1. Introduction. In his studies in ergodic theory Hopf [3] introduced an equivalence relation, which in the language of von Neumann algebras, is an equivalence relation on the projections in an abelian von Neumann algebra acted upon by a group of *-automorphisms. He then showed that, with some extra assumptions, "finiteness" of the partial ordering defined by the equivalence was equivalent to the existence of an invariant normal state. Later on the "semi-finite" case was taken care of by Kawada [6] in a well ignored paper, and then independently by Halmos [2]. In the theory of von Neumann algebras, Murray and von Neumann introduced their celebrated equivalence relation on the projections in [8] and again showed (at least for factors) the equivalence of finiteness (resp. semi-finiteness) and the existence of normal finite (resp. semi-finite) traces. It is the purpose of the present paper to introduce and study an equivalence relation which includes in the countably decomposable case the one by Hopf and in the general case the one by Murray and von Neumann. It is defined as follows. Let \( \mathcal{A} \) be a von Neumann algebra acting on a
Hilbert space $\mathcal{H}$. Let $G$ be a group and let $t \to U_t$ be a unitary representation of $G$ on $\mathcal{H}$ such that $U_t^*RU_t = R$ for all $t \in G$. Then we say two projections $E$ and $F$ in $\mathcal{R}$ are $G$-equivalent if there is for each $t \in G$ an operator $T_t \in \mathcal{R}$ such that $E = \sum_{t \in G} T_t T_t^*$, $F = \sum_{t \in G} U_t^* T_t^* T_t U_t$.

Our main results now state that this relation is indeed an equivalence relation (Thm. 1), that "semi-finiteness" is equivalent to the existence of a faithful normal semi-finite $G$-invariant trace on $\mathcal{R}^+$ (Thm. 2), and that "finiteness" together with countable decomposability of $\mathcal{R}$ is equivalent to the existence of a faithful normal finite $G$-invariant trace on $\mathcal{R}$ (Thm. 3). In the proofs we shall not follow the apparently natural approach of developing a comparison theory for the projections in $\mathcal{R}$ and then to construct the traces. We shall instead consider the cross product $\mathcal{R} \times G$, and then show that the canonical imbedding of $\mathcal{R}$ into the von Neumann algebra $\mathcal{R} \times G$ is close to being an isomorphism of $\mathcal{R}$ with the structure of $G$-equivalence into $\mathcal{R} \times G$ with the equivalence relation of Murray and von Neumann. In the last two sections of the paper we shall study the relation of $G$-equivalence to $G$-finite von Neumann algebras, and to the equivalence relation of Hopf.

We refer the reader to the book of Dixmier [1] for the theory of von Neumann algebras.

2. Statements of results. In the present section we state the main results and definitions. The proofs will be given in section 3.

**Theorem 1.** Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert
space $\mathcal{H}$. Let $G$ be a group and $t \to U_t$ a unitary representation of $G$ on $\mathcal{H}$ such that $U_t^* R U_t = R$ for all $t \in G$. If $E$ and $F$ are projections in $\mathcal{R}$ we write $E \sim G F$ if for each $t \in G$ there is an operator $T_t \in \mathcal{R}$ such that

$$E = \sum_{t \in G} T_t^T T_t^* , \quad F = \sum_{t \in G} U_t^* T_t^T T_t U_t .$$

Then $\sim_G$ is an equivalence relation on the projections in $\mathcal{R}$.

**Remark 1.** If $G$ is the one element group then the equivalence relation $\sim_G$ is the same as the usual equivalence relation $\sim$ for projections in a von Neumann algebra.

**Remark 2.** If $G$ is the additive group of $\mathcal{R}$ and the representation $t \to U_t$ is the trivial representation, so $U_t = I$ for $t \in G$, then the equivalence relation $\sim_G$ is the one defined by Kadison and Pedersen [4, Def.A].

**Remark 3.** If $\mathcal{R}$ is abelian and countably decomposable the equivalence relation $\sim_G$ coincides with the one defined by Hopf [3] in ergodic theory. For this see Theorem 5.

**Remark 4.** If $E$ and $F$ are equivalent projections in $\mathcal{R}$, i.e. there is a partial isometry $V \in \mathcal{R}$ such that $E = V V^*$, $F = V^* V$, then $E \sim_G F$. This is clear from the definition of $\sim_G$, putting $T_e = V$, $T_t = 0$ for $t \neq e$.

**Definition 1.** With notation as in Theorem 1 we say two projections $E$ and $F$ in $\mathcal{R}$ are $G$-equivalent if $E \sim_G F$. We write $E \simeq_G F$ if $E \sim_G F$ and $E \preceq F$. A projection $F$ is said to be $G$-finite if $E \preceq F$ and $E \simeq_G F$ implies $E = F$. $\mathcal{R}$ is said to be $G$-finite.
if the identity operator \( I \) is \( \sim \)-finite. \( \mathcal{R} \) is said to be \( \sim \)-semi-finite if every non-zero projection in \( \mathcal{R} \) majorizes a non-zero \( \sim \)-finite projection.

**Theorem 2.** With notation as in Theorem 1 there exists a faithful normal semi-finite \( G \)-invariant trace on \( \mathcal{R}^+ \) if and only if \( \mathcal{R} \) is \( \sim \)-semi-finite.

**Theorem 3.** With notation as in Theorem 1 there exists a faithful \( G \)-invariant trace on \( \mathcal{R} \) if and only if \( \mathcal{R} \) is \( \sim \)-finite and countably decomposable.

