

ON THE LENGTH OF FAITHFUL MODULES OVER ARTINIAN LOCAL RINGS.

by

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Introduction.

Let  $R$  be an Artinian local ring with residue field  $k = R/\mathfrak{m}$ . Let  $M$  be any faithful  $R$ -module, i.e.  $rM = 0$  implies  $r = 0$  for all  $r \in R$ . Then for a large class of rings  $R$  one has the inequality

$$(*) \quad \ell(M) \geq \ell(R)$$

$\ell$  denoting classical length. It is easily seen that the inequality is valid whenever  $R$  is self injective, that is when  $\dim_k \text{Hom}_R(k, R) = 1$ ; see (2.8) in [1]. The purpose of the present note is to generalize this fact by showing that (\*) is valid for all faithful  $R$ -modules  $M$  whenever  $\dim_k \text{Hom}_R(k, R) \leq 3$ . This result is in a way the best possible, in fact for each integer  $s \geq 4$  we can give an example of a local ring  $R$  and a faithful  $R$ -module  $M$  such that

$$\ell(M) < \ell(R) \quad \text{and} \quad \dim_k \text{Hom}_R(k, R) = s.$$

Notations and definitions.

$R$  will always be an Artinian local ring with maximal ideal  $\mathfrak{m}$ .  $R$ -modules are assumed to be unitary and finitely generated. If  $M$  is an  $R$ -module we define the annihilator

$$\text{an}(M) = \{r \in R \mid rM = 0\}$$

and the socle

$$s(M) = \{x \in M \mid \mathfrak{m}x = 0\}.$$

Observe that  $s(M) \approx \text{Hom}_R(R/\mathfrak{m}, M)$ .

$\ell(M)$  denotes the length of  $M$ . If  $\text{an}(M) = \mathfrak{m}$  then  $\dim M$  will denote the dimension of  $M$  as a vectorspace over  $R/\mathfrak{m}$ .  $E$  denotes the injective hull of the  $R$ -module  $R/\mathfrak{m}$ . We let  $M^*$  denote the dual of  $M$ , that is

$$M^* = \text{Hom}_R(M, E).$$

Recall that the functor  $\text{Hom}_R(-, E)$  defines a duality on the category of finitely generated  $R$ -modules, cf. [2]. Note that

$$\text{an}(M) = \text{an}(M^*), \quad s(M^*) \approx M/\mathfrak{m}M.$$

$M$  will be called a faithful  $R$ -module if  $\text{an}(M) = 0$ . Observe that  $E$  is, up to isomorphism, the only faithful  $R$ -module with one-dimensional socle.

Lemma 1. Let  $M$  be a faithful  $R$ -module. Suppose that  $M/N$  is not faithful for any submodule  $N \neq 0$ . Then  $s(M) = s(R)M$ .

Proof. Let  $N$  be a submodule of  $M$  such that

$$s(M) = s(R)M \oplus N.$$

We are going to show that  $N = 0$ . Suppose  $N \neq 0$ . Then by the minimality of  $M$  there exists an element  $r \neq 0$  in  $R$  such that  $rM \subset N$ . We may as well assume that  $r \in s(R)$ . It follows that  $rM \subset s(R)M \cap N = 0$ . Hence  $r = 0$ , which is a contradiction.

Lemma 2. Let  $M$  be a faithful  $R$ -module. Assume that neither  $N$  nor  $M/N$  is faithful for any submodule  $N$  such that  $0 \neq N \neq M$ . Then we have

$$(i) \quad \dim M/\mathfrak{m} M \leq \dim s(R)$$

$$(ii) \quad \dim s(M) \leq \dim s(R).$$

Moreover, if  $M \neq R$  then at least one of the inequalities is strict.

Proof. We will first prove (i). Let  $m = \dim M/\mathfrak{m} M$  and let  $g_1, \dots, g_m$  be a minimal set of generators for  $M$ . Since (i) is obvious if  $m = 1$ , we may assume that  $m \geq 2$ .

For  $1 \leq i \leq m$  let  $M_i$  be the submodule generated by all  $g_1, \dots, g_m$  except  $g_i$ . Put  $C_i = \text{an}(M_i)$ . By the minimality of  $M$  we have  $C_i \neq 0$  hence  $C_i \cap s(R) \neq 0$  for all  $i$ . Choose one non-zero element  $u_i$  in  $C_i \cap s(R)$  for each  $i$ . Since  $M$  is faithful, the elements  $u_i$  are clearly linearly independent over the field  $R/\mathfrak{m}$ . It follows that  $m \leq \dim s(R)$ .

