞ partition of unity on $\mathbb{R}$, with $\operatorname{supp} \theta_\nu \subset I_\nu$, and define the functionals f_ν on $C_0^\infty(I_\nu) \times H$ by

$$\langle f_\nu,\phi,\psi\rangle = \langle f,\phi\cdot\theta_\nu,\psi\rangle \qquad \phi \in C_0^\infty(I_\nu), \quad \psi \in H.$$

By Lemma 3.1, each f_ν can be extended to a distribution F_ν on $\mathbb{R}^{1+2n}$, with $\operatorname{supp} F_\nu \subset I_\nu \times K_\nu$. Since the family of supports of the F_ν is locally finite, the sum $\sum_\nu F_\nu$ converges in $\mathcal{D}'(\mathbb{R}^{1+2n})$, and it is clear that its sum F has the properties stated in the lemma.

Corollary. Let f be as in the lemma, and suppose that f vanishes on $\mathbb{R}^- = \{(t,x),\ t<0\}$ that is

$$\langle f,\phi,\psi\rangle = 0 \quad \text{if} \quad \phi \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}\phi \subset \mathbb{R}^-, \quad \psi \in H.$$

Then $\quad \operatorname{supp} F \subset \{(t,x,y);\quad t \geq 0\}.$

Lemma 3.3. Let $U \in \mathcal{D}'(\mathbb{R}^{1+2n})$ be given, and suppose that for any compact $I \subset \mathbb{R}$ the set

$$K_I = (I \times \mathbb{R}^{2n}) \cap \operatorname{supp} U$$

is compact. Then the "restriction" u of U to $C_0^\infty(\mathbb{R}) \times H(\mathbb{C}^n)$, defined by

$$\langle u, \phi, \psi \rangle = \langle U, \phi \cdot \psi \rangle$$

is in B, and for $\phi \in C_0^\infty(I)$ the analytic functionals $u_\phi : \psi \to \langle u, \phi, \psi \rangle$ can be carried by any compact convex neighbourhood of K_I.

Proof. For $\phi \in C_0^\infty(I)$, we have

$$\langle u, \phi, \psi \rangle \leq C\left(\sum_{j=1}^{n} \sup_I |\phi^{(j)}|\right) \cdot \sum \sup \left| \frac{\partial^\alpha}{\partial x} \frac{\partial^\beta}{\partial y} \psi(x+iy) \right|$$

and the result follows from Cauchy's integral formulae.

4. End of the proof, and a supplement.

The proof is built upon the fact that in B equation (*) has exactly the same solutions as

$$(4.1) \qquad I \frac{\partial u}{\partial t} + \sum_{j=1}^{n} \left[\tfrac{1}{2}(A_j + \bar{A}_j') \frac{\partial u}{\partial x_j} + \frac{1}{2i}(A_j - \bar{A}_j') \frac{\partial u}{\partial y_j} \right] + Bu = f$$

where $\bar{A}_j'$ denote the transposed and complex conjugate of the matrix A_j.

This is seen by adding to (*) the equation

$$(4.2) \qquad \sum_{j=1}^{n} \bar{A}_j' \frac{\partial u}{\partial \bar{z}_j} = 0$$

which is an immediate consequence of the Cauchy-Riemann equations (3.2).

To solve (4.1) we consider it as a system of equations in $\mathbb{R}^{1+2n}$, replacing the bilinear functional f by a distribution F, according to lemma 3.2. Then we have $\operatorname{supp} F \subset \{(t,x,y);\ t \geq 0\}$ and $\operatorname{supp} F \cap (I \times \mathbb{R}^{2n})$ is compact whenever $I \subset \mathbb{R}$ is compact.

Since the matrices $\frac{1}{2}(A_j + \bar{A}_j')$ and $\frac{1}{2i}(A_j - \bar{A}_j')$ are Hermitian, it follows from lemma 2.2 that (4.1) has a unique fundamental solution $E = (E_1,\cdots,E_N) \in [\mathcal{D}'(\mathbb{R}^{1+2n})]^N$ with $\operatorname{supp} E \subset \{(t,x,y);\ t \geq 0\}$. The properties of F then ensure that the distribution

$$(4.3) \qquad U = E * F \quad (= (E_1 * F_1,\cdots,E_N * F_N))$$

is defined and that $\operatorname{supp} U \subset \{(t,x,y),\ t \geq 0\}$.
Since $\operatorname{supp} E$ is contained in a proper convex cone in $\mathbb{R}^{1+2n}$, we also have that $\operatorname{supp} U \cap (I \times\times \mathbb{R}^{2n})$ is compact whenever $I \subset \mathbb{R}$ is. Therefore U is defined on functions of the form $\phi \cdot \psi$, $\phi \in C_0^\infty(\mathbb{R})$, $\psi \in C^\infty(\mathbb{R}^{2n})$, and we can define $u \in B$ by

$$\langle u,\phi,\psi\rangle = \langle U,\phi\cdot\psi\rangle \qquad \phi \in C_0^\infty(\mathbb{R}),\quad \psi \in H$$

according to lemma 3.3.
It is then easily verified that u solves (4.1), and in view of (4.2) also (*); and that the last statement in theorem 1 holds

To show that u is unique, it is sufficient to show that if F vanishes on test functions of the form

$$(t,x,y) \to \phi(t)\psi(x+iy),\quad \phi \in C_0^\infty(\mathbb{R}),\quad \psi \in H$$

then so does U. But this follows from (4.3).

This ends the proof of theorem 1.

With this method of proof, it is possible to transfer much more of the information from [3] Section 5,6 to the situation described in Theorem 1.

We shall, however, content ourselves with sketching one such result, which has been proved earlier by Persson, in [4].

Theorems 2,3,4 and 5,6,3 in [3] imply that the components E_j of the fundamental solution E to (4.1) is in $B_{\infty,p}(\mathbb{R}^{1+2}$ with

$$p(\tau,\xi) = (\det P)(\tau,\xi)\left|\left(\sum_{k=1}^{N} {}^{co}P_{jk}\right)(\tau,\xi)\right|^{-1}$$

(notations from [3] section 2.2). Since $(\det P)(\tau,\xi)$ and $(\sum_k {}^{co}P_{jk})(\tau,\xi)$, considered as polynomials in τ, are monic, and of degree N and $N-1$ respectively, theorem 2,2,8 in [3] imply the following proposition

Proposition 4.1. If the right-hand side f of (*) is a C^n-function from $\mathbb{R}$ to H', then the solution u is a C^{n+1}-function.

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