CRAIG'S INTERPOLATION THEOREM
FOR THE INTUITIONISTIC LOGIC
OF CONSTANT DOMAINS

by

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In this paper I will prove the Craig interpolation theorem for the intuitionistic logic of constant domains and hence give a positive solution to a problem posed by D.M. Gabbay [1].

The Craig interpolation theorem for a logic \( \mathcal{L} \) says that:

If \( \vdash_{\mathcal{L}} A \supset B \), then there exists formula \( C \) in the language common to \( A \) and \( B \) with \( \vdash_{\mathcal{L}} A \supset C \) and \( \vdash_{\mathcal{L}} C \supset B \).

It has been proved for classical logic, intuitionistic logic and in his papers D.M. Gabbay [1] proved it for some extensions of intuitionistic logic.

The intuitionistic logic of constant domains is the logic which is complete for Kripke models with constant domain function. S. Gürnemann proved in her thesis [2] that this logic can be characterized syntactically by intuitionistic predicate logic + the schema \( \forall x(A \vee Bx) \supset A \vee \forall x Bx \) where \( x \) is not free in \( A \).

The method used in this paper is the following:
Below I will describe this logic by a sequential calculus \( \mathcal{LG} \), prove cut-elimination for \( \mathcal{LG} \), and then prove a sequential formulation of the interpolation theorem by induction over the cut-free derivations in \( \mathcal{LG} \).
II

We fix our language:

Logical symbols $\land, \land, \lor, \Rightarrow, \lor, \exists$

We have relations but no functions symbols.

Build up formulae as usual.

A sequent is a pair, $\Gamma \rightarrow \Delta$, of two finite sets of formulae $\Gamma, \Delta$. As usual we use abbreviations $\Gamma \rightarrow, \rightarrow, \Delta$, $\Gamma, \Delta \rightarrow \Delta, \Delta$, etc.

AXIOMS: are sequents of the forms

$\Gamma, P \rightarrow P, \Delta$ or $\Gamma, \neg \rightarrow \Delta$

with $P$ an atomic formula.

RULES:

Weakening

\[ \frac{\Gamma \rightarrow \Delta}{\Gamma, \Gamma' \rightarrow \Delta, \Delta'} \]

$\land \rightarrow$

\[ \frac{\Gamma, A \rightarrow \Delta}{\Gamma, A \land B \rightarrow \Delta} \quad \text{or} \quad \frac{\Gamma, B \rightarrow \Delta}{\Gamma, A \land B \rightarrow \Delta} \]

$\rightarrow \land$

\[ \frac{\Gamma \rightarrow \Delta}{\Gamma, \Gamma, A \land B \rightarrow \Delta} \]

$\lor \rightarrow$

\[ \frac{\Gamma, A \rightarrow \Delta}{\Gamma, A \lor B \rightarrow \Delta} \quad \text{or} \quad \frac{\Gamma, B \rightarrow \Delta}{\Gamma, A \lor B \rightarrow \Delta} \]

$\Rightarrow \lor$

\[ \frac{\Gamma \rightarrow \Delta}{\Gamma, A \Rightarrow \rightarrow \Delta} \quad \text{or} \quad \frac{\Gamma \rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \]

$\Rightarrow \Rightarrow$

\[ \frac{\Gamma, A \rightarrow B}{\Gamma, A \Rightarrow \rightarrow B} \]

$\neg \rightarrow$

\[ \frac{\Gamma, A \rightarrow B}{\Gamma \rightarrow A \Rightarrow B} \]
This concludes the description of our formal system LG. There are a few things to notice:

i) We do not need other structural rules than weakening since we assume the sequents to consist of sets of formulae.

ii) We get the intuitionistic sequential calculus if we add the restriction that all sequents shall have at most one formula in the succedent (i.e. after $\rightarrow$). It is straightforward that we only need to assume $\Delta$ empty in applications of $\rightarrow V$ to get intuitionistic logic. Not assuming $\Delta$ to be empty in applications of $\rightarrow V$ we get a logic which is stronger than intuitionistic logic.

iii) On the other hand we do not get classical logic. The essential restriction is that succedents in $\rightarrow \exists$ are assumed to have only one formula.

iv) Our formulation of $\rightarrow \rightarrow$ is special. We could have changed it to:

$$\Gamma \rightarrow \exists \Delta, \exists, \Gamma, \exists, \exists \rightarrow \Delta$$

without changing the logic; but we would then get into troubles in the induction proof of the interpolation theorem.
III

THEOREM A. LG is equivalent to intuitionistic predicate logic + the schema \( \forall x (A \lor Bx) \rightarrow A \lor \forall x Bx \).

