

On the classification of Anosov Flows.

by

Per Tomter.

University of Oslo

Sept. 1971.

INTRODUCTION.

In his survey article [23] Smale suggested the importance of obtaining progress on the problem of finding all Anosov flows on compact manifolds. We refer to this article for the role played by Anosov flows in the more general theory of differentiable dynamical systems. Until then the known examples of Anosov flows were the geodesic flows on the unit tangent bundles of compact Riemannian manifolds of negative curvature and the suspensions of Anosov diffeomorphisms on infranilmanifolds defined by hyperbolic automorphisms. In [24] a new type of Anosov flow was constructed on a compact 7-dimensional manifold.

This paper contains a general investigation of Anosov flows satisfying an additional symmetry condition (here called symmetric Anosov flows). A procedure for constructing a new class of Anosov flows is given (generalizing the above example); and we obtain a complete description of all symmetric Anosov flows on compact manifolds. It turns out that the parameters which determine a symmetric Anosov flow of the most general type are: i) a spin group  $S_1$  defined over a totally real number field  $F$  s.t.  $S = R_{F/\mathbb{Q}}(S_1)$  (restriction of scalars to  $\mathbb{Q}$ ) is anisotropic over  $\mathbb{Q}$  and has rank one over  $\mathbb{R}$ , and ii) a finite number of highest weights  $\lambda_i$ , which are of "spin type". To describe the associated flow, one then computes, for each index  $i$ , the fixed field  $F_i$  of the isotropy subgroup at  $\lambda_i$  of the Galois group

of  $E/\mathbb{Q}$  ( $E$  is a splitting field for  $S$ , and there is a standard action of the Galois group on the weight space), and a central division algebra  $\kappa_i$  over  $F_i$  s.t. its Hasse-Brauer invariant is the image of a certain invariant of  $S$  in  $H^2(F_i, \mathbb{Z}(S))$  (see Satake [21]) under the induced map  $\lambda_i^*: H^2(F_i, \mathbb{Z}(S)) \rightarrow H^2(F_i, \mathbb{C}^*)$ .

This raises the question: Is an arbitrary Anosov flow on a compact manifold topologically conjugate to a symmetric one?

This is a generalization of the well-known conjecture on Anosov diffeomorphisms: An Anosov diffeomorphism on a compact manifold is topologically conjugate to a hyperbolic infra-nilmanifold diffeomorphism. There has been some recent progress here; the answer is positive in the codimension one case (see Newhouse [18]). On the other hand; all explicitly known examples belongs to the class of symmetric Anosov flows considered here.

In §1 some preliminary remarks on symmetric Anosov flows are made; and some results from [24] are summarized (polished). In §2 the structure on the covering manifold  $G/K$  is determined. In §3 the question of the embedding of the fundamental group in the symmetry group of the flow is considered in detail; and the above parameters are shown to characterize the symmetric Anosov flow completely. Of course, one of the motivations for these group theoretical methods is for the applications to the ergodic properties of Anosov flows. The methods in [24] readily carry over to the situation considered here.

I would like to thank Professors C.C. Moore, M.S. Raghunathan and I. Satake for helpful conversations.

Notation.

$\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  have the usual meaning of integers, rationals, reals and complex numbers;  $\mathbb{C}^*$  the multiplicative group of non-zero complex numbers.

$Z(G)$  denotes the center of the group  $G$ .

$G \rtimes H$  denotes a semidirect product of the groups  $G$  and  $H$ ;  
similarly for the Lie algebras,  $\mathfrak{g} \dot{+} \mathfrak{h}$ .

### §I. Symmetric Anosov Flows.

An Anosov flow on a complete Riemannian manifold  $M$  is a flow  $\{\phi_t\}$  whose induced flow on the unit tangent bundle  $T(M)$  is hyperbolic; i.e.  $T(M) = E_1 \oplus E_2 \oplus E_3$  (continuous Whitney sum of invariant subbundles), where (i)  $D\phi_t$  is contracting on  $E_1$ , i.e.  $\exists c, \lambda > 0$  s.t.  $\|D\phi_t(v)\| \leq c \exp(-\lambda t) \|v\|$  for  $v \in E_1$ , (ii)  $D\phi_t$  is expanding on  $E_2$ ; i.e.  $\exists d, \mu > 0$  s.t.  $\|D\phi_t(v)\| \geq d \exp(\mu t) \|v\|$  for  $v \in E_2$ , and  $E_3$  is the one-dimensional bundle defined by the velocity vector (which is thus assumed non-vanishing at each point).

If  $M$  is compact, the Anosov condition is independant of the Riemannian metric. Then an Anosov flow is structurally stable; i.e. a sufficiently small perturbation  $(\psi_t)$  of  $(\phi_t)$  is topologically conjugate to  $(\phi_t)$ ; (i.e. there is a homeomorphism of  $M$  sending orbits of  $(\phi_t)$  into orbits of  $(\psi_t)$ ). Also, an Anosov flow is known to be ergodic for any invariant measure on  $M$ .

Now, let  $M$  be any connected differentiable manifold with a flow  $\{\phi_t\}$ .

#### Definition.

A Lie transformation group  $G$  on  $M$  is called a symmetry group for  $(M, \{\phi_t\})$  if it acts with compact isotropy subgroups and centralizes  $\{\phi_t\}$  in  $\text{Diff}(M)$ .

We begin with the following observation, which can easily be given in more general versions:

#### Prop. I

Let  $H = \{\phi_t\}$  be a flow on  $M$ , and let  $G$  be a transitive

symmetry group for  $(M, \{\phi_t\})$  with isotropy group  $K$ . Let  $\mathfrak{g}, \mathfrak{h}$ ,  $\kappa$  be the respective Lie algebras. Then there is an element  $\alpha$  in  $\mathfrak{g}$  s.t. the flow  $\phi_t$  is given by:  $gK \rightarrow g(\exp t\alpha)K$ . Furthermore,  $\alpha$  may always be chosen in the centralizer of  $\kappa$ .

Proof: Since  $G$  and  $H$  commute, there is a natural action of  $G \times H$  on  $M$  (use the Bochner-Montgomery theorem for joint smoothness). Clearly this is transitive, so  $M \cong G/K = G \times H / K^1$ , with  $K = K^1 \cap G$ ,  $\dim K^1 = \dim K + 1$ . On  $G \times H / K^1$  the flow is right multiplication by  $H$ : let  $H = \{\exp(t\beta)\}$  then  $(\exp t\beta) \cdot (g, h)K^1 = (g, (\exp t\beta)h)K^1 = (g, h(\exp t\beta))K^1 = (g, h)(\exp t\beta)K^1$ . Let  $p$  be the projection from  $\mathfrak{g}$  to  $\mathfrak{h}$ . Then, by dimension,  $p(\kappa^1) = \mathfrak{h}$ . Choose  $k$  in  $\kappa^1$  with  $k = \alpha + \beta$ ,  $\alpha \in \mathfrak{g}$  then  $\exp t\beta = (\exp t\alpha)(\exp tk)$ , and the flow is:  $(g, h)K^1 \rightarrow (g, h)(\exp t\alpha)K^1 = (g(\exp t\alpha), h)K^1$ . Hence, by the natural identification  $M \cong G/K: gK \rightarrow g(\exp t\alpha)K$ .

For the last statement of the proposition, notice that  $\exp(t\alpha) = \exp(t\beta) \exp(-tk)$  normalizes  $K^1$  and hence also  $K$  and its identity component  $K_0$ .  $K_0$  is finitely covered by  $T \times S$ , where  $T$  is a torus and  $S$  is compact, semisimple. A one-parameter group of automorphisms must be trivial on  $T$ , and given by inner conjugation on  $S$ . Hence, for the corresponding decomposition of  $\kappa$ :  $\kappa = \mathcal{T} + \mathcal{S}$ , we have:  $\text{ad}\alpha|_{\mathcal{T}} = 0$ , and  $\text{ad}\alpha|_{\mathcal{S}} = \text{ad}s|_{\mathcal{S}}$  for some  $s$  in  $\mathcal{S}$ . But then  $g(\exp t\alpha)K = g \exp(t(\alpha-s)) \exp(ts)K = g \exp(t(\alpha-s))K$ , and  $\alpha-s$  is in the centralizer of  $\kappa$ . q.e.d.