To prove (ii) we just have to apply (i) to the dual  $M^*$ , observing that  $M^*$  satisfies the same minimality conditions as  $M$ . We get

$$\dim s(M) = \dim \frac{M^*}{\mathcal{M}M^*} \leq \dim s(R).$$

We will now assume that we have equality in both (i) and (ii), and we assume that  $M$  is not isomorphic to  $R$ . We are going to show that this is impossible.

Since  $M$  is faithful, but not isomorphic to  $R$ , we have  $\dim \frac{M}{\mathcal{M}M} \geq 2$ . Let  $g_1, \dots, g_m$  and  $u_1, \dots, u_m$  be as above. The equality in (i) gives that  $u_1, \dots, u_m$  is a basis for  $s(R)$ . Hence by lemma 1 we obtain

$$s(M) = (u_1, \dots, u_m)(g_1, \dots, g_m) = (u_1g_1, u_2g_2, \dots, u_mg_m).$$

Let  $\mathcal{C}$  be the annihilator of the element  $g_1 + \dots + g_m$ . By minimality of  $M$  we have  $\mathcal{C} \neq 0$  and hence  $\mathcal{C} \cap s(R) \neq 0$ . Let  $u$  be a non-zero element in  $\mathcal{C} \cap s(R)$ . Let  $r_1, \dots, r_m$  be elements in  $R$  such that  $u = \sum_{i=1}^m r_i u_i$ . We have

$$0 = u(g_1 + \dots + g_m) = \sum_{i=1}^m r_i u_i g_i.$$

Since not all  $r_i$  are in  $\mathcal{M}$ , the equation above shows that  $\dim s(M) < m$  contradicting the equality in (ii).

Corollary. Let  $M$  be as in lemma 2 and suppose that  $\dim s(R) \leq 2$ . Then  $M \approx R$  or  $M \approx E$ .

Proof. If  $M \neq R$  then by lemma 2 we have  $\dim s(M) = 1$ , hence  $M \approx E$ .

Theorem 1. Let  $R$  be an Artinian local ring with  $\dim_{R/\mathfrak{M}} \text{Hom}_R(R/\mathfrak{M}, R) \leq 3$ . Let  $M$  be a faithful  $R$ -module. Then we have  $\ell(M) \geq \ell(R)$ .

Proof. Clearly we may assume that  $M$  is a faithful module of minimal length, so that  $M$  as well as  $M^*$  satisfies the assumption in lemma 2. If  $\dim s(R) \leq 2$  then the theorem follows from the above corollary. We may therefore assume that  $\dim s(R) = 3$ . Moreover we may assume that  $M$  is not isomorphic to  $R$ . Hence using lemma 2 and the relation

$$\dim_{M/\mathfrak{M}} M = \dim s(M^*)$$

we have either

$$\dim s(M^*) \leq 2 \quad \text{or} \quad \dim s(M) \leq 2.$$

There is no loss of generality in assuming that  $\dim s(M) \leq 2$ . If  $\dim s(M) = 1$  then  $M \approx E$ , and if  $\dim_{M/\mathfrak{M}} M = 1$  then  $M \approx R$ . Hence in the rest of the proof we may work under the following assumptions:

$$\dim s(R) = 3, \quad \dim s(M) = 2 \quad \text{and} \quad \dim_{M/\mathfrak{M}} M \geq 2.$$

By the second of these assumptions we can find non-zero irreducible submodules  $M_1, M_2$  in  $M$  such that  $0 = M_1 \cap M_2$ ; see § 2 in [1].

Put  $\mathcal{O}_i = \text{an}(M/M_i)$  for  $i = 1, 2$ . We will first show that

$$(1) \quad \ell(M/M_i) = \ell(R/\mathcal{O}_i) \quad \text{for } i = 1, 2.$$

Since  $M_i$  is irreducible we have  $\dim s(M/M_i) = 1$ . It follows that  $(M/M_i)^*$  is a homomorphic image of  $R$ . Moreover we have

$$\text{an}((M/M_i)^*) = \text{an}(M/M_i) = \mathcal{O}_i$$

hence

$$(M/M_1)^* \approx R/\sigma_1$$

so (1) follows.