Proof:
In LG we derive the schema

\[
\begin{align*}
A & \rightarrow A, Ba \\
B & \rightarrow A, Ba \\
A \lor B & \rightarrow A, Ba \\
\forall x(A \lor Bx) & \rightarrow A, Ba \\
\forall x(A \lor Bx) & \rightarrow A, \forall x Bx \\
\forall x(A \lor Bx) & \rightarrow A \lor \forall x Bx, \forall x Bx \\
\forall x(A \lor Bx) & \rightarrow A \lor \forall x Bx
\end{align*}
\]

As intuitionistic predicate logic we use LG with the restriction that \( \Delta \) is empty in \( - \lor - \). We show how to get the rule

\( \Gamma \rightarrow A, Ba \rightarrow \Gamma \rightarrow A, \forall x Bx \), a not in \( \Gamma \rightarrow A, \forall x Bx \), from the schema.

\[
\begin{align*}
\Gamma & \rightarrow A, Ba \\
\Gamma & \rightarrow A \lor B, Ba \\
\Gamma & \rightarrow A \lor B \\
\Gamma & \rightarrow \forall x(A \lor Bx) \\
\forall x(A \lor Bx) & \rightarrow A \lor \forall x Bx \\
\forall x Bx & \rightarrow A, \forall x Bx
\end{align*}
\]

In the same way we get the rule \( \Gamma \rightarrow \Delta, Ba \rightarrow \Gamma \rightarrow \Delta, \forall x Bx \), a not in \( \Gamma \rightarrow \Delta, \forall x Bx \). QED

THEOREM B. In LG we can eliminate CUT.

Proof:
We only need to check that in the usual proof for LK, the classical sequential calculus, we transform derivations in LG into derivations in LG. We take the cases involving \( \lor \).
i) From:

\[\Gamma_1 \rightarrow A \quad \Gamma_1, B \rightarrow D\]

we get:

\[\Gamma_1, A \supset B \rightarrow D \quad \Gamma_2, D \rightarrow \Delta\]

\[\Gamma_1, A \supset B, \Gamma_2 \rightarrow \Delta\]

ii) From:

\[\Gamma_2, B, D \rightarrow \Delta \quad \Gamma_2, D \rightarrow A\]

we get:

\[\Gamma_1 \rightarrow D \quad \Gamma_2, B, D \rightarrow \Delta \]

\[\Gamma_1, \Gamma_2, B \rightarrow \Delta \]

\[\Gamma_1, \Gamma_2, A \supset B \rightarrow \Delta\]

iii) From:

\[\Gamma_2, D, A \rightarrow B\]

we get:

\[\Gamma_1 \rightarrow D \quad \Gamma_2, D, A \rightarrow B\]

\[\Gamma_1, \Gamma_2, A \supset B\]
iv) From:

\[
\frac{
\Gamma_1, A \rightarrow B \\
\Gamma_1 \vdash A \supset B
}
{\Gamma_1 \vdash A \supset B, A \rightarrow \Delta}
\]

\[
\frac{
\Gamma_2, A \supset B \rightarrow A \\
\Gamma_2, A \supset B
}
{\Gamma_2, A \supset B, B \rightarrow \Delta}
\]

we get:

\[
\frac{
\Gamma_1, A \rightarrow B \\
\Gamma_1, \Gamma_2 \vdash A
}
{\Gamma_1, \Gamma_2 \vdash B}
\]

\[
\frac{
\Gamma_1, \Gamma_2 \vdash A \\
\Gamma_1 \vdash A \supset B, \Gamma_2 \vdash B \rightarrow \Delta
}
{\Gamma_1 \vdash A \supset B, \Gamma_2, A \supset B, B \rightarrow \Delta}
\]

\[
\frac{
\Gamma_1 \rightarrow B \\
\Gamma_1, \Gamma_2 \rightarrow A
}
{\Gamma_1, \Gamma_2 \rightarrow \Delta}
\]

\[
\frac{
\Gamma_1, \Gamma_2 \rightarrow B \\
\Gamma_1 \rightarrow A \supset B, \Gamma_2 \rightarrow A, \Gamma_2 \rightarrow A
}
{\Gamma_1, \Gamma_2 \rightarrow \Delta}
\]

i - iii are the cases where immediately above a CUT we have application of \( \rightarrow \supset \) or \( \supset \rightarrow \) not involving the cut formula.

iv is the case where immediately above a cut we have applications of \( \rightarrow \supset \) and \( \supset \rightarrow \) involving the cut formula.