Now, let  $(M, \{\phi_t\})$  be as before, and let  $\tilde{M} \rightarrow M$  be a normal covering with  $\Gamma$  as the group of deck transformations. Then  $\Gamma$  is a symmetry group for the lifted flow  $\{\tilde{\phi}_t\}$  on  $\tilde{M}$ .

Definition. The flow  $(M, \{\phi_t\})$  is called symmetric if there exists a normal covering  $\tilde{M} \rightarrow M$  such that the group of deck transformations can be extended to a transitive symmetry group for  $(\tilde{M}, \{\tilde{\phi}_t\})$ .

Thus, a symmetric flow on a compact manifold is given by the following data:

- i) A Lie group  $G$  with a finite number of components
- ii) A compact subgroup  $K$  which intersects all components of  $G$ .
- iii) A uniform, discrete subgroup  $\Gamma$  which acts freely on  $G/K$ . (This means that if one conjugate of an element of  $\Gamma$  lies in  $K$ , then all its conjugates lie in  $K$  and the element acts trivially on  $G/K$ .)
- iv) A one-parameter subgroup  $\exp(t\alpha)$  in the normalizer of  $K$ .

Then the flow is given on the manifold  $\Gamma \backslash G/K$  by:  $\Gamma gK \rightarrow \Gamma g(\exp t\alpha)K$ . (It is clear that one could normalize by lifting to the universal covering manifold  $\tilde{M}$ , however, f.ex. in the case of geodesic flows on surfaces of negative curvature it is customary to lift to an intermediate covering).

Let  $\mathfrak{g}$  and  $\kappa$  be the Lie algebras of  $G$  and  $K$  respectively. Let  $I(\alpha)$  be the subalgebra of corresponding to eigenvalues on the imaginary axis in the primary decomposition of  $\mathfrak{g}$  under  $a\alpha$ . From the proof of the above proposition it is clear that  $\kappa + Ra \subseteq I(\alpha)$ . By the following theorem, the Anosov case is characterized as precisely the case where this inclusion becomes an equality.

Theorem 1.

Let  $M$  be a compact, connected manifold with a symmetric flow  $\{\phi_t\}$ . Let  $G, K, \Gamma, \mathfrak{g}, \kappa, \alpha$  be as above. Then  $\{\phi_t\}$  is an Anosov flow on  $M = \Gamma \backslash G/K$  iff  $\kappa + Ra$  is the subalgebra cor-

responding to the imaginary eigenvalues (including zero) in the primary decomposition of  $\mathcal{O}$  under  $a\alpha$ .

Proof.

A  $G$ -invariant Riemannian structure is defined on  $G/K$  in the standard way; this projects to a Riemannian structure on  $\Gamma \backslash G/K$ . The Anosov condition only has to be checked on  $G/K$ ; for the details of this, see [24], where sufficiency is proved.  $\exp(-t a\alpha)$  is an exponentially contracting (expanding) one-parameter family on the parts of the tangent-space of  $G/K$  corresponding to positive real part (resp. negative) eigenvalues of  $a\alpha$ . On the other hand, if there were imaginary eigenvalues of  $a\alpha$  outside  $\kappa + \mathbb{R}\alpha$ , there could not be exponential growth corresponding to those directions.

Remark 1.

In [24] it was assumed that  $a\alpha|_K = 0$ . According to prop. 1 this is no loss of generality; and we make this assumption from now on.

Remark 2.

The flow  $(M, \{\phi_t\})$  is finitely covered by  $(\bar{M}, \{\phi_t\})$  if there exists a finite, normal flow-equivariant covering map from  $\bar{M}$  to  $M$ . Two flows are said to be commensurable if they admit a common covering, finite over both. Then, if one flow is symmetric Anosov, the other one must also be.

Let  $G_0$  be the identity component of  $G$ .

Then  $G/K = G_0/K \cap G_0$ ;  $\Gamma \cap G_0$  has finite index in  $\Gamma$ , and  $\Gamma G_0 \backslash G_0/K \rightarrow \Gamma \backslash G/K$  is finite, flow equivariant covering. We are only interested in classifying Anosov flows up to commensurability; hence we assume  $G$  connected.

Remark 3.

In [24] the unstable ideal  $\mathcal{U}$  was defined as the subideal of  $\mathcal{O}$  generated by the subspaces of  $\mathcal{O}$  corresponding to eigen-

values of  $\alpha$  outside the imaginary axis. Let  $\mathcal{V} = \mathcal{U} + \mathbb{R}\alpha$ , and let  $V$  be the corresponding subgroup; then  $V$  acts transitively on  $G/K$ . However, as will be clear from the construction in §3, it will not in general be possible to obtain  $M = \Gamma \backslash G/K$  as a quotient by a discrete subgroup of  $V$ ; so the restriction would mean some loss of generality. On the other hand we do make the assumption that no ideal of  $\mathcal{O}$  is contained in  $\kappa$ . For in that case  $K$  would contain a closed subgroup  $L$  normal in  $G$ , and  $\Gamma \backslash G/K$  would be flow-equivariantly diffeomorphic to  $\Gamma \backslash G/L / K/L$  (the projection of  $\Gamma$  into  $G/L$  is obviously injective).

The following theorems were essentially proved in [24]:

Theorem 2.

The symmetric Anosov flows with solvable Lie group are precisely those which are finitely covered by the suspension of Anosov diffeomorphisms given by hyperbolic automorphisms of nilmanifolds.

Here we recall the construction of such automorphisms: Let  $N$  be a simply connected, nilpotent Lie group with uniform, discrete subgroup  $\Delta$ ;  $\phi$  an automorphism of  $N$  such that  $d\phi$  has no eigenvalues on the unit circle and  $\phi(\Delta) = \Delta$ ; then  $\phi$  induces an Anosov diffeomorphism  $\bar{\phi}$  on  $\Delta^N$ . Then it can be proved that  $\phi = JA = AJ$ , with  $J^N = \text{identity}$ ,  $A = \psi_1$  where  $\{\psi_t\}$  is a one-parameter group in  $\text{Aut}(N)$ . Let  $K$  be the cyclic group generated by  $J$ . Let  $G = (N \times K) \rtimes R$  be the semidirect product,  $\mathbb{R}$  acting through  $\{\psi_t\}$ . Let  $\Gamma$  be the group generated by  $(e, J, 1)$ , and  $\alpha$  the generator of  $\{\psi_t\}$ . Then the corresponding symmetric flow is the suspension of  $\bar{\phi}$ .

Conversely, if  $\mathcal{O}$  is assumed solvable,  $\kappa$  must be Abelian, and  $\alpha \notin \mathcal{U}$ . The analytic subgroup corresponding to

is the nilradical  $N$ ; by a theorem of Mostow  $\Gamma \cdot N$  is closed, and  $\Gamma \cdot N$  is uniform in  $N$ . Then the projection of  $\Gamma$  on  $K \times R = G/N$  is discrete, uniform, and by Ausländer's Bieberbach theorem it follows that  $\Gamma$  has a finite index subgroup which is included in  $N \times R$ . For other details, and the infranil-manifold case, see [24].

Theorem 3.