Since  $M$  is faithful we have  $\sigma_1 \cap \sigma_2 = 0$ . Since  $\dim s(R) = 3$ , at least one of the two vectorspaces  $s(\sigma_1)$  and  $s(\sigma_2)$  is one-dimensional.

We will assume that  $\dim s(\sigma_1) = 1$ .

In view of (1) it now suffices to show that  $\ell(M_1) \geq \ell(\sigma_1)$ . Since  $\sigma_1 M \subseteq M_1$  it will be sufficient to prove the following:

$$(2) \quad \ell(\sigma_1 M) \geq \ell(\sigma_1).$$

Let  $g_1, g_2, \dots, g_m$  be a minimal set of generators for  $M$ . Put  $\mathfrak{b}_i = \text{an}(g_i)$  for  $1 \leq i \leq m$ . Then  $\bigcap_{i=1}^m \mathfrak{b}_i = 0$ . Hence one of the  $\mathfrak{b}_i$ , say  $\mathfrak{b}_1$ , does not contain  $s(\sigma_1)$ . Since  $\dim s(\sigma_1) = 1$  we conclude that  $\sigma_1 \cap \mathfrak{b}_1 = 0$ . We obtain  $\sigma_1 M \supseteq \sigma_1 g_1 \approx \sigma_1 (R/\mathfrak{b}_1) \approx \sigma_1 / \sigma_1 \cap \mathfrak{b}_1 = \sigma_1$  which yields (2).

Theorem 2. Let  $s \geq 4$  be an integer. Then there exists a local Artinian ring  $R$  and a faithful  $R$ -module  $M$  such that

$$(i) \quad \dim_{R/\mathfrak{m}} \text{Hom}_R(R/\mathfrak{m}, R) = s$$

$$(ii) \quad \ell(M) < \ell(R).$$

Proof. Let  $m \geq 2$  be an integer and let  $k$  be an arbitrary field. Let  $R_m$  be the  $k$ -algebra of  $(m+2) \times (m+2)$ -matrices of the form

$$(3) \quad \left( \begin{array}{ccc|cc} \lambda I_{m,m} & & & & O_{m,2} \\ \hline a_1 & \cdots & a_m & & \\ b_1 & \cdots & b_m & & \lambda I_{2,2} \end{array} \right)$$

where  $\lambda, a_1, \dots, a_m, b_1, \dots, b_m$  run through  $k$  and  $I_{p,q}$  and  $O_{p,q}$  denotes the identity matrix and the zero-matrix of size  $p \times q$ . Clearly  $R_m$  is a commutative local Artinian ring of length  $\ell(R_m) = 2m+1$ . In fact the socle of  $R_m$  coincides with the maximal ideal which consists of all matrices of the form (3) in which  $\lambda = 0$ . Hence  $\dim s(R_m) = 2m$ .

Now let  $M$  be the  $k$ -vectorspace  $k^{m+2}$ . Clearly  $M$  becomes a faithful  $R_m$ -module in the obvious way. We have

$$\ell(M) = \dim_k M = m+2 < 2m+1 = \ell(R_m).$$

This proves the theorem in the case where  $s$  is even.

Let us now assume that  $s$  is odd. Write  $s = 2m-1$  where  $m \geq 3$ . Consider  $R_m$  and  $M$  as before. Let  $R'_m$  be the subring consisting of all matrices of the form (3) in which  $a_m = 0$ . Clearly  $R'_m$  is a local ring of length  $2m$  and  $\dim s(R'_m) = 2m-1 = s$ . Moreover  $M$  is a faithful  $R'_m$ -module with

$$\ell(M) = \dim_k M = m+2 < 2m = \ell(R'_m).$$

The proof is now complete.

Remark. Let  $R = \mathbf{C}[X,Y]/(X,Y)^4$ . It can be shown that  $\ell(M) \geq \ell(R)$  for any faithful  $R$ -module, inspite of the fact that  $\dim s(R) = 4$ .

REFERENCES

- [1] Bass, H.: On the ubiquity of Gorenstein rings. Math. Zeitschr. 82, 8-28 (1963).
- [2] Matlis, E.: Injective modules over noetherian rings. Pacific J. Math. 8, 511-528 (1958).