3. **Proofs.** We first introduce some notation and follow [1, Ch.I, § 9] closely. Following the notation in Theorem 1 \( \mathcal{R} \) acts on a Hilbert space \( \mathcal{H} \), \( G \) is a group, considered as a discrete group, and \( t \rightarrow U_t \) is a unitary representation of \( G \) on \( \mathcal{H} \) such that \( U_t^* \mathcal{R} U_t = \mathcal{R} \) for all \( t \in G \). For \( t \in G \) let \( \mathcal{H}_t \) be a Hilbert space of the same dimension as \( \mathcal{H} \) and \( J_t \) an isometry of \( \mathcal{H} \) onto \( \mathcal{H}_t \). Let \( \mathcal{H} = \sum_{t \in G} \mathcal{H}_t \). We write an operator \( R \in \mathcal{B}(\mathcal{H}) \) - the bounded operators on \( \mathcal{H} \) - as a matrix

\[
(R_s,t)_{s,t \in G},
\]

where \( R_s,t = J_t^* R J_t \in \mathcal{B}(\mathcal{H}) \). For each \( T \in \mathcal{R} \) let \( \mathcal{T}(T) \) denote the element in \( \mathcal{B}(\mathcal{H}) \) with matrix \( (R_s,t) \), where \( R_s,t = 0 \) if \( s \neq t \), and \( R_s,t = T \) for all \( s \in G \). Then \( \mathcal{T} \) is a *-isomorphism of \( \mathcal{R} \) onto a von Neumann subalgebra \( \tilde{\mathcal{R}} \) of \( \mathcal{B}(\mathcal{H}) \). For \( y \in G \) let \( \tilde{U}_y \) be the operator in \( \mathcal{B}(\mathcal{H}) \) with matrix \( (R_s,t) \), where \( R_s,t = 0 \) if \( st^{-1} \neq y \), \( R_y t,t = U_y \) for all \( t \in G \). Then (see [1, Ch.I, § 9]) \( y \rightarrow \tilde{U}_y \) is a unitary representation of \( G \) on \( \tilde{\mathcal{R}} \) such that

\[
\tilde{U}_y^* \mathcal{T}(T) \tilde{U}_y = \mathcal{T}(U_y^* T U_y), \quad y \in G, \ T \in \mathcal{R}.
\]
If \( B \) denotes the von Neumann algebra generated by \( \tilde{R} \) and the \( \tilde{U}_y, y \in G \), then each operator in \( B \) is represented by a matrix 
\[
(R_{s,t}) \quad \text{where} \quad R_{s,t} = T_{st-1}U_{st-1}, \quad T_{st-1} \in \tilde{R}.
\]

We denote by \( R^G \) the von Neumann subalgebra of \( R \) consisting of the \( G \)-invariant operators in \( R \). \( C \) shall denote the center of \( R \), and \( D \) shall denote \( C \cap R^G \). Whenever we write \( P \sim Q \) for two projections in \( B \) we shall mean they are equivalent as operators in \( B \), i.e. there is a partial isometry \( V \in B \) such that \( VV^* = P, V^*V = Q \), and we shall not consider \( P \) and \( Q \) as equivalent in a von Neumann subalgebra of \( B \). The next lemma includes Theorem 1 and shows more, namely that \( \sim_G \)-equivalence is the same as equivalence in \( B \).

**Lemma 1.** Let \( E \) and \( F \) be projections in \( R \). Then \( E \sim F \) if and only if \( \tilde{\iota}(E) \sim \tilde{\iota}(F) \). Hence \( \sim_G \) is an equivalence relation on the projections in \( B \).

**Proof:** Suppose \( E \sim_G F \). Then for each \( t \in G \) there is \( T_t \in R \) such that
\[
E = \sum_{t \in G} T_t T_t^*, \quad F = \sum_{t \in G} U_t^* T_t T_t^* U_t.
\]

Then we have
\[
\tilde{\iota}(E) = \sum T_t T_t^* \tilde{U}_t = \sum \tilde{\iota}(T_t) \tilde{\iota}(T_t)^* \tilde{U}_t = \sum \tilde{\iota}(T_t) \tilde{U}_t \tilde{\iota}(T_t)^* \tilde{U}_t
\]
and
\[
\tilde{\iota}(F) = \sum U_t^* T_t T_t^* U_t = \sum \tilde{U}_t^* \tilde{\iota}(T_t) \tilde{U}_t = \sum \tilde{\iota}(T_t) \tilde{U}_t^* \tilde{\iota}(T_t) \tilde{U}_t.
\]

Thus by a result of Kadison and Pedersen [4, Thm. 4.1] \( \tilde{\iota}(E) \sim \tilde{\iota}(F) \).
Conversely assume $\xi(E) \sim \xi(F)$. Then there is a partial isometry $V \in \mathcal{B}$ such that $VV^* = \xi(E)$, $V^*V = \xi(F)$. Say $V = (T_{st-1}U_{st-1})$. Then an easy calculation shows

$$E = \sum_{t \in G} T_t T_t^*, \quad F = \sum_{t \in G} U_t T_t^* T_t U_t,$$

hence $E \sim F$. The proof is complete.

**Lemma 2.** Let $S = (T_{st-1}U_{st-1})$ belong to the center of $\mathcal{B}$. Then for each $s \in G$ we have

i) $TTS = T_S U_S T_U^*$ for all $T \in \mathcal{R}$,

ii) $T Sy = U_T Y S U_Y^*$ for all $y \in G$.

In particular $T_e \in \mathcal{D}$. Furthermore, if $R \in \mathcal{D}$ then $\xi(R)$ belongs to the center of $\mathcal{B}$.

**Proof.** Let $T \in \mathcal{R}$. Then

$$(T_{st-1}U_{st-1})^* = \xi(T) S = S \xi(T) = (T_{st-1}U_{st-1} T_{st-1} U_{st-1})$$

and i) follows. Let $y \in G$. Then an easy computation shows

$$(T_{st-1} U_{st-1})^* = S \tilde{U}_y = \tilde{U}_y S = (U_T Y_{st-1} U_{st-1}^*)$$

Replacing $y$ by $y^{-1}$ and letting $t = e$, ii) follows. By i) $T_e T = T_T e$, so $T_e \in \mathcal{G}$. By ii) if $s = y^{-1}$ we find $T_e = U_T Y_{st-1}^*$, so $T_e \in \mathcal{G}$, hence $T_e \in \mathcal{D}$.