The symmetric Anosov flows with semisimple Lie groups are precisely those which are finitely covered by the geodesic flows on the unit tangent bundles of compact Clifford-Klein forms of the symmetric spaces of negative curvature; i.e. real, complex, and quaternionic hyperbolic space, together with the Cayley hyperbolic plane. The corresponding Lie algebras are  $S_0(n,1)$ ,  $Su(n,1)$ ,  $Sp(n,1)$  and the real form of  $F_4$  which has split rank 1.

We recall that the generator  $\alpha$  for the geodesic flow defines a real, split Cartan subalgebra of  $\mathfrak{g}$ ;  $\mathfrak{n}$  is the centralizer of  $\alpha$  in the compact part of the Cartan decomposition (in the Iwasawa decomposition this is usually denoted  $\mathfrak{m}$ ). Borel has shown the existence of infinitely many Clifford-Klein forms ([7] and [10]). If  $G$  is the isometry group and  $\Gamma$  has no torsion, we have the geodesic flow on a unit tangent bundle. By a result of A. Selberg,  $\Gamma$  always has a torsion-free subgroup of finite index. Explicit examples are found by taking  $\Gamma$  to be the arithmetic subgroup corresponding to a rational structure on  $G$  (see § 3).

We will need the following:

Remark. We can generalize the conditions as follows:

For theorem 2: If the ideal  $\mathcal{V}$  is solvable, we get the suspended flows.

For theorem 3: If the ideal  $\mathcal{V}$  is semisimple, we get the geodesic flows.

This is easily verified. F.ex., if  $\mathcal{V}$  is semisimple,  $\mathcal{U} = \mathcal{V}$  is a semifimple ideal; if  $\mathcal{R}$  is the radical of  $\mathcal{J}$ ,  $[\mathcal{R}, \mathcal{U}] \subseteq \mathcal{R} \cap \mathcal{U} = (0)$ . It follows that  $\mathcal{R} \subseteq \mathfrak{n}$ ; and hence  $\mathcal{R} = (0)$  by Remark 3.

## § 2. The Covering Flows of Mixed Type.

In [24] a new example of Anosov flows was discovered by realizing  $SL(2, \mathbb{R})$  as the group of unit quaternions in the even-dimensional Clifford algebra  $C^+$  corresponding to an anisotropic quadratic form on  $\mathbb{R}^3$ ; and taking the corresponding representation on  $C^+$ .  $\Gamma$  was constructed as the semidirect product of the arithmetic subgroup of  $SL(2, \mathbb{R})$  with the lattice in  $C^+$ . We will give an ultimate generalization of this construction which exhausts the class of symmetric Anosov flows. In this chapter we determine the  $(G, K, \alpha)$  which satisfy the conditions of §1. Some explicit computations are avoided by applying a representation-theoretical result due to Kostant and Rallis.

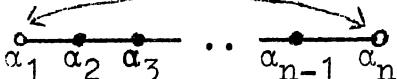
For the remaining cases of symmetric Anosov flows we may now assume that the ideal  $\mathcal{V}$  is neither solvable nor semisimple. Let  $\mathcal{R}$  be the radical of  $\mathcal{G}$ , and  $\pi$  the projection  $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{R} = \mathcal{S}$ . Then we have the following observations, which follow from easy computations similar to [24]:  $\pi(\alpha)$  is a compactly embedded subalgebra of  $\mathcal{S}$ ,  $\pi(\alpha)$  acts trivially on  $\pi(\mathcal{u})$  with non-imaginary eigenvalues outside  $\pi(\mathcal{u}) + \pi(\mathbb{R}\alpha)$ . Then  $\mathcal{S} = \mathcal{S}_1 + \mathcal{u}_1 + \dots + \mathcal{u}_p$  where  $\mathcal{S}_1$  is a simple, real rank 1 Lie algebra, and  $\mathcal{u}_1, \dots, \mathcal{u}_p$  are compact simple. It follows that there must exist elements  $e_\lambda$  and  $e_{-\lambda}$  in the subspaces of  $\mathcal{G}$  corresponding to eigenvalues  $\lambda$  and  $-\lambda$  respectively, such that  $[e_\lambda, e_{-\lambda}] = \alpha + k$ ,  $k \in \mathcal{R} \cap \mathcal{u}$ . Obviously, we may substitute  $\alpha + k$  as new flow generator, hence we may assume  $[e_\lambda, e_{-\lambda}] = \alpha$ . The subalgebra  $[\mathcal{G}, \mathcal{G}]$  contains  $\alpha$  and is algebraic; hence  $\alpha$  has a Jordan decomposition in  $[\mathcal{G}, \mathcal{G}]$  where the nilpotent part is seen to vanish. It follows that  $\{\alpha, e_\lambda, e_{-\lambda}\}$  span an  $S_1(2, \mathbb{R})$  - subalgebra of  $\mathcal{G}$ ; and by a

known corollary to the Levi-Malcev theorem (Bourbaki [13]), this may be embedded in a Levi subalgebra  $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{n}_1 \oplus \dots \oplus \mathcal{n}_p$  of  $\mathcal{G}$ . Now  $\mathcal{R} = \mathcal{N} + \mathcal{R} \cap \mathcal{n}$ , where  $\mathcal{N}$ , the nilpotent radical of  $\mathcal{G}$ , corresponds to non-zero eigenvalues of  $\text{ad } \alpha$ . The restriction of the action of  $\mathcal{S}$  to the invariant subspace  $\mathcal{N}$  now defines a representation of  $\mathcal{S}$  s.t. the split Cartan algebra  $\mathbb{R}\alpha$  acts without zero eigenvalues. By standard representation theory; it is sufficient to consider irreducible, complex representations of the first factor  $\mathcal{S}_1$ .

In general, let  $\mathcal{O}$  be a real, split Cartan subalgebra of a semisimple Lie algebra, and consider an irreducible representation. The restriction of the weights to  $\mathcal{O}$  defines the "restricted weight system". The "restricted root system" is a root system; but not generally reduced; i.e. twice a root may be a root. Thus we have a "restricted weight lattice" and a "restricted root lattice". We need to compute these explicitly for our real rank 1 simple groups.

$A_n$ : The simple roots  $\alpha_1, \dots, \alpha_n$  form a basis for the Cartan subalgebra. By inverting the Cartan matrix one finds the fundamental highest weights:

$$\lambda_i = \frac{n+1-i}{n+1}(\alpha_1 + 2\alpha_2 + \dots + i\alpha_i) + \frac{i}{n+1}((n-i)\alpha_{i+1} + \dots + 2\alpha_{n-1} + \alpha_n)$$

The real form of split rank one is given by the following Satake-Dynkin diagram: 

with the corresponding conjugation  $\sigma$ :  $\sigma(\alpha_1) = \alpha_2 + \dots + \alpha_n$ .

Then  $\alpha_1 + \sigma(\alpha_1)$  is  $\sigma$ -invariant; and the corresponding split Cartan subalgebra is spanned by  $\alpha = 4h[\alpha_1 + \dots + \alpha_n]$ . (Here, for  $\beta$  in the dual Cartan subalgebra  $\mathfrak{g}^*$ ,  $h[\beta]$  denotes the corresponding element in  $\mathfrak{g}$  under the Cartan-Killing form.)

Fundamental restricted roots:

$$\begin{aligned}\alpha_i(\alpha) &= 4 \sum_j \alpha_i(h[\alpha_j]) = 0 \quad \text{for } i \neq 1, n \\ &\quad = 2 \quad \text{for } i = 1, n.\end{aligned}$$

Fundamental restricted weights:  $\lambda_i(\alpha) = 2$ .

Conclusion: For  $Su(n,1)$  the restricted weight lattice coincides with the restricted root lattice.

