Sections of functors and the problem of lifting algebraic structures.

by

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Contents

Introduction.

Let $\pi: R \to S$ be a surjective homomorphism of rings and suppose $(\ker \pi)^2 = 0$.

Let A be an S-algebra and let A' be an R-algebra such that $A' \otimes S \simeq A$. A' is called a <u>lifting of A to R</u> , if $\mathrm{Tor}_1^R(A',S) = 0$.

In particular, if A is S-flat then A' is a lifting of A to R if, and only if, $A' \otimes_R S \simeq A$ and A' if R-flat. We may then ask the following question.

When do liftings exist, and if there are some, how many liftings will there be?

If A is S-flat the answer was given by Schlessinger and Lichtenbaum [S].

Using their cohomology theory of algebras, they proved that there exists an obstruction $\sigma \in H^2(S,A;\, A \otimes \ker \pi)$ such that $\sigma = 0$ if and only if there exists a lifting, and the set of liftings, modulo isomorphisms reducing to the identity, is then a principal homogenous space over $H^1(S,A;A \otimes \ker \pi)$.

This is the kind of problem we shall be concerned with in this paper.

We shall eventually consider a variety of algebraic objects defined over S , such as an algebra, a morphism of algebras, a diagram of morphisms of algebras, a bialgebra etc. In each case we will study the corresponding lifting problem.

A good starting point for the theory of lifting seems to be to consider the following general problem.

Let $\pi: \underline{C} \to \underline{c}$ be any functor. When does π admit a section (i.e. a functor $\sigma: \underline{c} \to \underline{C}$ such that $\sigma\,\pi = 1_{\underline{c}}$) ?

Chapter 1 is concerned with this general problem. We prove that under certain conditions on π (we need coefficients for a cohomology theory) there exists a sequence of cohomology groups $H^i(\underline{c},\pi)$ and an obstruction $\sigma \in H^2(\underline{c},\pi)$ such that $\sigma = 0$ if and only if π has a section. Moreover, if there is one section, then the set of all sections is a union of principal homogenous spaces over $H^1(\underline{c},\pi)$.

In Chapter 2 we shall use the methods of Chapter 1 to give a new proof of a slightly improved version of the result of Lichtenbaum and Schlessinger. The cohomology involved here will be the cohomology of André [A].

Finally in Chapter 3 we shall consider diagrams of morphisms of algebras.

The main result is not too startling. If $\psi: A \to B$ is a homomorphism of S-algebras, and if A' and B' are liftings of A resp. B to R, then there exist an obstruction

$$\sigma(A'.B') \in H'(S,A;B \otimes \ker \pi)$$

such that $\sigma(A',B') = 0$ if and only if there exists a homomorphism of R-algebras $\psi': A' \to B'$ such that $\psi' \otimes_R S \simeq \psi$. The set of such liftings is a principal homogenous space over $\mathrm{Der}_S(A,B \otimes \ker \pi)$.

This paper grew out of a seminar given at the Department of Mathematics at the University of Oslo through the spring and fall of 1970. The author wishes to thank the audience for its unfailing patience.

Chapter 1. Sections of functors.

(1.1) Derivation functors associated to a functor

Let $\pi\colon \underline{C} \to \underline{c}$ be a functor of small categories. We shall consider the category $\underline{\mathrm{Mor}}\ \underline{c}$, for which

1. The objects are the morphisms of $\underline{c}$.
2. If φ, φ' are objects in $\underline{\mathrm{Mor}}\ \underline{c}$ then the set of morphisms $\mathrm{Mor}(\varphi,\varphi')$ is the set of commutative diagrams

$$\begin{tikzcd} * \arrow[r, "\psi"] \arrow[d, "\varphi"'] & * \arrow[d, "\varphi'"] \\ * & * \arrow[l, "\psi'"] \end{tikzcd}$$

We write $(\psi,\psi')\colon \varphi \to \varphi'$ for such a morphism.

Let $\varphi \in \underline{\mathrm{Mor}}\ \underline{c}$ be an object (i.e. a morphism of $\underline{c}$) and let $\pi^{-1}(\varphi) = \{\lambda \in \underline{\mathrm{Mor}}\ \underline{C} \mid \pi(\lambda) = \varphi\}$.

If φ_1 and φ_2 are morphisms in $\underline{c}$ which can be composed then we have a partially defined map:

$$m : \pi^{-1}(\varphi_1) \times \pi^{-1}(\varphi_2) \to \pi^{-1}(\varphi_1 \circ \varphi_2)$$

defined by composition of morphisms in $\underline{C}$.