Finally let $R \in \mathcal{D}$, and let $S' = (S_{st-1}U_{st-1}) \in \mathcal{B}$. Then we have

$$\xi(R) S' = (RS_{st-1}U_{st-1}) = (S_{st-1}R U_{st-1})$$

$$= (S_{st-1} R U_{st-1}) = S' \xi(R),$$

hence $\xi(R)$ belongs to the center of $\mathcal{B}$. The proof is complete.
Lemma 3. Let $E$ be a projection in $\mathcal{B}$. Let $D_E$ be the smallest operator in $\mathcal{D}$ majorizing $E$. Then $D_E$ is a projection, and $\mathcal{C}(D_E)$ is the central carrier of $\mathcal{C}(E)$ in $\mathcal{B}$.

Proof. Since $\mathcal{D}$ is an abelian von Neumann algebra its positive operators form a complete lattice under infs and sups. Thus $D_E = \text{g.l.b.} \{ A \in \mathcal{D} : E \leq A \leq I \}$, and $D_E$ is well defined. Since $E \leq D_E$ and both operators commute we have $E = E^2 \leq D_E^2$. But $D_E \leq I$, so $D_E^2 \leq D_E$. Hence by minimality of $D_E$, $D_E = D_E^2$, so it is a projection. By Lemma 2 $\mathcal{C}(D_E)$ is a central projection in $\mathcal{B}$, hence if $\mathcal{C}(E)$ denotes the central carrier of $\mathcal{C}(E)$ in $\mathcal{B}$, then $\mathcal{C}(D_E) \geq \mathcal{C}(E)$. Now let $\mathcal{C}(E) = (T_{st^{-1}}U_{st^{-1}})$. By Lemma 2 $T_e E \leq E$, and since $\mathcal{C}(E) \geq \mathcal{C}(E)$, $T_e \geq E$. By definition of $D_E$, $T_e \geq D_E$. But $\mathcal{C}(T_e) \geq \mathcal{C}(D_E)$, so $D_E \geq T_e$, hence $T_e = D_E$. The operator $\mathcal{C}(D_E) - \mathcal{C}(E)$ is positive and has zeros on the main diagonal. Therefore it is $0$, and $\mathcal{C}(D_E) = \mathcal{C}(E)$ as asserted.

Lemma 4. Let $E$ be a projection in $\mathcal{B}$. Let $C_E$ be its central carrier in $\mathcal{B}$, and let $D_E$ be as in Lemma 4. Then $D_E = D_{C_E}$.

Proof. Since $E \leq C_E$, $D_E \leq D_{C_E}$. But $D_E \in \mathcal{E}$ and $C_E \geq D_E$, hence $D_E \geq C_E$. Therefore by definition of $D_{C_E}$, $D_E \geq D_{C_E}$, and they are equal.

Lemma 5. Let $E$ be a countably decomposable projection in $\mathcal{B}$. Then $\mathcal{C}(E)$ is countably decomposable in $\mathcal{B}$.

Proof. Let $x$ be a vector in $E \mathcal{H}$. Then $x$ considered as a vector in $\bigoplus_{t \in \mathcal{G}} \mathcal{H}_t$ belongs to $\mathcal{H}_e$. Let $F$ be the support...
of $w_x$ in $E \mathcal{R} E$. Then $F$ is countably decomposable, and $w_x$ is a faithful normal state of $F \mathcal{R} F$. Let $\{ \alpha \} \alpha \in J$ be an orthogonal family of projections in $\mathcal{B}$ such that $\sum_{\alpha} F_{\alpha} = \#(F)$. Let $F_{\alpha} = (T_{\alpha}^{-1} U_{\alpha} T_{\alpha}^{-1})$. Then $F_{\alpha} \leq \#(F)$, so $T_{\alpha}^{-1} U_{\alpha} T_{\alpha} \leq F$, hence $T_{\alpha} \in F \mathcal{R} F$. Furthermore, since $x \in \mathcal{K}_e$, $w_x(T_{\alpha}) = w_x(T_{\alpha}^e)$. Thus we have

$$1 = w_x(F) = w_x(\#(F)) = \sum w_x(F_{\alpha}) = \sum w_x(T_{\alpha}).$$

Therefore $w_x(T_{\alpha}^e) = 0$ except for a countable number of $\alpha \in J$. But then $T_{\alpha}^e = 0$ and hence $F_{\alpha} = 0$ except for a countable number of $\alpha \in J$. Thus $\#(F)$ is countably decomposable in $\mathcal{B}$. Now $E$ is a countable sum of orthogonal cyclic projections, hence $\#(E)$ is a countable sum of orthogonal countably decomposable projections. Hence $\#(E)$ is countably decomposable.

The proof is complete.

**Definition 2.** We say a projection $E$ in $\mathcal{R}$ is $G$-abelian if $E \mathcal{R} E = E \mathcal{R}$. Clearly a $G$-abelian projection is abelian.

**Lemma 6.** There is a projection $P \in \mathcal{B}$ such that there exists a $G$-abelian projection $E \leq P$ with $D_E = P$, and $I - P$ has no non-zero $G$-abelian subprojection.

**Proof.** Partially order the $G$-abelian projections in $\mathcal{R}$ by $E \ll F$ if $E \leq F$ and $D_{F-E} \leq I - D_E$. Then in particular $D_{E-F} = E$. Let $\{ E_\alpha \}$ be a totally ordered set of $G$-abelian projections, and let $E = \sup E_\alpha$, so $E_\alpha \ll E$ strongly. Then

$$D_E E_\alpha = D_E \alpha \lim_{\beta > \alpha} E_\beta = \lim_{\beta > \alpha} D_E E_\beta = E_\alpha,$$