Now, let  $\{h[\alpha_i], e[\gamma]\}$  be a basis for  $\mathcal{O}$  satisfying the first Weyl normalization; i.e.  $[e[\gamma], e[-\gamma]] = -h[\gamma]$ . ( $\gamma$  varies over the set of roots.) Let  $X = 2(e[\alpha_1 + \dots + \alpha_{n-1}] + e[\alpha_n])$ ,  $Y = -2(e[-\alpha_1 - \dots - \alpha_{n-1}] + e[-\alpha_n])$ . Then  $\{X, Y, \alpha\}$  span an (2) subalgebra whose root-lattice coincides with the restricted root lattice above:  $[X, Y] = \alpha$ ;  $[\alpha, X] = 2X$ ,  $[\alpha, Y] = -2Y$ . For  $Sl(2)$  zero is a weight in an irreducible representation iff the highest weight is in the root lattice. Hence, for an irreducible representation of  $\mathcal{O}$ , by decomposing the restriction of the representation to  $Sl(2)$ , we see that  $\alpha$  acts without zero eigenvalues precisely if the restriction of the highest weight does not lie in the restricted root lattice. But this is an impossible situation here.

Proposition: No representation of  $Su(n,1)$  can give an Anosov flow of mixed type.

Note. One must chose the generators with some care;

$X = 2e[\alpha_1 + \dots + \alpha_n]$ ,  $Y = -2e[-\alpha_1 - \dots - \alpha_n]$  would give a root lattice for the corresponding  $Sl(2)$  coarser than the restricted root lattice from  $\mathcal{O}$ , and hence would be useless.

Instead of performing this latter computation for all the groups, we establish a general criterion, which follows at once

from a theorem by Kostant and Rallis [15].

Prop. An irreducible representation of a real, semisimple Lie algebra has a zero restricted weight iff the highest restricted weight  $\lambda$  belongs to the restricted root lattice.

Proof. The result is well known in the absolute case. The condition is necessary, since any restricted weight is obtained from  $\lambda$  by subtracting elements in the root lattice. We quote the needed result from [15]: Let  $\Sigma$  be the restricted root system, consider the reduced root system  $\Lambda$  on the split Cartan subalgebra  $\mathcal{O}$  obtained by removing every restricted root  $\Phi$  s.t.  $\frac{\Phi}{2}$  is a restricted root. Then there exists a semisimple subalgebra, split over the reals, for which  $\mathcal{O}$  is a Cartan subalgebra; and whose root system w.r.t.  $\mathcal{O}$  is precisely  $\Lambda$ . --- Now, by restricting the representation to this subalgebra, we are in the split case, and the result follows by decomposing this restriction.

The computations for the remaining cases are now routine.

The results are:

$$B_n : \alpha_1 \overset{\circ}{\alpha}_2 \overset{\bullet}{\alpha}_3 \dots \overset{\bullet}{\alpha}_{n-1} \overset{\rightarrow}{\alpha}_n$$

Fundamental roots:  $\alpha_1, \dots, \alpha_n$ .

Fundamental highest weights:  $\lambda_i = \alpha_1 + 2\alpha_2 + \dots + i(\alpha_i + \dots + \alpha_n)$

for  $i = 1, 2, \dots, n-1$ .  $\lambda_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \dots + n\alpha_n)$ .

Conjugation:  $\sigma(\alpha_1) = \alpha_1 + 2(\alpha_2 + \alpha_3 + \dots + \alpha_n)$ .

$\alpha = h[\alpha_1 + \sigma(\alpha_1)] = 2h[\alpha_1 + \dots + \alpha_n]$ .

Restricted fundamental roots:  $\alpha_1(\alpha) = 2$ .  $\alpha_i(\alpha) = 0$  for  $i > 1$ .

Restricted fundamental weights:  $\lambda_i(\alpha) = 2$  for  $i < n$ .  $\lambda_n(\alpha) = 1$ .

The restricted root lattice is of index two in the restricted

weight lattice. Let  $\lambda$  be the highest weight of an irreducible representation, then  $\lambda = \sum_{i=1}^n n_i \lambda_i$ ,  $n_i$  positive integers. There is no zero restricted weight precisely in the case where the coefficient of the spin representation  $\lambda_n$  is odd. These are precisely the representations which are faithful on the spin group of the quadratic form of index 1; i.e. they do not project to representations of the corresponding orthogonal groups.

$$C_n : \alpha_1 \ \overset{\circ}{\alpha}_2 \ \overset{\bullet}{\alpha}_3 \ \overset{\circ}{\alpha}_4 \ \cdots \ \overset{\bullet}{\alpha}_{n-1} \overset{\leftarrow}{\alpha}_n$$

Fundamental roots:  $\alpha_1, \dots, \alpha_n$ .

Fundamental highest weights:  $\lambda_i = \alpha_1 + 2\alpha_2 + \dots + (i-1)\alpha_{i-1} + i(\alpha_i + \dots + \alpha_{n-1} + \frac{1}{2}\alpha_n)$

Conjugation:  $\sigma(\alpha_2) = \alpha_1 + \alpha_2 + 2(\alpha_3 + \dots + \alpha_{n-1}) + \alpha_n$ .

$\alpha = h[\alpha_2 + \sigma(\alpha_2)] = h[\alpha_1 + 2(\alpha_2 + \dots + \alpha_{n-1}) + \alpha_n]$

Restricted fundamental roots:  $\alpha_i(\alpha) = 0$  for  $i \neq 2$ .  $\alpha_2(\alpha) = \frac{1}{2}$ .

Restricted fundamental weights  $\lambda_i(\alpha) = \frac{1}{2}$ ,  $i = 1, 2, \dots, n$ .

The restricted lattices coincide; and all representations must have zero restricted weights.

$$D_n : \alpha_1 \ \overset{\circ}{\alpha}_2 \ \overset{\bullet}{\alpha}_3 \ \cdots \ \overset{\bullet}{\alpha}_{n-1} \overset{\leftarrow}{\alpha}_n$$

Fundamental roots:  $\alpha_1, \dots, \alpha_n$ .

Fundamental highest weights:  $\lambda_i = \alpha_1 + 2\alpha_2 + \dots + i(\alpha_i + \alpha_{i+1} + \dots + \alpha_{n-2}) + \frac{1}{2}i(\alpha_{n-1} + \alpha_n)$  for  $i = 1, 2, \dots, n-2$ .

$$\lambda_{n-1} = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2}) + \frac{n}{4}\alpha_{n-1} + \frac{n-2}{4}\alpha_n$$

$$\lambda_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2}) + \frac{n-2}{4}\alpha_{n-1} + \frac{n}{4}\alpha_n$$

Conjugation:  $\sigma(\alpha_1) = \alpha_1 + 2(\alpha_2 + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$

$$\alpha = h[\alpha_1 + \sigma(\alpha_1)] = h[2(\alpha_1 + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n].$$

Restricted fundamental roots:  $\alpha_i(\alpha) = 1$  for  $i = 1$ ,  $\alpha_i(\alpha) = 0$  for  $i \neq 1$

Restricted fundamental weights:  $\lambda_i(\alpha) = 1$  for  $i \leq n-2$ .

$$\lambda_{n-1}(\alpha) = \frac{1}{2}, \quad \lambda_n(\alpha) = \frac{1}{2}.$$

Again the quotient of the lattices is  $\mathbb{Z}_2$ . Let  $\lambda = \sum n_i \lambda_i$  be the highest weight of an irreducible representation. Then there is no zero restricted weight precisely in the case where the sum of the indexes of the two half-spin representations is odd.

For  $n = 2m+1$ : the center of the real spin group is lying as  $\mathbb{Z}_2 = \{1, z^2\}$  in the center of the complex spin group  $\mathbb{Z}_4 = \{1, z, z^2, z^3\}$ . Both the half-spin representations are  $-1$  on  $z^2$ .

For  $n = 2m$  the situation is that  $\mathbb{Z}_2$  is lying as  $\{1, zz'\}$  in the center of the complex spin group,  $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{1, z\} \oplus \{1, z'\}$ .

Again both half-spin representations are  $-1$  on  $zz'$ .