We shall suppose that there exists a contravariant functor

$\mathrm{Der}\colon \underline{\mathrm{Mor}}\ \underline{c} \to \underline{\mathrm{Ab}}$

with the properties:

(Der 1) There exists a map:

$$\mu : \pi^{-1}(\varphi) \times \mathrm{Der}(\varphi) \to \pi^{-1}(\varphi)$$

and a partially defined map

$$\nu : \pi^{-1}(\varphi) \times \pi^{-1}(\varphi) \to \mathrm{Der}(\varphi)$$

defined on the subset of those pairs (λ_1,λ_2) having same "source" and same "aim". These maps satisfy the following relations

$$\mu(\lambda,\alpha+\beta) = \mu(\mu(\lambda,\alpha),\beta)$$
$$\nu(\lambda_1,\lambda_2) = \alpha \quad \text{is equivalent to} \quad \lambda_1 = \mu(\lambda_2,\alpha) .$$

(Der 2) Suppose φ_1 and φ_2 can be composed in $\underline{c}$, then the diagram

$$\begin{tikzcd}
\pi^{-1}(\varphi_1)\times\pi^{-1}(\varphi_2) \arrow[r, "m"] & \pi^{-1}(\varphi_1\circ\varphi_2) \\
(\pi^{-1}(\varphi_1)\times \mathrm{Der}(\varphi_1))\times(\pi^{-1}(\varphi_2)\times \mathrm{Der}(\varphi_2)) \arrow[u, "\mu\times\mu"] \arrow[r, "\delta"'] & \pi^{-1}(\varphi_1\circ\varphi_2)\times \mathrm{Der}(\varphi_1\circ\varphi_2) \arrow[u, "\mu"]
\end{tikzcd}$$

commutes, with δ defined by:

$$\delta((\lambda_1,\alpha),(\lambda_2,\beta)) = (m(\lambda_1,\lambda_2),\ \mathrm{Der}(id,\varphi_2)(\alpha) + \mathrm{Der}(\varphi_1,id)(\beta))$$

<u>Note</u> that $(id,\varphi_2): \varphi_1\circ\varphi_2 \to \varphi_1$ and $(\varphi_1,id): \varphi_1\circ\varphi_2 \to \varphi_2$ are morphisms in <u>Mor</u> $\underline{c}$, since the diagrams

$$\begin{tikzcd}
* \arrow[r, "\varphi_1"] \arrow[d, "\varphi_1\circ\varphi_2"'] & * \arrow[d, "\varphi_2"] \\
* & * \arrow[l, "1"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
* \arrow[r, "1"] \arrow[d, "\varphi_1\circ\varphi_2"'] & * \arrow[d, "\varphi_1"] \\
* & * \arrow[l, "\varphi_2"]
\end{tikzcd}$$

commute.

We shall from now on use the following notations:

$$\varphi_1\beta = \mathrm{Der}(\varphi_1,id)(\beta)$$
$$\alpha\varphi_2 = \mathrm{Der}(id,\varphi_2)(\alpha)$$
$$\lambda_1-\lambda_2 = \nu(\lambda_1,\lambda_2)$$

A functor with these properties will be called a <u>derivation functor associated to π</u>

There are some obvious examples.

Ex.1. Let $\pi: R \to S$ be a surjective homomorphism of rings. Let $I = \ker\pi$ and suppose $I^2 = 0$. Consider the category $\underline{C}$ of flat R-algebras and the category $\underline{c}$ of flat S-algebras. Tensorization with S over R defines a functor

$$\pi\colon \underline{C} \to \underline{c}$$

and the ordinary derivation functor

$$\mathrm{Der}\colon \underline{\mathrm{Mor}}\ \underline{c} \to \underline{\mathrm{Ab}}$$

given by:

$$\mathrm{Der}(\varphi) = \mathrm{Der}_S(A, B \underset{S}{\otimes} I)$$

where $\varphi\colon A \to B$ defines the A-module structure on $B \underset{S}{\otimes} I$, is a <u>derivation functor for π</u>.

Ex.2. Let $\underline{C}_o$ be the full subcategory of $\underline{C}$ defined by the free R-algebras (i.e. the polynomial rings over R in any set of variables), and let $\underline{c}_o$ be the full subcategory of $\underline{c}$ defined by the free S-algebras. As above the ordinary derivation functor induces a derivation functor for the restriction π_o of π to $\underline{C}_o$.