The remaining cases involve neither \( \rightarrow \supset \) nor \( \supset \rightarrow \) and can be carried over from LK without change. The ordinal calculations are as in the proof of LK. QED.
Now we come to the interpolation theorem. There is a slight trick in the formulation of it - in that we do not split up the succedent. Note also how our formulation of \( \rightarrow \Rightarrow \) comes in case ix b.

THEOREM C. If we can derive \( \Gamma_1, \Gamma_2 \rightarrow \Delta \) in \( LG \), we can find a formula \( C \) in the common language of \( \Gamma_1 \rightarrow \Delta \) and \( \Gamma_2 \rightarrow \) with both \( \Gamma_1, C \rightarrow \Delta \) and \( \Gamma_2 \rightarrow C \) derivable in \( LG \).

Proof:
The proof is by induction over a cut-free derivation of \( \Gamma_1, \Gamma_2 \rightarrow \Delta \) in \( LG \). We must consider the following cases:

i) We have an axiom \( \Gamma, P \rightarrow P, \Delta \)
   a) We split up \( (\Gamma_1, P \rightarrow P, \Delta) \) \( (\Gamma_2 \rightarrow ) \)
      Take \( C \) to be \( \wedge \Rightarrow \wedge \)
   b) We split up \( (\Gamma_1 \rightarrow P, \Delta) \) \( (\Gamma_2, P \rightarrow ) \)
      Take \( C \) to be \( P \)

ii) We have an axiom \( \Gamma, \ll \rightarrow \Delta \)
    a) We split up \( (\Gamma_1, \ll \rightarrow \Delta) \) \( (\Gamma_2 \rightarrow ) \)
       Take \( C \) to be \( \ll \Rightarrow \ll \)
    b) We split up \( (\Gamma_1 \rightarrow \Delta) \) \( (\Gamma_2, \ll \rightarrow ) \)
       Take \( C \) to be \( \ll \)

iii) We have a not quantifier one premiss rule.
    All cases are equal. We treat \( \rightarrow \Rightarrow \) where we split up \( \Gamma \rightarrow A \Rightarrow B \) as \( (\Gamma_1 \rightarrow A \Rightarrow B) \) \( (\Gamma_2 \rightarrow ) \).
    Since the theorem is true for the premiss \( \Gamma, A \rightarrow B \), we can find \( C \) in the common language of \( \Gamma_1, A \rightarrow B \) and \( \Gamma_2 \rightarrow \) with \( \Gamma_1, C, A \rightarrow B \) and \( \Gamma_2 \rightarrow C \) derivable in \( LG \). By appli-
cation of \( \rightarrow \) we have also \( \Gamma_1, C \rightarrow \Lambda \rightarrow B \) derivable in LG. C is also an interpolation formula for \( \Gamma_1 \rightarrow A \rightarrow B \) and \( (\Gamma_2 \rightarrow \cdot) \).

iv) \( \forall \rightarrow \cdot \\

a) Split up \( (\Gamma_1, \forall xA \rightarrow \Lambda) \) and \( (\Gamma_2 \rightarrow \cdot) \)

By hypothesis we have an interpolation formula C of \( (\Gamma_1, Aa \rightarrow \Lambda) \) and \( (\Gamma_2 \rightarrow \cdot) \) which may or may not contain a. If it does not or if a occurs in \( \Gamma_1, \forall xA \rightarrow \Delta \), then C is also an interpolation formula of \( (\Gamma, \forall xA \rightarrow \Lambda) \) and \( (\Gamma_2 \rightarrow \cdot) \). If we have \( C = C_a \) and a does not occur in \( \Gamma_1, \forall xA \rightarrow \Delta \), then \( \exists xC_{x} \) is an interpolation formula of \( (\Gamma_1, \forall xA \rightarrow \Delta) \) and \( (\Gamma_2 \rightarrow \cdot) \).

b) Split up \( (\Gamma_1 \rightarrow \Delta) \) and \( (\Gamma_2, \forall xA \rightarrow \cdot) \).