From this it follows that the above representations are again precisely those which are faithful on the real spin group.

$F_4$ . The simply connected complex group is centerfree, so the root - and weight lattices coincide already over the complex numbers, hence any representation has a zero weight.

Theorem 4. Symmetric Anosov flows of "mixed type" can be constructed only from faithful representations of the real rank one spin groups.

Proposition. The nilradical  $\mathcal{N}$  must be Abelian.

Proof. Consider the ascending central series of  $\mathcal{N}$ :

$\mathcal{N} = \mathcal{N}_k \supseteq \dots \supseteq \mathcal{N}_1 = \mathcal{Z}(\mathcal{N}) \supseteq (0)$ . Since the  $\mathcal{N}_i$ 's are characteristic ideals, the filtration is preserved under the action of  $S$ . Consider this action restricted to  $\mathcal{N}_2$ ; then  $\mathcal{N}_2 = \mathcal{N}' + \mathcal{N}_1$ ,

where  $\mathcal{N}'$  is an  $S$ -module complement of  $\mathcal{N}_1$  in  $\mathcal{N}_2$ . The Lie-algebra structure is defined by an  $S$ -invariant element  $\Phi$  of  $\text{Hom}(\Lambda^2(\mathcal{N}'), \mathcal{N}_1)$ . If  $e_\lambda, e_\mu \in \mathcal{N}'$  with  $\text{ad } \alpha(e_\lambda) = \lambda e_\lambda$ ,  $\text{ad } \alpha(e_\mu) = \mu e_\mu$ , then  $\text{ad } \alpha(\Phi(e_\lambda \wedge e_\mu)) = \Phi(\text{ad } \alpha(e_\lambda) \wedge e_\mu) + \Phi(e_\lambda \wedge \text{ad } \alpha(e_\mu)) = (\lambda + \mu)\Phi(e_\lambda \wedge e_\mu)$ . But then  $\Phi(e_\lambda \wedge e_\mu) = 0$ , since otherwise  $\lambda + \mu$  would be a restricted weight belonging to the restricted root lattice. Hence the central series has only one term, and  $\mathcal{N}$  is Abelian. q.e.d.

The only remaining unknown part of  $\mathcal{N}$  is  $\mathcal{n} \cap \mathcal{R}$ . In the next paragraph it will be proved that this can be ignored w.l.o.g. Hence we have determined  $G, K, \alpha$ ; i.e. the flow on the covering manifold  $G/K$ .

### § 3. The Embedding of the Fundamental Group

We continue with the assumptions of § 2.

The Abelian nilradical is now denoted by  $\mathcal{V}$ .

So  $\mathcal{O} = (\mathcal{V} + \mathcal{R} \cap \kappa) + (S_1 + \kappa_1 + \dots + \kappa_p)$ . The corresponding simply connected group is  $\tilde{G} = (V \times E) \times (S_1 \times K_1 \times \dots \times K_p)$ , where  $V$  and  $E$  are vector groups,  $K_1, \dots, K_p$  compact, simple, and  $S_1$  the spin group of a quadratic form of index 1. Let  $K_0$  be the compact subgroup of  $S_1$  corresponding to  $\kappa \cap S_1$ . Then  $G = p(\tilde{G}) = \tilde{G}/\tilde{D}$ , where  $\tilde{D}$  is a discrete central subgroup of  $\tilde{G}$ . Since  $\mathcal{R} \cap \kappa$  is central in  $\kappa$ ,  $\tilde{K} = E \times K_0 \times K_1 \times \dots \times K_p$  is a subgroup of  $\tilde{G}$ , and  $p(\tilde{K}) = \tilde{K}/\tilde{K} \cap \tilde{D} = K$  is compact by hypothesis. From Ausländer's Bieberbach theorem [5],  $E \cap \tilde{D}$  is uniform in  $E$  and of finite index in  $\tilde{K} \cap \tilde{D}$ . Let  $\bar{D} = E \cap \tilde{D}$ , then  $\bar{G} = \tilde{G}/\bar{D} = (V \times T) \times (S_1 \times K_1 \times \dots \times K_p)$  with  $T$  a torus is a covering group of  $G$ ,  $\bar{K} = \tilde{K}/\bar{D}$  is compact; then we may obviously use  $\bar{G}, \bar{K}, \bar{\Gamma}$  (lift of  $\Gamma$ ) for  $G, K, \Gamma$ .

Lemma 1.  $\bar{\Gamma} \cap V$  is a lattice in  $V$ .

This follows from work of Ausländer [6]. Let  $\tilde{\Gamma} = p^{-1}(\Gamma)$ , a uniform, discrete subgroup of  $\tilde{G}$ . By Ausländer's extension of a theorem of Zassenhaus [6], the identity component of  $\overline{\tilde{\Gamma} \cdot (V \times E)}$  is solvable. By applying Borel's density theorem, it follows that  $\overline{\tilde{\Gamma} \cdot (V \times E) \cdot (K_1 \times \dots \times K_p)} / (V \times E) \cdot (K_1 \times \dots \times K_p)$  is discrete.

Hence  $\tilde{\Gamma} \cap (V \times E) \times (K_1 \times \dots \times K_p)$  is uniform, discrete in the latter group, and by projection  $\tilde{\Gamma} \cap (V \times T) \times (K_1 \times \dots \times K_p)$  is also uniform, discrete. By applying Ausländer's Bieberbach theorem again,  $\Gamma \cap V$  is a lattice in  $V$ .

We let  $G = (V \times T) \times (S_1 \times K_1 \times \dots \times K_p) = (V \times T) \times S$ . Let

$q$  be the projection  $G \rightarrow G/V$ , and  $\pi$  the representation  $G/V \rightarrow \text{Aut}(V)$ .  $\Gamma \cap V$  is a lattice in  $V$ , and  $q(\Gamma)$  is uniform discrete in  $G/V$ .

Lemma 2.  $\ker \pi$  is a finite subgroup of  $T \times Z(S_1) \times Z(K_1) \times \dots \times Z(K_p) \subseteq G/V$ .

Proof: Let  $(v, t, s^{-1}, k_1, \dots, k_p) \in \ker \pi$ ; i.e.  $\pi(s) = \pi(v, t, e, k_1, \dots, k_p)$  which centralizes  $\pi(S_1)$ . Since  $\pi$  is faithful on  $S_1$ , it follows that  $s \in Z(S_1)$ . Similarly,  $k_i \in Z(K_i)$  (we can assume  $\pi$  is almost faithful on the  $K_i$ 's). So  $\ker \pi \subseteq T \times Z(S_1) \times \dots \times Z(K_p)$ . A one-parameter subgroup would have to be in  $T$ , its Lie algebra would bracket  $\mathcal{O}$  into  $\mathcal{V}$ , by semi-simplicity it would be central in  $\mathcal{O}$ , a contradiction. Hence  $\ker \pi$  is discrete and finite.

Let  $\varphi, \psi$  be the restrictions of  $\pi$  to  $T$  and  $S$ .

Lemma 3.  $\pi(q(\Gamma))$  belongs to the arithmetic subgroup of  $\text{Aut}(V)$  corresponding to the lattice  $\Gamma_V = \Gamma \cap V$ .

Proof: Let  $\gamma = (a, b, c) \in [(V \times T) \times S] \cap \Gamma$ , then  $\gamma^{-1} = (\psi_c^{-1}[\varphi_b^{-1}(-a), b^{-1}], c^{-1})$ . Let  $(v, e, e) \in \Gamma_V$ . Then  $\gamma^{-1}(v, e, e)\gamma = (\psi_c^{-1}[\varphi_b^{-1}(-a), b^{-1}], c^{-1})(v+a, b, c) = (\psi_c^{-1}[\varphi_b^{-1}(-a), b^{-1}], \psi_c^{-1}(v+a, b), e) = (\psi_c^{-1}(\varphi_b^{-1}(v)), e, e)$ . Hence  $\psi_c^{-1}(\varphi_b^{-1}(v)) \in \Gamma_V$ .