Ex.3. Let $\pi\colon R \to S$ be as before and let $\underline{C}$ be the category of R-flat affine group schemes over R and $\underline{c}$ the category of S-flat affine groups schemes over S.

Tensorization by S over R defines a functor

$$\pi\colon \underline{C} \to \underline{c}$$

Let φ be an object in $\underline{\mathrm{Mor}}\ \underline{c}$ (i.e. $\varphi\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a homomorphism of S-flat affine group schemes over S) and consider

$$\mathrm{Der}(\varphi) = \{\xi \in \mathrm{Der}_S(A, B \otimes \ker \pi) \mid \xi \circ \mu_B = \mu_A \circ (\varphi \otimes \xi + \xi \otimes \varphi)\}$$

where $\mu_A\colon A \to A \otimes A$ and $\mu_B\colon B \to B \otimes B$ are the comultiplications defining the group scheme structure on $\mathrm{Spec}(A)$ and

Spec(B) respectively.

Then Der is a <u>derivation functor for π</u>.

<u>Remark</u>. If $\pi^{-1}(\varphi)$ is empty then the conditions (Der 1) and (Der 2) are vacuous.

(1.2.) <u>Obstructions for the existence of sections of functors</u>

Given a functor π with a derivation functor Der: $\underline{\text{Mor}}\ \underline{c} \to \underline{\text{Ab}}$, let us try to find conditions on $\underline{c}$ and π under which there exists a section σ for π, i.e. a functor $\sigma: \underline{c} \to \underline{C}$ such that

$$\sigma \circ \pi = 1_{\underline{c}}$$

We observe immediately that if such a σ exists then certainly we must have

$$\pi^{-1}(\varphi) \neq \emptyset \qquad \text{for all} \quad \varphi \in \underline{\text{Mor}}\ \underline{c} \ ,$$

and moreover there must exist a <u>quasisection</u> i.e. a map σ': Mor $\underline{c} \to$ Mor $\underline{C}$ such that if φ_1 and φ_2 can be composed then $\sigma'(\varphi_1)$ and $\sigma'(\varphi_2)$ can be composed and $\sigma'(\varphi_1) \circ \sigma'(\varphi_2)$ have the same "source" and "aim" as $\sigma'(\varphi_1 \circ \varphi_2)$. Given such a quasisection σ' we deduce a map σ_o: ob $\underline{c} \to$ ob $\underline{C}$, which we shall call the <u>stem</u> of the quasisection σ' .

Now, with all this we may prove:

<u>Theorem 1.2.1)</u> Suppose given a quasisection σ' of π . Then there exists an obstruction

$$\mathcal{O}(\sigma') \in \varprojlim_{\underline{\text{Mor}}\ \underline{c}}{}^{(2)} \text{Der}$$

such that $\mathcal{O}(\sigma') = 0$ if and only if there exists a section σ of π with the same stem σ_o as σ'. Moreover, if $\mathcal{O}(\sigma') = 0$ then the set of sections having the stem σ_o, modulo isomorphisms reducing to the identity, is a principal homogenous space over

$$\varprojlim_{\underline{\mathrm{Mor}}\ \underline{c}}{}^{(1)} \mathrm{Der}$$

<u>Proof</u>. Consider the complex $D^{\bullet} = D^{\bullet}(\mathrm{Der})$ of abelian groups defined by

$$D^{o}(\mathrm{Der}) = \prod_{c \in \mathrm{ob}\,\underline{c}} \mathrm{Der}(1_c)$$

$$D^{n}(\mathrm{Der}) = \prod_{c_o \xrightarrow[\psi_1]{} c_1 \to \cdots \to c_{n-1} \xrightarrow[\psi_n]{} c_n} \mathrm{Der}(\psi_1 \circ \psi_2 \circ \cdots \circ \psi_n) \qquad n \geq 1$$

where the indices are chains of morphisms in $\underline{c}$, and where

$$d^n : D^n \to D^{n+1}$$

is defined by:

$$(d^o\xi)(\psi_1) = \psi_1\, \xi_{c_1} - \xi_{c_o}\, \psi_1$$

$$(d^n\xi)(\psi_1, \ldots \psi_{n+1}) = \psi_1\, \xi(\psi_2, \ldots, \psi_{n+1}) +$$

$$\sum_{i=1}^{n} (-1)^i\, \xi(\psi_1, \ldots, \psi_i \circ \psi_{i+1}, \ldots \psi_{n+1}) + (-1)^{n+1}\, \xi(\psi_1, \ldots, \psi_n)\, \psi_{n+1}$$

for $n \geq 1$.