We have interpolation formula C of \( (\Gamma_1 \rightarrow \Delta) \) and \( (\Gamma_2, Aa \rightarrow \cdot) \). Then as above either C or \( \exists xC_{x} \) will be an interpolation formula of \( (\Gamma_1 \rightarrow \Delta) \) and \( (\Gamma_2, \forall xA \rightarrow \cdot) \).

v) \( \rightarrow \forall \) or \( \exists \rightarrow \cdot \). Say \( \rightarrow \forall \)

We have split up \( (\Gamma_1 \rightarrow \forall xA, \Delta) \) and \( (\Gamma_2 \rightarrow \cdot) \). The interpolation formula C of \( (\Gamma_1 \rightarrow Aa, \Delta) \) and \( (\Gamma_2 \rightarrow \cdot) \) cannot contain a since a does not occur in \( \Gamma_2 \rightarrow \cdot \). Hence C is an interpolation formula of \( (\Gamma_1 \rightarrow \forall xA, \Delta) \) and \( (\Gamma_2 \rightarrow \cdot) \). \( \forall \rightarrow \cdot \) is similar.

vi) \( \rightarrow \exists \)

Split up \( (\Gamma_1 \rightarrow \exists xA, \Delta) \) and \( (\Gamma_2 \rightarrow \cdot) \).

We have interpolation formula C of \( (\Gamma_1 \rightarrow Aa, \Delta) \) and \( (\Gamma_2 \rightarrow \cdot) \). Then as in case iv a, either C or \( \exists xC_{x} \) is an interpolation formula of \( (\Gamma_1 \rightarrow \exists xA, \Delta) \) and \( (\Gamma_2 \rightarrow \cdot) \).

vii) \( \rightarrow \land \)

Split up \( (\Gamma_1 \rightarrow A \land B, \Delta) \) and \( (\Gamma_2 \rightarrow \cdot) \).
We have interpolationformula $C_1$ of $(\Gamma_1 \rightarrow A, \Delta)$ and $(\Gamma_2 \rightarrow )$ and $C_2$ of $(\Gamma_1 \rightarrow B, \Delta)$ and $(\Gamma_2 \rightarrow )$. Then $C_1 \land C_2$ is an interpolationformula of $(\Gamma_1 \rightarrow A \land B, \Delta)$ and $(\Gamma_2 \rightarrow )$.

viii) $\lor \rightarrow$

a) Split up $(\Gamma_1, A \lor B \rightarrow \Delta)$ $(\Gamma_2 \rightarrow )$.

We have interpolationformula $C_1$ of $(\Gamma_1, A \rightarrow \Delta)$ $(\Gamma_2 \rightarrow )$ and $C_2$ of $(\Gamma_1, B \rightarrow \Delta)$ $(\Gamma_2 \rightarrow )$. Then $C_1 \land C_2$ is an interpolationformula of $(\Gamma_1, A \lor B \rightarrow \Delta)$ $(\Gamma_2 \rightarrow )$.

b) Split up $(\Gamma_1 \rightarrow \Delta)$ $(\Gamma_2, A \lor B \rightarrow )$.

We have interpolationformula $C_1$ of $(\Gamma_1 \rightarrow \Delta)$ $(\Gamma_2, A \rightarrow )$ and $C_2$ of $(\Gamma_1 \rightarrow \Delta)$ $(\Gamma_2, B \rightarrow )$. Then $C_1 \lor C_2$ is an interpolationformula of $(\Gamma_1 \rightarrow \Delta)$ $(\Gamma_2, A \lor B \rightarrow )$.

ix) $\Rightarrow \rightarrow$

a) Split up $(\Gamma_1, A \Rightarrow B \rightarrow \Delta)$ $(\Gamma_2 \rightarrow )$.

We have interpolationformula $C_1$ of $(\Gamma_1 \rightarrow A)$ $(\Gamma_2 \rightarrow )$ and $C_2$ of $(\Gamma_1, B \rightarrow \Delta)$ $(\Gamma_2 \rightarrow )$. Then $C_1 \land C_2$ is an interpolationformula of $(\Gamma_1, A \Rightarrow B \rightarrow \Delta)$ $(\Gamma_2 \rightarrow )$.

b) Split up $(\Gamma_1 \rightarrow \Delta)$ $(\Gamma_2, A \Rightarrow B \rightarrow )$.

We have interpolationformula $C_1$ of $(\Gamma_2 \rightarrow A)$ $(\Gamma_1 \rightarrow )$ and $C_2$ of $(\Gamma_1 \rightarrow \Delta)$ $(\Gamma_2, B \rightarrow )$. Then $C_1 \Rightarrow C_2$ is an interpolationformula of $(\Gamma_1 \Rightarrow \Delta)$ $(\Gamma_2, A \Rightarrow B \rightarrow )$.

QED.

As an immediate corollary we have the interpolation theorem for $\text{LG}$.

THEOREM D. If $A \rightarrow B$ is derivable in $\text{LG}$, then we can find $C$ in the common language of $A$ and $B$ with $A \rightarrow C$ and $C \rightarrow B$ derivable in $\text{LG}$.
REFERENCES