Notice that  $H = G/V = T \times S_1 \times K_1 \times \dots \times K_p$  is a direct product, although  $T$  and  $S_1$  do not need to commute in  $G$ .

The following theorem justifies the remark in § 2 that  $\kappa \cap \mathcal{R}$  may be assumed (0).

Theorem 5.  $\Gamma \cap (V \times S)$  is uniform, discrete in the subgroup  $V \times S$ , hence we may assume  $G = V \times S$  without loss of generality.

Proof:  $H$  is a "Tannaka" group since  $H/[H,H]$  is compact. Let  $H^+$  be the "universal complexification" of  $H$ , let  $\pi^+$  be the extension of  $\pi$  mapping  $H^+$  into  $\text{Aut}(V_{\mathbb{C}})$ . Then  $\pi^+(H^+)$  is an algebraic subgroup of  $\text{Aut}(V_{\mathbb{C}})$  (Hochschild and Mostow [14]), and  $\pi(H)$  must be the identity component of its real points. The lattice  $\Gamma_V$  defines a  $\mathbb{Q}$ -structure; and the reductive group  $\pi^+(H^+)$  is the almost direct product of its connected center and the semisimple part (= the commutator subgroup), both defined over  $\mathbb{Q}$ . Since  $\pi(H)$  is the almost direct product, its arithmetic subgroup  $\Delta$  is commensurable with  $\Delta_1 \cdot \Delta_2$ ;  $\Delta_1 = \Delta \cap \pi(T)$ ,  $\Delta_2 = \Delta \cap \pi(S)$ . By lemma 3  $\pi(q(\Gamma))$  is also commensurable with  $\Delta_1 \cdot \Delta_2$ ; with lemma 2 one can then show that  $q(\Gamma)$  is commensurable with  $\varphi^{-1}(\Delta_1) \cdot \psi^{-1}(\Delta_2)$ ; hence with  $\psi^{-1}(\Delta_2)$ . Then  $\Gamma \cap V \times S$  is uniform discrete in  $V \times S$ .  $(V \times S / \Gamma \cap (V \times S))$  is a fibration over  $S / q(\Gamma \cap (V \times S))$  with fibre  $V / \Gamma_V$ . q.e.d.

We have now realized  $S = S_1 \times K_1 \times \dots \times K_p$  as the identity component of a simply connected, semisimple algebraic  $\mathbb{Q}$ -group; it must then split as the direct product of  $\mathbb{Q}$ -simple groups.  $S_1$  belongs to one such factor  $S_2$ , and since  $q(\Gamma) \cap S_2$  is uniform, discrete in  $S_2$ ; we may assume that  $S$  is  $\mathbb{Q}$ -simple. Viewed as an algebraic group,  $S$  is now the direct product of  $\mathbb{C}$ -simple factors. Here  $S_1$  is defined over an algebraic number field  $F$  of finite degree. The Galois-automorphisms of  $F/\mathbb{Q}$  exchange the factors, so they are all  $\mathbb{C}$ -isomorphic.  $S = R_{F/\mathbb{Q}}(S_1)$  (restriction of the scalars from  $F$  to  $\mathbb{Q}$ ). Over the reals, the factors of  $S$  consist of the  $(S_1^\sigma)_{\mathbb{R}}$ 's for the Galois-automorphisms  $\sigma$  with

$\sigma = \bar{\sigma}$ , and one  $(S_1^\sigma)_{\mathbb{Q}}$  (viewed as a real group) corresponding to each pair of Galois-automorphisms  $(\sigma, \bar{\sigma})$  with  $\sigma \neq \bar{\sigma}$ . (Borel-Harish-Chandra [11]). In our case, since  $K_1, \dots, K_p$  are all compact, it follows that  $F$  is totally real; i.e.  $F = \mathbb{Q}[u]$ ,  $u \in \mathbb{R}$ .  $u > 0$  with all Galois-conjugates of  $u$  negative.

We review when such an  $F$ -structure on  $S_1$  gives rise to a uniform, arithmetic subgroup.

Case 1.  $F \neq \mathbb{Q}$ . Then compact factors occur, and the arithmetic subgroup will always be uniform. For any element of  $K_i$  is semisimple, hence any element of  $(S)_\mathbb{Q}$  is semisimple (its  $S_1$ -component is the Galois-conjugate of a  $K_i$ -component).  $S_\mathbb{Q}$  has no unipotent elements, and the conclusion follows from the "Godement conjecture". (Borel-Harish-Chandra [11]).

Case 2.  $F = \mathbb{Q}$ . The quadratic form is defined over  $\mathbb{Q}$ , and the arithmetic subgroup is uniform precisely if it is  $\mathbb{Q}$ -anisotropic; (i.e. the diophantine equation has no integer solution). The example in [24] was of this kind, but such examples are possible only in low dimensions.

To a general symmetric Anosov flow we have now associated the following data: A spin group  $S_1$  defined over a totally real number field  $F$  s.t.  $R_{F/\mathbb{Q}}(S_1)$  has  $\mathbb{Q}$ -rank zero, real rank one; and a faithful rational representation. Conversely, given these data, we construct an Anosov flow in the obvious manner, choosing for  $\Gamma$  the semidirect product of the arithmetic group in  $S$  and the lattice in  $V$ , (possibly dropping to a subgroup of finite index in  $S$ , to avoid intersections with conjugates of  $K$ ). As was already remarked, Borel has constructed such  $F$ -structures for all non-compact, semisimple Lie groups (linear), using the existence

of a Chevalley-basis for the Lie algebra. For a simple example in our case, let  $F = \mathbb{Q}[\sqrt{3}]$ ,  $\sigma$  be the non-trivial Galois-automorphism; and  $x_1^2 + \dots + x_n^2 - \sqrt{3} x_{n+1}^2$  the quadratic form. Then  $S_1^\sigma$  is the spin group corresponding to the form  $x_1^2 + \dots + x_n^2 + \sqrt{3} x_{n+1}^2$ , and  $S = S_1 \times S_1^\sigma = \text{Spin}(n, 1) \times \text{Spin}(n+1)$  over the reals. Before showing that these data characterize the Anosov flow completely (determine  $\Gamma$ ), we determine these rational representations.

$$S = S_1 \times K_1 \times \dots \times K_p .$$

We only have to consider simple  $\mathbb{Q}$ -modules, the relevant invariants to describe these are motivated by the following (see Borel-Tits [12] and Satake [21]): Let  $V$  be a  $\mathbb{Q}$ -simple  $S$ -module, let  $U$  be a  $\mathbb{C}$ -simple factor.  $\Delta = \text{Gal}(\mathbb{C}/\mathbb{Q})$  acts on  $V$ , let  $\sigma \in \Delta$ , then  $V = \sum_{\sigma \in \Delta} U^\sigma$ . Let  $\Delta_1 = \{\sigma \in \Delta, U^\sigma = U\}$ , let  $F_1$  be the corresponding fixed field; then  $U$  is defined over  $F_1$ . Let  $\Delta_2 = \{\sigma \in \Delta, U^\sigma \cong U\}$ , let  $F_2$  be the fixed field,  $F_2 \subseteq F_1$ ; then  $\hat{U} = \sum_{\sigma \in \Delta_2} U^\sigma = R_{F_1/F_2}(U)$ , (restriction of scalars) is defined over  $F_2$ , and  $V = R_{\mathbb{Q}/F_2}(\hat{U})$ . Now, if  $\lambda$  is the highest weight of the module  $U$ , then  $\lambda^{[\sigma]}$  is the highest weight of  $U^\sigma$ , here  $\lambda^{[\sigma]}$  is the image of  $\lambda$  under a well-known action of  $\Delta$  on the weight space. (This is given as follows: If  $T$  is a maximal torus def. over  $\mathbb{Q}$  and  $X$  is the character module of  $T$ ; let  $\pi$  be a fundamental root system, then for  $\sigma \in \Delta$ , there is a unique  $w_\sigma$  in the Weyl group s.t.  $\Delta^\sigma = w_\sigma \Delta$ . For  $\chi \in X$ , define  $\chi^{[\sigma]} = w_\sigma^{-1} \chi$ .) So  $\Delta_2$  is the isotropy group of  $\lambda$ , and the last transition from  $\hat{U}$  to  $V$  is easily determined by the explicit knowledge of the action of the Galois-group. The first transition from  $U$  to  $\hat{U}$  is not detected by the Galois-invariance of the highest weight, and a more delicate invariant  $\kappa$  is needed:

For  $\sigma \in \Delta_2$ , let  $\Phi_\sigma: U \rightarrow U^\sigma$  be an  $S$ -isomorphism. Taking the corresponding projective maps we get a 1-cocycle of  $\Delta_2$  in the sense of Galois cohomology; this defines an  $F_2$ -form  $\mathcal{O}$  of  $\mathcal{J}^1(U)$ , which makes the representation corresponding to  $U$  an  $F_2$ -morphism. Since  $\mathcal{O}$  is a central, simple algebra, it is of the form  $\mathcal{J}^1(W)$ , where  $W$  is a right vector space over a central division algebra  $\kappa$  over  $F_2$ . By Schur's lemma the  $\Phi_\sigma$ 's are determined up to a scalar, i.e.  $\Phi_\sigma^\top \Phi_\tau = \lambda_{\sigma, \tau} \Phi_{\sigma\tau}$  with  $\lambda_{\sigma\tau} \in \mathbb{C}^*$ .  $\lambda_{\sigma, \tau}$  is a 2 cocycle of  $\Delta_2$ , its cohomology class in  $H^2(\Delta_2, \mathbb{C}^*)$  is the Brauer-Hasse invariant of  $\kappa$ ,  $c(\kappa)$ . Since  $H^1(\Delta_2, PL(n)) \rightarrow H^2(\Delta_2, \mathbb{C}^*)$  is injective (generalized Hilbert lemma); we see that  $\hat{U}$  is now completely determined by  $\kappa$ , the number of  $U$ -factors in  $\hat{U}$  equals the degree of  $\kappa$  over  $F_2$ .

In practice, observe that  $S$  splits over a finite extension  $E$  of  $\mathbb{Q}$  (since the character module is finitely generated). So we only have the action of  $\Delta = \text{Gal}(E/\mathbb{Q})$  to consider. Let  $\lambda$  be the highest weight for an irreducible representation  $\rho$  of  $S$  s.t.  $\rho$  and all its Galois-conjugates have no zero restricted weight on  $\mathbb{R}\alpha \subseteq \mathcal{S}$ . If  $\rho = \rho_0 \otimes \rho_1 \otimes \dots \otimes \rho_p$ , with  $\rho_0$  and  $\rho_i$  irreducible representations of  $S_1$  and  $K_i$  respectively; then this means that  $\rho_0$  and  $\rho_i$  must all be of "spin" type.  $\Delta_2$  and  $K_2$  are computed as above. To compute  $\kappa$ , Satake has defined an invariant  $\gamma_{F_2}(S) \in H^2(F_2, Z(S))$ .  $\rho$  induces a natural homomorphism  $\rho^*$  from  $H^2(F_2, Z(S))$  to  $H^2(F_2, \mathbb{C}^*)$ ; then  $c(\kappa) = \rho^*(\gamma_{F_2}(S))$ . This Satake-invariant (Satake [21]) satisfies natural properties; i.e.  $\gamma_{F_2}(S)$  is obtained from  $\gamma_{\mathbb{Q}}(S)$  by restriction of the Galois-group. Also,  $S = R_{\mathbb{Q}/F}(S_1)$ ,  $Z(S) = R_{\mathbb{Q}/F}(Z(S_1))$ , and  $H^2(\mathbb{Q}, Z(S)) \cong H^2(F, Z(S_1))$  canonically. Under this isomorphism  $\gamma_{\mathbb{Q}}(S)$  corresponds to  $\gamma_F(S_1)$ . Hence it suffices to consider

$\gamma_F(S_1)$ ; and in our case this is obtained from the Minkowski-Hasse invariant of the quadratic form.

This completes our procedure for constructing symmetric Anosov flows. It remains to show that the construction is general; i.e. that  $\Gamma$  must always (up to commensurability) split as the semidirect product above (i.e.  $\Gamma$  is "arithmetic" in  $G$  w.r.t. the  $\mathbb{Q}$  structure defined by the choice of a Levi complement).

We have the commutative diagram:  $\{1\} \rightarrow V \rightarrow G \rightarrow S \rightarrow \{1\}$

$$\begin{array}{ccccccc} & & & & \uparrow & \uparrow & \uparrow \\ & & & & \{1\} & \rightarrow & \Gamma_V \rightarrow \Gamma \rightarrow \Gamma_S \rightarrow \{1\} \end{array}$$

We want to prove that the lower exact sequence splits; and that the splitting homomorphisms  $\varphi_1: S \rightarrow G$  and  $\varphi_2: \Gamma_S \rightarrow \Gamma$  may be chosen such that the backwards diagram commutes.

Theorem 6.  $\Gamma$  is a split extension of  $\Gamma_V$  by  $\Gamma_S$  (up to commensurability).

$\Gamma$  preserves  $V_{\mathbb{Q}}$  under conjugation, hence we have the following commutative diagram:

$$\begin{array}{ccccccc} (a) & \{1\} & \rightarrow & V & \rightarrow & \Gamma \cdot V & \rightarrow & \Gamma_S & \rightarrow & \{1\} \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ (b) & \{1\} & \rightarrow & V_{\mathbb{Q}} & \rightarrow & \Gamma \cdot V_{\mathbb{Q}} & \rightarrow & \Gamma_S & \rightarrow & \{1\} \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ (c) & \{1\} & \rightarrow & \Gamma_V & \rightarrow & \Gamma & \rightarrow & \Gamma_S & \rightarrow & \{1\} \end{array}$$

Proof: It is sufficient to prove that the middle sequence (b) splits.

For  $\Gamma_S$  is finitely generated, let  $\gamma_1, \dots, \gamma_r$  be the generators. Let  $\varphi: \Gamma_S \rightarrow \Gamma \cdot V_{\mathbb{Q}}$  be the splitting homomorphism,  $\varphi(\gamma_i) = (v_i, \gamma_i)$  with  $v_i$  rational points in  $V$  (coordinates relative to  $\Gamma_V$ ). Let  $w$  be the least common multiple of all denominators,

and let  $\Gamma'_V$  be the lattice obtained from  $\Gamma_V$  by multiplying all base vectors by  $\frac{1}{w}$ . Then  $\Gamma'_V$  is also  $\Gamma$ -variant,  $\Gamma$  is of finite index in  $\Gamma' = \Gamma \cdot \Gamma'_V$ , and  $\varphi$  takes values in  $\Gamma'$ . Now, to prove that (b) splits:

Let  $u: \Gamma_S \rightarrow \Gamma \cdot V_{\mathbb{Q}}$  be a section (not necessarily a homomorphism). Then  $u(x)u(y) = f(x,y)u(xy)$ ; with  $f(x,y) \in V_{\mathbb{Q}}$ , so  $f$  is a 2-cocycle corresponding to the given extension. This extension splits over the reals (sequence (a)), so  $f$  is cohomologous to zero over  $\mathbb{R}$ ; i.e. there is a function  $g: \Gamma_S \rightarrow V$  s.t.  $f(x,y) = \delta g(x,y) = x \cdot g(y) - g(xy) + g(x)$ . By taking a Hamel basis for  $\mathbb{R}$  over  $\mathbb{Q}$  we consider  $V_{\mathbb{Q}}$  as a subspace of the  $\mathbb{Q}$ -vector-space  $V$ . By taking components in  $V_{\mathbb{Q}}$  on both sides, using that  $\Gamma_S$  preserves  $V_{\mathbb{Q}}$ , we see that  $f$  is also cohomologous to zero over  $\mathbb{Q}$ .