One easily verifies that $d^n \circ d^{n+1} = 0$ for all $n \geq 0$.

<u>Lemma (1.2.2)</u> $\quad H^n(D^{\bullet}) \simeq \varprojlim_{\underline{\mathrm{Mor}}\ \underline{c}}{}^{(n)} \mathrm{Der}$

The proof will be given in (1.3).

Now consider the quasisection σ' and define the element $\mathcal{O}(\sigma')$ of D^2 by:

$$\mathcal{O}(\sigma')(\psi_1,\psi_2) = \sigma'(\psi_1 \circ \psi_2) - \sigma'(\psi_1) \circ \sigma'(\psi_2) .$$

By assumption $\mathcal{O}(\sigma')(\psi_1,\psi_2) \in \mathrm{Der}(\psi_1 \circ \psi_2)$.

In fact $\mathcal{O}(\sigma') \in \ker d^2$ since

$$(d^2\, \mathcal{O}(\sigma'))(\psi_1,\psi_2,\psi_3) = \psi_1 \mathcal{O}(\sigma')(\psi_2,\psi_3) - \mathcal{O}(\sigma')(\psi_1 \circ \psi_2,\psi_3)$$

$$+ \mathcal{O}(\sigma')(\psi_1,\psi_2 \circ \psi_3) - \mathcal{O}(\sigma')(\psi_1,\psi_2)\psi_3$$

$$= \psi_1(\sigma'(\psi_2 \circ \psi_3) - \sigma'(\psi_2) \circ \sigma'(\psi_3)) - (\sigma'(\psi_1 \circ \psi_2 \circ \psi_3) - \sigma'(\psi_1 \circ \psi_2) \circ \sigma'(\psi_3))$$

$$+ (\sigma'(\psi_1 \circ \psi_2 \circ \psi_3) - \sigma'(\psi_1) \circ \sigma'(\psi_2 \circ \psi_3)) - (\sigma'(\psi_1 \circ \psi_2) - \sigma'(\psi_1) \circ \sigma'(\psi_2))\psi_3$$

$$= (\sigma'(\psi_1) \circ \sigma'(\psi_2 \circ \psi_3) - \sigma'(\psi_1) \circ \sigma'(\psi_2) \circ \sigma'(\psi_3))$$

$$- (\sigma'(\psi_1 \circ \psi_2 \circ \psi_3) - \sigma'(\psi_1 \circ \psi_2)\ \sigma'(\psi_3))$$

$$+ (\sigma'(\psi_1 \circ \psi_2 \circ \psi_3) - \sigma'(\psi_1) \circ \sigma'(\psi_2 \circ \psi_3))$$

$$- (\sigma'(\psi_1 \circ \psi_2)\sigma'(\psi_3) - \sigma'(\psi_1) \circ \sigma'(\psi_2) \circ \sigma'(\psi_3))$$

$$= 0 .$$

It follows that $\mathcal{O}(\sigma')$ defines an element $\mathcal{O}(\sigma') \in H^2(D^\bullet)$. Suppose $\mathcal{O}(\sigma') = 0$, then there is a $\xi \in D^1$ such that $d\xi = \mathcal{O}(\sigma')$.

Now put

$$\sigma(\varphi) = \sigma'(\varphi) + \xi(\varphi)$$

Then $\sigma(\psi_1 \circ \psi_2) - \sigma(\psi_1) \circ \sigma(\psi_2)$

$$= (\sigma'(\psi_1 \circ \psi_2) + \xi(\psi_1 \circ \psi_2)) - (\sigma'(\psi_1) + \xi(\psi_1)) \circ (\sigma'(\psi_2) + \xi(\psi_2))$$

$$= \sigma'(\psi_1 \circ \psi_2) - \sigma'(\psi_1) \circ \sigma'(\psi_2) - (\sigma'(\psi_1)\xi(\psi_2) - \xi(\psi_1 \circ \psi_2)$$

$$+ \xi(\psi_1)\ \sigma'(\psi_2)) = \mathcal{O}(\sigma')_{(\psi_1,\psi_2)} - (d\xi)_{(\psi_1,\psi_2)} = 0 .$$

i.e. σ is a functor, (we easily find that $\sigma(1_c) = 1\sigma_o(c)$).