hence if $A \in \mathcal{R}$ then
EAEDE\alpha = E\alpha AE\alpha = A\alpha E\alpha ,
where \( A\alpha \in \mathcal{D}E\alpha \). Now it is well known that if \( Q\alpha \) is an increasing net of projections, and \( Q\alpha \to Q \) strongly, then \( CQ\alpha \to CQ \) strongly. Thus
\[
\hat{\xi}(DE\alpha) = C\hat{\xi}(E\alpha) - C\hat{\xi}(E) = \hat{\xi}(D_E)
\]
by Lemma 3, hence \( DE\alpha \to D_E \) strongly. The same argument also shows
\[
DE-E\alpha = \lim_{\beta > \alpha} DE\beta-E\alpha \leq I-DE\alpha .
\]
Thus \( E = E(I-DE\alpha) + E\alpha \), and since \( A\alpha = A\alpha DE\alpha \) we have
\[
EAEDE\alpha = A\alpha E \in \mathcal{E} \mathcal{D} .
\]
Since \( DE\alpha \to DE \) it follows that \( EAE = \lim_{\alpha} EAEDE\alpha \in \mathcal{E} \mathcal{D} \). Therefore \( E \) is \( \mathcal{G} \)-abelian. Now let \( E \) be a maximal \( \mathcal{G} \)-abelian projection in \( \mathcal{R} \). Let \( P = DE \). Suppose \( F \) is a \( \mathcal{G} \)-abelian subprojection of \( I-P \). Then \( E+F \) is \( \mathcal{G} \)-abelian. Indeed, if \( A \in \mathcal{R} \) then there are \( A_E \in DE \mathcal{D} \) and \( A_F \in DF \mathcal{D} \) such that
\[
(E+F)A(E+F) = EAE+FAF = EA_E + FA_F = (E+F)(A_E+A_F) \in (E+F) \mathcal{D} .
\]
Thus \( E+F \) is \( \mathcal{G} \)-abelian. Since \( E << E+F \), the maximality of \( E \) implies \( F = 0 \). The proof is complete.

Thus in order to prove theorems 2 and 3 we may consider two cases separately, namely the case when \( \mathcal{R} \) has a \( \mathcal{G} \)-abelian projection \( E \) with \( DE = I \), and the case when \( \mathcal{R} \) has no non-zero \( \mathcal{G} \)-abelian projection. We first treat the case with a \( \mathcal{G} \)-abelian projection.

**Lemma 7.** Let \( E \) be a \( \mathcal{G} \)-abelian projection in \( \mathcal{R} \). Then \( CE \) is not \( \mathcal{G} \)-equivalent to a proper central projection. Furthermore
if \( Q \) is a central projection such that \( Q \leq C_E \) then \( Q = D_Q C_E \).

**Proof.** Let \( Q \) be as in the statement of the lemma. Since \( E \) is \( G \)-abelian there is an operator \( D \in \mathcal{D} \) such that \( Q E = DE \), hence, since \( E \mathcal{C} \simeq C_E \mathcal{C} \), \( Q = Q C_E = D C_E \), and \( D \geq Q \). By definition of \( D_Q \), \( D \geq D_Q \). But \( D_Q \geq Q \), so \( Q C_E \leq D_Q C_E \leq D C_E \) = \( Q \), so that \( Q = D_Q C_E \). Now suppose \( P \) is a projection in \( \mathcal{C} \) such that \( P \leq C_E \) and \( P \not\sim G C_E \). Then in particular by Lemma 1 \( \Phi(P) \sim \Phi(C_E) \), so they have the same central carrier in \( \mathcal{C} \), hence \( D_P = D_{C_E} = D_E \) by Lemma 4. By the preceding, \( P = D_P C_E = C_E \). The proof is complete.

**Lemma 8.** Let \( E \) be a \( G \)-abelian projection in \( \mathcal{Q} \). Let \( s \in G \) and let \( Q \) be a central projection orthogonal to \( C_E \). Then if \( E \mathcal{C} \) and \( E \mathcal{C} + Q \) are \( G \)-equivalent relative to \( \mathcal{C} \), i.e. the operators \( T_t \) defining the equivalence belong to \( \mathcal{C} \), then \( Q = 0 \).

**Proof.** Let \( P = C_E \) and assume \( P \not\sim G P + Q \) relative to \( \mathcal{C} \). Then since \( \mathcal{C} \) is abelian, for each \( t \in G \) there is \( A_t \in \mathcal{C}^+ \) such that \( P = \sum_{t \in G} A_t \), \( P + Q = \sum_{t \in G} U^* A_t U \). Since \( E \mathcal{C} = E \mathcal{D} \) and \( P \mathcal{C} \not\sim E \mathcal{C} \), we have \( P \mathcal{C} = P \mathcal{D} \). Since \( A_t \leq P \) there is \( D_t \in \mathcal{D}^+ \) such that \( A_t = P D_t \). Thus we have

\[
\sum P D_t = P = P(P + Q) = \sum P U^* A_t U = \sum P U^* P D_t U = \sum P D_t U^* P U_t.
\]

Now \( P D_t U^* P U_t \leq P D_t \) for all \( t \), hence we have \( P D_t U^* P U_t = P D_t \) for all \( t \) and \( E_t \in \mathcal{D} \). Since \( U^* P U_t P D_t = P D_t \), \( U^* P U_t P E_t = P E_t \). Thus \( U^* P U_t \geq P E_t \), and thus \( U^* P E_t U = U^* P U_t E_t \geq P E_t \). Consequently \( P E_t \geq U^* P E_t U \). By Lemma 7 \( C_E \) is \( G \)-finite relative to \( \mathcal{C} \), hence so is \( P E_t \).
Therefore \( PE_t = U_t^*PE_tU_t^* \), and \( U_t^*PE_tU_t = PE_t \). Therefore we have

\[
U_t^*A_tU_t = U_t^*PD_tU_t = U_t^*PE_tU_tD_t = PE_tD_t = PD_t = A_t,
\]

and \( P = P + Q \), so that \( Q = 0 \). The proof is complete.

**Lemma 9.** Suppose \( E \) is a \( \sim_G \)-abelian projection in \( \mathcal{R} \) with \( D_E = I \). Then \( \mathcal{R} \) is of type \( I \), and there exists a faithful normal semi-finite \( G \)-invariant trace on \( \mathcal{R}^+ \).

**Proof.** Since \( E \) is abelian \( C_E \mathcal{R} \) is of type \( I \). Since every \(*\)-automorphism of \( \mathcal{R} \) preserves the type \( I \) portion of \( \mathcal{R} \), and \( D_E = I \), \( \mathcal{R} \) is of type \( I \).