Theorem 7. Let  $\varphi_1$  be a splitting homomorphism for the exact sequence (c). Then the Levi complement of  $V$  in  $G$  may be chosen to include  $\varphi_1(\Gamma_S)$ .

Proof: Consider the commutative diagram

$$\begin{array}{ccccccc} \{1\} & \rightarrow & V & \rightarrow & V \times S & \rightarrow & S \rightarrow \{1\} \\ & & \uparrow & & \uparrow & & \uparrow \\ \{1\} & \rightarrow & V & \rightarrow & \Gamma \cdot V & \xrightarrow{\varphi_1} & \Gamma_S \rightarrow \{1\} \end{array}$$

Any section of the upper sequence (homomorphism) will induce another splitting homomorphism  $\Gamma_S \rightarrow \Gamma \cdot V$ . Now define  $\varphi: \Gamma_S \rightarrow V$  by  $\varphi_1(\gamma_s) = (\varphi(\gamma_s), \gamma_s)$ .  $\varphi_1(\gamma_s \gamma_t) = (\varphi(\gamma_s \gamma_t), \gamma_s \gamma_t) = \varphi_1(\gamma_s) \varphi_1(\gamma_t) = (\varphi(\gamma_s) + \gamma_s \cdot (\varphi(\gamma_t)), \gamma_s \gamma_t)$ . Hence  $\varphi(\gamma_s \gamma_t) = \varphi(\gamma_s) + \gamma_s \cdot (\varphi(\gamma_t))$ , which is the cocycle identity. So  $\varphi$  is a 1-cocycle of  $\Gamma_S$  with coefficients in the module  $V$ . Now, for such representations  $\rho$  Matsushima and Murakami have defined associated cohomology groups  $H^p(\Gamma_S, S/C, \rho)$  ( $C = \max.$  com-

pact subgroup of  $S$ ) which may be interpreted as the cohomology groups of the locally symmetric space  $\Gamma_s \backslash S/C$  with values in the sheaf of germs of locally constant sections of the vector bundle attached to the universal covering  $S/C \rightarrow \Gamma_s \backslash S/C$  by this representation  $\rho$ . ( $\Gamma_s$  chosen without torsion.) They coincide with  $H^p(\Gamma_s, V)$ . The vector fields in  $S$  project to vector-fields on  $\Gamma_s \backslash S$ ; let  $A^p(\Gamma_s \backslash S, C, \rho)$  be the  $V$ -valued  $p$ -forms  $\eta$  on  $\Gamma_s \backslash S$  satisfying: (i)  $i_X \eta = 0$ , (ii)  $\theta_X \eta = -\rho(X)$  for  $X$  in the Lie algebra of  $C$ . Here  $i_X$  is interior product,  $\theta_X$  is Lie derivation along the vector field  $X$ . They define a Laplacian  $\Delta_\rho$  s.t.  $H^p(\Gamma_s, S/C, \rho)$  is isomorphic to the corresponding harmonic  $p$ -forms. A vanishing theorem for cohomology is obtained if a certain quadratic form ([16]) on the space of exterior  $p$ -forms on the  $\mathcal{F}$ -part of the Cartan decomposition of  $G$  with values in  $V$  is positive definite. In [19] and [20] Raghunathan has computed many examples explicitly. In particular,  $H^1(\Gamma_s, V) = 0$  here, since  $\rho$  is of spin type. Hence the cocycle  $\varphi$  is cohomologous to zero; i.e. there is an element  $v \in V$  s.t.  $\varphi(\gamma_s) = \gamma_s(v) - v$ . Conjugation by  $(v, e)$  defines a "special automorphism" of  $G$  which sends the Levi complements  $S$  into another Levi complement  $S' = \{(-v, e)(0, s)(v, e) | s \in S\} = \{(s(v) - v, s) | s \in S\}$ . Hence  $\varphi_1(\Gamma_s) \subseteq S'$ .

q.e.d.

References

1. D.V. Anosov, "Roughness of geodesic flows on compact Riemannian manifolds of negative curvature". Soviet Math. Dokl. 3 (1962), 1068-1070.
2. D.V. Anosov, "Geodesic flows on compact Riemannian manifolds of negative curvature", Trudy Mat. Inst. Steklov. 90 (1967).
3. L. Ausländer, L. Green and F. Hahn. "Flows on homogeneous spaces", Ann. of Math. Studies No.53, Princeton University Press, Princeton, N.J., 1963.
4. L. Ausländer and L. Green. "G-induced flows". Amer.J.Math, 88 (1966), 43-60.
5. L. Ausländer, "Bieberbach's theorem on space groups and discrete uniform subgroups of Lie groups". Ann. of Math. 71, (1960), 579-590.
6. L. Ausländer, "On radicals of discrete subgroups of Lie groups". Amer. J. Math. 85 (1963), 145-150.
7. A. Borel, "Compact Clifford-Klein forms of symmetric spaces". Topology 2 (1963), 111-122.
8. A. Borel, "Ensembles fondamentaux pour les groupes arithmetiques et formes automorphes". Secr. Math. Ecole Norm, Sup., Paris, 1967.
9. A. Borel, "Density properties for certain subgroups of semi-simple Lie groups without compact factors". Ann. of Math. 72 (1960), 179-188.
10. A. Borel, "On the automorphisms of certain subgroups of semi-simple Lie groups". Proc. of the Bombay Colloquim on Algebraic Geometry, 1968, p. 43-73.
11. A. Borel and Harish-Chandra, "Arithmetic subgroups of algebraic groups". Ann. of Math. 75 (1962), 485-535.

12. A. Borel and J. Tits, "Groupes réductifs". Publ. Math. I.H.E.S. 27 (1965), 55-150.
13. N. Bourbaki, "Groupes et algèbres de Lie". Paris, 1966.
14. G. Hochschild and G.D. Mostow. "Representations and representative functions on Lie groups". Annals of Math. Vol.66 (1957), 495-542.
15. B. Kostant and S. Rallis, "On representations associated with symmetric spaces". Bull. Amer. Math. Soc. 75 (1969), 884-888.
16. Y. Matsushima and S. Murakami, "On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds". Ann. of Math. 78 (1963), 365-416.
17. G. D. Mostow, "Factor spaces of solvable groups". Ann. of Math. 60 (1954), 1-27.
18. S.E. Newhouse, "On codimension one Anosov diffeomorphisms". Am. Journal of Math. 92 (1970), 761-769.
19. M.S. Raghunathan, "On the first cohomology of discrete subgroups of semisimple Lie groups". Am. Journal of Math. 87 (1965), 103-139.
20. M.S. Raghunathan, "Vanishing theorems for cohomology groups associated to discrete subgroups of semisimple Lie groups". Osaka J. Math. Vol.3 (1966), 243-256.
21. I. Satake, "Representations of Algebraic Groups". Adv. Sci. Sem. in Alg. Groups, Bowdoin College, 1968. (Notes by F. Grosshans).
22. A. Selberg, "On discontinuous groups in higher dimensional symmetric spaces". Contributions to function theory, Tata Inst. of fundamental research. Bonbay, 1960.
23. S. Smale, "Differentiable dynamical systems". Bull. Amer. Math. Soc. 73 (1967), 747-817.
24. P. Tomter, "Anosov flows on infra-homogeneous spaces". A.M.S.Proc. Sump. Pure Math. Vol.XIV. Global Analysis (1970).