Obviously the stem of σ is equal to the stem of σ' (i.e. $= \sigma_o$).

Now let σ_1 and σ_2 be two sections of π with the same stem σ_o. Then $(\sigma_1 - \sigma_2)$ defines an element in D^1, by:

$$(\sigma_1 - \sigma_2)(\psi) = \sigma_1(\psi) - \sigma_2(\psi) .$$

Since σ_1 and σ_2 both are sections $(d_1(\sigma_1 - \sigma_2))(\psi_1, \psi_2)$ $= \psi_1(\sigma_1 - \sigma_2)(\psi_2) - (\sigma_1 - \sigma_2)(\psi_1 \circ \psi_2) + (\sigma_1 - \sigma_2)(\psi_1)\, \psi_2 = 0$, and therefore $(\sigma_1 - \sigma_2)$ defines an element in $H^1(D^\bullet)$.

Suppose this element is zero, then there exists an element $\zeta \in D^0$ such that

$$\sigma_1(\psi) - \sigma_2(\psi) = \psi\,\zeta - \zeta\,\psi$$

i.e.

$$\sigma_1(\psi) \circ (1_{\sigma_o(c_1)} - \zeta c_1) = (1_{\sigma_o(c_o)} - \zeta c_o) \circ \sigma_2(\psi)$$

for all

$$\psi \colon c_o \to c_1 .$$

Conversely, suppose $s \in H^1(D^\bullet)$ is represented by $\xi \in D^1$ then given any section σ of π, $\xi + \sigma$ is another section with the same stem as σ.

QED.

(1.3.) Proof of lemma 1.2.2. In this section we shall prove lemma (1.2.2) by proving a more general theorem.

Theorem (1.3.1) The functor

$D^\bullet \colon \underline{Ab}^{\underline{Mor}\ \underline{c}^o} \to \underline{Complexes}$ is a resolving functor for $\lim\limits_{\overleftarrow{\underline{Mor}\ \underline{c}^o}}$.

Proof. Let L be the constant functor on $\underline{Mor}\ \underline{c}$ with $L(\varphi) = \mathbb{Z}$

for all φ .

We shall construct a projective resolution of L in $\underline{Ab}^{\underline{Mor}\ \underline{c}^{o}}$.

Let $\varphi: x \to y$ be any object of $\underline{Mor}\ \underline{c}$ and consider the sets

$$\Delta^{o}(\varphi) = \{x \xrightarrow{\varepsilon} c_o \xrightarrow{\rho} y \mid \varepsilon \circ \rho = \varphi\} .$$

$$\Delta^{n}(\varphi) = \{x \xrightarrow{\varepsilon} c_o \xrightarrow{\psi_1} c_1 \to \cdots \to c_{n-1} \xrightarrow{\psi_n} c_n \xrightarrow{\rho} y \mid \varepsilon \circ \psi_1 \circ \cdots \circ \psi_n \circ \rho = \varphi\}$$

There exist maps:

$$\eta^{n}_{i} : \Delta^{n}(\varphi) \to \Delta^{n+1}(\varphi)$$

$$\delta^{n}_{i} : \Delta^{n}(\varphi) \to \Delta^{n-1}(\varphi)$$

defined by:

$$\eta^{n}_{i}(x \xrightarrow{\varepsilon} c_o \xrightarrow{\psi_1} c_1 \to \cdots \to c_{n-1} \xrightarrow{\psi_n} c_n \xrightarrow{\rho} y) = (x \xrightarrow{\varepsilon} c_o \to \cdots \to c_i \xrightarrow{id} c_i \to \cdots \to c_n \xrightarrow{\rho} y)$$

$$\delta^{n}_{i}(x \xrightarrow{\varepsilon} c_o \xrightarrow{\psi_1} c_1 \to \cdots \to c_{n-1} \xrightarrow{\psi_n} c_n \xrightarrow{\rho} y) = \begin{cases} (x \xrightarrow{\varepsilon \circ \psi_1} c_1 \xrightarrow{\psi_2} c_2 \to \cdots \to c_n \xrightarrow{\rho} y) & i=0. \\ (x \xrightarrow{\varepsilon} c_o \to \cdots \to c_{i-1} \xrightarrow{\psi_i \circ \psi_{i+1}} c_{i+1} \to \cdots \to c_n \xrightarrow{\rho} y) & \\ \qquad \text{for } o < i < n & \\ (x \xrightarrow{\varepsilon} c_o \to \cdots \to c_{n-1} \xrightarrow{\psi_n \circ \rho} y) & i=n \end{cases}$$

giving $\Delta^{n}(\varphi)$, $n \geq 0$ the structure of a simplicial set.