\( E \) is a sum of orthogonal cyclic projections \( E_\alpha \). If we can show the lemma for each \( E_\alpha \) then it holds for \( E \). Therefore we may assume \( E \) is cyclic, say \( E = [\mathcal{R}'x] \). Then \( \omega_x \) is faithful on \( E \mathcal{R}E \), hence faithful on \( E \mathcal{E} \). If \( A > 0 \) belongs to \( C_E \mathcal{E} \) and \( \omega_x(A) = 0 \), then \( 0 = \omega_x(A) \) \( E \mathcal{E} \), so \( EA = 0 \). Hence \( A = AC_E = 0 \). Thus \( \omega_x \) is faithful on \( C_E \mathcal{E} \), so \( C_E \) is a countably decomposable projection in \( \mathcal{E} \).

We shall now apply the previous theory to \( \mathcal{A} = \mathcal{E} \times G \) instead of \( \mathcal{B} = \mathcal{R} \times G \). We use the same notation as before. By Lemma 7 \( C_E \) is \( \sim_G \)-finite. If \( C_E = D_E = I \) then by Lemma 7 \( \mathcal{E} = \mathcal{E} \), and it is trivial that there exists a faithful normal semi-finite \( G \)-invariant trace on \( \mathcal{E}^+ \). Assume \( C_E \neq I \). Then there is \( s \in G \) such that \( U_{s}^*C_{E}s \neq C_E \). Since by Lemma 7 \( C_E \) is \( \sim_G \)-finite, and \( U_{s}^*C_{E}s \sim_G C_E \), \( U_{s}^*C_{E}s \) is not a subprojection of \( C_E \). Thus \( Q = U_{s}^*C_{E}s(I - C_E) \neq 0 \). Since \( C_E \) is countably decomposable, so is \( Q \), and hence \( C_E + Q \). By Lemma 5 \( \Phi(C_E + Q) \) is countably decomposable in \( \mathcal{A} \). Since \( I = D_E \leq D_{C_E + Q} \), the central carriers of
\(\hat{\Phi}(C_E)\) and \(\hat{\Phi}(C_E + Q)\) are by Lemma 3 equal to \(I\). If \(\hat{\Phi}(C_E)\) is properly infinite then by [1, Ch. III, §8, Cor. 5] \(\hat{\Phi}(C_E) \sim \hat{\Phi}(C_E + Q)\), so by Lemma 1 \(C_E \nsim_G C_E + Q\), contradicting Lemma 8. Thus \(\hat{\Phi}(C_E)\) is not properly infinite, and there is a non-zero central projection \(P\) in \(\mathcal{A}\) such that \(P\hat{\Phi}(C_E)\) is non-zero and finite. Since the central carrier of \(\hat{\Phi}(C_E)\) is \(I\), \(P\mathcal{A}\) is semi-finite. Let \(\varphi\) be a normal semi-finite trace on \(\mathcal{A}^+\) with support \(P\). For \(A \in \mathcal{B}^+\) define \(\tau(A) = \varphi(\hat{\Phi}(A))\). Then \(\tau\) is a normal \(G\)-invariant trace because \(\tau(U_s^*AU_s) = \varphi(U_s^*\hat{\Phi}(A)U_s) = \varphi(\hat{\Phi}(A)) = \tau(A)\). Since \(\tau(C_E) < \infty\) and \(D_{C_E} = I\), \(\tau\) is semi-finite, hence \(\tau\) is a normal semi-finite \(G\)-invariant trace on \(\mathcal{B}^+\). Let \(D\) be the support of \(\tau\). Then \(0 \neq D \in \mathcal{B}\). Now apply the preceding to \((I - D)\mathcal{E}\) and \(E(I - D)\), and use Zorn's lemma to obtain a family \(D_\alpha\) of orthogonal projections in \(\mathcal{E}\) with sum \(I\), and a normal semi-finite \(G\)-invariant trace \(\tau_\alpha\) of \(\mathcal{B}^+\) with support \(D_\alpha\). Let \(\tau = \sum \tau_\alpha\). Then \(\tau\) is a faithful normal semi-finite \(G\)-invariant trace on \(\mathcal{B}^+\).

Now since \(\mathcal{R}\) is of type I there is a faithful normal center valued trace \(\psi\) on \(\mathcal{R}^+\) such that \(U_s^*\psi(U_sAU_s^*)U_s = \psi(A)\) for each \(s \in G, A \in \mathcal{R}^+\), see [11, p. 3]. Then \(\tau \ast \psi\) is a faithful normal semi-finite \(G\)-invariant trace on \(\mathcal{R}^+\), see [1, Ch. III, §4, Prop. 2]. The proof is complete.

**Lemma 10.** Suppose \(\mathcal{R}\) is \(\mathcal{G}\)-semi-finite and there are no non-zero \(\mathcal{G}\)-abelian projections in \(\mathcal{R}\). Then there is a faithful normal semi-finite \(G\)-invariant trace on \(\mathcal{R}^+\).

**Proof.** Let \(E\) be a non-zero countably decomposable \(\mathcal{G}\)-finite projection in \(\mathcal{R}\). Since \(E\) is not \(\mathcal{G}\)-abelian there is a projection \(H \in E\mathcal{R}E\) such that \(H \neq ED_H\). Let \(F = H + (I - D_H)E\). Th
Then $F \leq E$, $F \not= E$, and $D_F = D_H + (I - D_H)D_E = D_E$. $\hat{\phi}(F)$ is not properly infinite in $\mathcal{B}$. Indeed, if it were, then since $\hat{\phi}(E)$ is countably decomposable by Lemma 5, [1, Ch. III, §8, Cor. 5] would imply $\hat{\phi}(F) \sim \hat{\phi}(E)$, hence by Lemma 1, $F \sim_G E$, contradicting the $\sim_G$-finiteness of $E$. Therefore there is a non-zero central projection $P$ in $\mathcal{B}$ such that $P\hat{\phi}(F)$ is finite and non-zero. Thus $P\hat{\phi}(D_E)\mathcal{B} = P\hat{\phi}(D_F)\mathcal{B}$ is semi-finite and non-zero.

Let $\varphi$ be a normal semi-finite trace on $\mathcal{B}$ with support $P\hat{\phi}(D_E)$. For $A \in \mathcal{R}^+$ define $\tau(A) = \varphi(\hat{\phi}(A))$. As in the proof of Lemma 9 $\tau$ is a normal $G$-invariant trace on $\mathcal{R}^+$. Since $\tau(F) < \infty$ there is a non-zero central projection $Q$ in $\mathcal{R}$ such that $\tau$ is faithful and semi-finite on $Q\mathcal{R}$. [1, Ch. I, §6, Cor. 2]. Since $\tau$ is $G$-invariant $Q \in \mathcal{S}$. Now a Zorn's Lemma argument completes the proof just as in Lemma 9.

Proof of Theorem 2. By Lemma 6 there is a projection $P \in \mathcal{S}$ such that there exists a $\sim_G$-abelian projection $E \in PR$ with $D_E = P$, and $I - P$ has no non-zero $\sim_G$-abelian subprojection. By Lemma 9 there is a faithful normal semi-finite $G$-invariant trace $\tau_1$ on $PR^+$. If $\mathcal{R}$ is $\sim_G$-semi-finite then by Lemma 10 there is a faithful normal semi-finite $G$-invariant trace $\tau_2$ on $(I - P)\mathcal{R}^+$. Thus $\tau = \tau_1 + \tau_2$ is a faithful normal semi-finite $G$-invariant trace on $\mathcal{R}^+$.

Conversely assume there exists a faithful normal semi-finite $G$-invariant trace $\tau$ on $\mathcal{R}^+$. Suppose $E$ is a projection in $\mathcal{R}$ such that $\tau(E) < \infty$. Since $E \sim_G F$ implies $\tau(E) = \tau(F)$ it is clear that $E$ is $\sim_G$-finite. Thus $\mathcal{R}$ is $\sim_G$-semi-finite. The proof is complete.
Lemma 11. Suppose $\mathcal{C}$ is countably decomposable and $\mathcal{R}$ is $\mathcal{G}$-finite. Then there is a faithful normal finite $\mathcal{G}$-invariant trace on $\mathcal{R}$.

Proof. Since $\mathcal{R}$ is $\mathcal{G}$-finite $\mathcal{R}$ is in particular finite. By [1, Ch. III, §4, Thm. 3] there is a unique center valued trace $\psi$ on $\mathcal{R}$ which is the identity on $\mathcal{C}$. By uniqueness $\psi$ is $\mathcal{G}$-invariant, so if $\tau$ is a faithful normal finite $\mathcal{G}$-invariant trace on $\mathcal{C}$, then $\tau \circ \psi$ is one on $\mathcal{R}$. Therefore we may assume $\mathcal{R} = \mathcal{C}$.

Now there exists a projection $P \in \mathcal{D}$ such that $P \mathcal{C} = P \mathcal{C}$, and $\mathcal{G}$ is freely acting on $(\mathcal{I} - P) \mathcal{C}$, i.e. for each projection $E \neq 0$ in $(\mathcal{I} - P) \mathcal{C}$ there is a non-zero subprojection $F$ of $E$ and $s \in \mathcal{G}$ such that $U_s^* F U_s \leq \mathcal{I} - F$, see e.g. [5]. Since $I$ is countably decomposable, so is $P$, and there is a faithful normal state on $P \mathcal{C}$, hence a faithful normal finite $\mathcal{G}$-invariant trace on $P \mathcal{C}$. We may thus assume $\mathcal{G}$ is freely acting. Let $F$ be a non-zero projection in $\mathcal{C}$ and $s$ an element in $\mathcal{G}$ such that $U_s^* F U_s \leq \mathcal{I} - F$. Let $E = \mathcal{I} - F$. Then $D_{\mathcal{E}} = \mathcal{I}$, and $F \prec_{\mathcal{G}} E$. As in the proof of Lemma 10 $\hat{\varphi}(E)$ is not properly infinite, so we can choose a central projection $P \neq 0$ in $\mathcal{B}$ such that $P \hat{\varphi}(E)$ is finite. Since $F \prec_{\mathcal{G}} E$, $\hat{\varphi}(F) \prec \hat{\varphi}(E)$, by Lemma 1, hence $P \hat{\varphi}(F) \prec P \hat{\varphi}(E)$, so $P \hat{\varphi}(F)$ is finite. Thus $P = P \hat{\varphi}(E) + P \hat{\varphi}(F)$ is finite in $\mathcal{B}$, and $P \mathcal{B}$ is finite. Since $I$ is countably decomposable in $\mathcal{C}(=\mathcal{R})$ $\hat{\varphi}(I)$ is countably decomposable in $\mathcal{B}$ by Lemma 5, hence so is $P$. Therefore by [1, Ch. I, §6, Prop. 9] there is a faithful normal finite trace $\varphi$ on $P \mathcal{B}$. Then $\tau$ defined by $\tau(A) = \varphi(\hat{\varphi}(A))$ is a normal finite $\mathcal{G}$-invariant trace on $\mathcal{C}$ with support $D \neq 0$ in $\mathcal{B}$. A Zorn's Lemma argument now gives a family $\tau_{\alpha}$ of normal finite $\mathcal{G}$-invariant traces on $\mathcal{C}$ with orthogonal supports $D_{\alpha}$ in $\mathcal{B}$. Since $I$ is countably decomposable
the family \( \{ \tau_\alpha \} \) is countable, and by multiplying each \( \tau_\alpha \) by a convenient positive scalar we may assume \( \Sigma \tau_\alpha(D_\alpha) = 1 \). Thus if \( \tau = \Sigma \tau_\alpha \), then \( \tau \) is a faithful normal finite \( G \)-invariant trace on \( \mathcal{G} \). The proof is complete.

**Proof of Theorem 3.** Suppose there is a faithful normal finite \( G \)-invariant trace \( \tau \) on \( \mathcal{R} \). Then \( I \) is \( \mathcal{G} \)-finite, for if \( E \) is a projection in \( \mathcal{R} \) which is \( G \)-equivalent to \( I \) then \( \tau(E) = \tau(I) \), hence \( \tau(I - E) = 0 \), hence \( I - E = 0 \), since \( \tau \) is faithful. Thus \( \mathcal{R} \) is \( \mathcal{G} \)-finite. Again since \( \tau \) is faithful, its support \( I \) is countably decomposable, i.e. \( \mathcal{R} \) is countably decomposable. The converse follows from Lemma 11.

**Corollary.** If \( \mathcal{R} \) is \( \mathcal{G} \)-semi-finite then \( \mathcal{B} \) is semi-finite. If \( \mathcal{R} \) is \( \mathcal{G} \)-finite and there is an orthogonal family of countably decomposable projections in \( \mathcal{D} \) with sum \( I \), then \( \mathcal{B} \) is finite.

**Proof.** If \( \mathcal{R} \) is \( \mathcal{G} \)-semi-finite, then by Theorem 2 there is a faithful normal semi-finite \( G \)-invariant trace on \( \mathcal{R} \). Thus there is a faithful normal semi-finite trace on \( \mathcal{B} \) by [1, Ch.I, §9, Prop.1], hence \( \mathcal{B} \) is semi-finite. If \( P \) is a projection in \( \mathcal{D} \) then by Lemma 2 \( \mathcal{B}(P) \) is a central projection in \( \mathcal{B} \). Thus in order to show the last part of the corollary we may assume \( I \) is countably decomposable. Then by Theorem 3 there is a faithful normal finite \( G \)-invariant trace on \( \mathcal{R} \), hence by [1, Ch.I, §9, Prop.1] there is a normal finite trace on \( \mathcal{B} \), so \( \mathcal{B} \) is finite. The proof is complete.
4. G-finite von Neumann algebras. Let notation be as in Theorem 1. Following [7] we say $R$ is G-finite if there is a family $F$ of normal G-invariant states which separate $R^+$, i.e. if $A \in R^+$, and $\omega(A) = 0$ for all $\omega \in F$, then $A = 0$. For semi-finite von Neumann algebras it would be natural to compare this concept with those of $\sim_G$-finite and $\sim_G$-semi-finite. Since a $\sim_G$-finite von Neumann algebra is necessarily finite we cannot expect a G-finite semi-finite von Neumann algebra to be $\sim_G$-finite. We say $G$ acts ergodically on $\mathcal{E}$ if $\mathcal{E} (= \mathcal{E} \cap R^G)$ is the scalars.

Theorem 4. Let $R$ be a semi-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let $G$ be a group and $t = U_t$ a unitary representation of $G$ on $\mathcal{H}$ such that $U_t^*RU_t = R$ for all $t \in G$. Assume either that $G$ acts ergodically on the center of $R$ or the center is elementwise fixed under $G$. Then $R$ is G-finite if and only if $R$ is $\sim_G$-semi-finite and there is an orthogonal family of finite $G$-invariant projections in $R$ with sum 1.

Proof. Assume $R$ is G-finite. Suppose first that $G$ acts ergodically on the center $\mathcal{E}$ of $R$, and suppose $\omega$ is a faithful normal $G$-invariant state on $R$. Then by [11] there is a faithful normal semi-finite $G$-invariant trace on $R^+$, hence by Theorem 2 $R$ is $\sim_G$-semi-finite. In general, by Zorn's Lemma there is a family $\{\omega_\alpha\}$ of normal $G$-invariant states with orthogonal supports $E_\alpha$ such that $\Sigma E_\alpha = \mathbb{1}$. Then each $E_\alpha$ is $G$-invariant, and by the first part of the proof $E_\alpha R E_\alpha$ is $\sim_G$-semi-finite. In particular, $E_\alpha$ is the sup of an increasing net of $\sim_G$-finite projections. Let $F$ be a projection in $R$. We show $F$ has a non-zero $\sim_G$-finite subprojection. By the above consider-
ations there is $E_\alpha$ and a $G$-finite subprojection $F_\alpha$ of $E_\alpha$ such that $C_{F_\alpha}F \neq 0$. Let $F_1 = C_{F_\alpha}F$. Then there is a non-zero subprojection $F_0$ of $F_1$ such that $F_0 \prec F_\alpha$. Say $F_0 \sim G_\alpha \leq F_\alpha$.

Since $F_\alpha$ is $G$-finite, so is $G_\alpha$. Indeed, if $G_\alpha \sim H \leq G_\alpha$ then by Lemma 1 $\xi(G_\alpha) \sim \xi(H)$, hence $\xi(F_\alpha) = \xi(G_\alpha) + \xi(F_\alpha - G_\alpha) \sim \xi(H) + \xi(F_\alpha - G_\alpha)$, so again by Lemma 1, $F_\alpha \sim G_\alpha H + F_\alpha - G_\alpha$, so that $H = G_\alpha$ by finiteness of $F_\alpha$. Thus $G_\alpha$ is $G$-finite.

Since $G_\alpha$ is in particular finite there is thus by [1,Ch.III,§2,Prop.6] a unitary operator $U \in \mathcal{R}$ such that $UF_0U^{-1} = G_\alpha$. But then $F_0$ is $G$-finite, for if $F_0 \sim F_2 \leq F_0$ then $UF_2U^{-1} \sim F_2 \sim G_\alpha$, so by transitivity $UF_2U^{-1} \sim G_\alpha$. Since $UF_2U^{-1} \leq G_\alpha$, they are equal by finiteness of $G_\alpha$, so $F_2 = F_0$, and $F_0$ is $G$-finite. Therefore the projection $F$ has a non-zero $G$-finite subprojection $F_0$, and $\mathcal{R}$ is $\sim G$-semi-finite.

Next assume $G = \mathcal{R}$. Then every normal semi-finite trace on $\mathcal{R}^+$ is $G$-invariant [10,Cor.2.2], so there exists a faithful normal semi-finite $G$-invariant trace on $\mathcal{R}^+$, hence by Theorem 2, $\mathcal{R}$ is $\sim G$-semi-finite.

Let $\tau$ be a faithful normal semi-finite $G$-invariant trace on $\mathcal{R}^+$. Let $\{\omega_\alpha\}$ be as before with orthogonal supports $\{E_\alpha\}$. Then there is a positive self-adjoint operator $H_\alpha \in L^1(\mathcal{R},\tau)$ affiliated with $\mathcal{R}^G$ such that $\omega_\alpha(T) = \tau(H_\alpha T)$ for $T \in \mathcal{R}$, see e.g. [1,Ch.I,§6,no.10]. Let $E$ be a finite spectral projection of $H_\alpha$. Then $E$ is $G$-invariant. A Zorn's Lemma argument now gives an orthogonal family of finite $G$-invariant projections in $\mathcal{R}$ with sum $I$.

Conversely assume $\mathcal{R}$ is $\sim G$-semi-finite and having an orthogonal family $\{E_\alpha\}$ of finite non-zero $G$-invariant projections with sum $I$. Let by Theorem 2 $\tau$ be a faithful normal semi-finite $G$-invariant trace on $\mathcal{R}^+$. Let $c_\alpha = \tau(E_\alpha)^{-1}$, and let
\( w_{\alpha}(T) = c_{\alpha}^T(E_{\alpha}T) \). Then \( \{w_{\alpha}\} \) is a separating family of normal \( G \)-invariant states on \( \mathcal{R} \), hence \( \mathcal{R} \) is \( G \)-finite. The proof is complete.

The above theorem is probably true without the assumptions of the action of \( G \) on \( \mathcal{C} \). A direct proof of this would be quite interesting.

5. Abelian von Neumann algebras. Assume \( \mathcal{R} \) is an abelian von Neumann algebra acting on a Hilbert space \( \mathcal{H} \). Let \( G \) be a group and suppose \( t \to U_t \) is a unitary representation of \( G \) on \( \mathcal{H} \) such that \( U_t^* \mathcal{R} U_t = \mathcal{R} \) for all \( t \in G \). We say two projections \( E \) and \( F \) in \( \mathcal{R} \) are equivalent in the sense of Hopf and write \( E \sim_H F \) if there is an orthogonal family of projections \( \{E_{\alpha}\}_{\alpha \in J} \) in \( \mathcal{R} \) and \( t_{\alpha} \in G \), for \( \alpha \in J \), such that

\[
E = \sum_{\alpha \in J} E_{\alpha}, \quad F = \sum_{\alpha \in J} U_{t_{\alpha}}^* E_{\alpha} U_{t_{\alpha}}.
\]

Since each \( U_{t_{\alpha}}^* E_{\alpha} U_{t_{\alpha}} \) is a projection, and their sum is a projection, they are all mutually orthogonal. Since we can collect the \( E_{\alpha}'s \) for which \( t_{\alpha} \) coincide the definition of equivalence in the sense of Hopf is equivalent to the existence of an orthogonal family of projections \( \{E_t\}_{t \in G} \) in \( \mathcal{R} \) such that

\[
E = \sum_{t \in G} E_t, \quad F = \sum_{t \in G} U_{t^*} E_t U_t.
\]

This ordering was introduced by Hopf [3]. Just as for \( \sim_G \) we define \( \sim_H \)-finite, \( \sim_H \)-semi-finite, and \( \prec_H \). Note that if \( E \sim_H F \) as above, if we let \( T_t = E_t \), then

\[
E = \sum_{t \in G} T_t T_t^*, \quad F = \sum_{t \in G} U_{t^*} T_t T_t^* U_t, \quad \text{so} \quad E \sim_H F.
\]

It is plausible that the converse is true too. If we assume \( \mathcal{R} \) is countably decomposable, we can prove this via a proof which makes use of the known results on invariant measures if \( \mathcal{R} \) is \( \sim_H \)-finite and \( \sim_H \)-semi-finite. A direct proof would be much more desirable.
Theorem 5. Assume $\mathcal{R}$ is countably decomposable, and let notation be as above. Then two projections $E$ and $F$ in $\mathcal{R}$ are $G$-equivalent if and only if they are equivalent in the sense of Hopf.

Outline of proof. It remains to be shown that if $E \sim G F$ then $E \sim H F$. Assume $E \sim G F$. By Lemma 1 $\xi(E) \sim \xi(F)$, so they have the same central carrier $C$. By Lemma 3 $\xi(D_E) = C = \xi(D_F)$, so $D_E = D_F$. Suppose first $E$ and $F$ are such that $E P$ and $F P$ are $H$-infinite for all non-zero projections $P \in \mathcal{D}$. In a von Neumann algebra two properly infinite countably decomposable projections with the same central carriers are equivalent [1, Ch. III, §8, Cor. 5]. Using the comparison theory for $\mathcal{R}$ with the Hopf ordering $\prec_H$, as developed in [6], see also [9], we can modify the proof of the quoted result for von Neumann algebras, to show $E \sim H F$. If $E$ is $H$-finite then since $D_E = D_F$, we may assume $\mathcal{R}$ is $H$-semi-finite, so by [6] there is a faithful normal semi-finite $G$-invariant trace $\tau$ on $\mathcal{R}^+$. From the comparison theorem on $\mathcal{R}$ [6, Lem. 16], or [9, Lem. 2.7], there exist two orthogonal projections $P$ and $Q$ in $\mathcal{D}$ with sum $I$ such that $PE \prec H PF$ and $QF \prec H QE$. Since $PE \sim G PF$ we have $\tau(PE) = \tau(PF)$. But if a proper subprojection $F_0$ of $PF$ is such that $PE \sim H F_0$ then $\tau(PE) = \tau(F_0) < \tau(PF) = \tau(PE)$, a contradiction. Thus $PE \sim H PF$, and similarly $QE \sim H QF$. Thus $E \sim H F$, and the proof is complete.
References