Moreover for each $n \geq 0$, $\Delta^{n}(\varphi)$ is functorial in φ defining a functor

$$\Delta: \underline{Mor}\ \underline{c} \to \underline{Simplicial}\ \underline{sets}$$

Composing Δ with the functor $C_{\bullet}(-, \mathbb{Z})$ we have constructed a complex of functors

$$C_{\bullet}: \underline{Mor}\ \underline{c} \to \underline{Ab}$$

Now, by a standard argument we construct a contracting homotopy for $C_{\bullet}$ thereby proving

$$H_i(C.) = \begin{cases} L & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

Moreover

$$C_n(\varphi) = \coprod_{\substack{(\epsilon,\rho):\varphi' \to \varphi \\ \text{in } \underline{\text{Mor}}\ \underline{c}^o}} \{ \coprod_{\substack{\psi_1,\ldots,\psi_n \\ \psi_1 \circ \cdots \circ \psi_n = \varphi'}} \mathbb{Z} \}$$

Using ([La],Prop.1.1.a) it follows that each C_n is projective as object of $\underline{\text{Ab}}^{\underline{\text{Mor}}\ \underline{c}^o}$

Therefore $C.$ is a projective resolution of L in $\underline{\text{Ab}}^{\underline{\text{Mor}}\ \underline{c}^o}$.
Since

$$\text{Mor}_{\underline{\text{Ab}}^{\underline{\text{Mor}}\ \underline{c}^o}}(C_n, F) = \prod_{c_0 \xrightarrow{\psi_1} c_1 \to \cdots \to c_{n-1} \xrightarrow{\psi_n} c_n} F(\psi_1 \circ \cdots \circ \psi_n)$$

we find by a dull computation that

$$D^{\cdot}(F) \simeq \text{Mor}_{\underline{\text{Ab}}^{\underline{\text{Mor}}\ \underline{c}}}(C., F)$$

thereby proving the theorem.

QED.

Chapter 2. Lifting of algebras.

(2.1) Leray spectral sequence for $\varprojlim$.

Let $\underline{c}$ be any small category and let c be an object of $\underline{c}$. Consider the contravariant functor $C(\mathbb{Z},c)$ defined by:

$$C(\mathbb{Z},c)(c') = \coprod_{c' \xrightarrow{\varphi} c} \mathbb{Z}$$

We know (see [La]) that these functors are projective objects in $\underline{Ab}^{\underline{c}^{o}}$.

Suppose $\underline{M}$ is a full subcategory of $\underline{c}$ and consider the restriction of $C(\mathbb{Z},c)$ to $\underline{M}$. Let F be any contravariant functor on $\underline{M}$ with values in $\underline{Ab}$, then we find,

$$\underline{Ab}^{\underline{M}^{o}}(C(\mathbb{Z},c),F) \simeq \varprojlim_{(\underline{M}/c)^{o}} F .$$

Now, suppose $c_o \xrightarrow[\varphi]{} c$ in $\underline{c}$ is an $\underline{M}$ epimorphism, i.e. $c_o \in \mathrm{ob}\,\underline{M}$ and the map

$$\mathrm{Mor}(c',c_o) \to \mathrm{Mor}(c',c)$$

is surjective for every $c' \in \mathrm{ob}\,\underline{M}$.

Suppose further that $\underline{c}$ has fibered products and consider the system of morphisms

$$c \xleftarrow{\varphi} c_o \leftleftarrows c_o \times_c c_o \;\overset{\leftarrow}{\leftleftarrows}\; \cdots\cdots \;\vdots\; \underbrace{c_o \times_c \cdots \times_c c_o}_{p} \;\vdots$$

Put $c_p = \underbrace{c_o \times_c \cdots \times_c c_o}_{p+1}$ and denote by

$$d_p^{\,i} : c_p \to c_{p-1} \qquad i = 0,\ldots,p$$

the $p+1$ projection morphisms.

Consider for each d_p^i the corresponding morphism $\partial_p^i : C(\mathbb{Z},c_p) \to C(\mathbb{Z},c_{p-1})$ and let $\partial_p = \sum_{i=o}^{p} (-1)^i \partial_p^i$. Then $\partial_p \partial_{p-1} = 0$ for all $p \geq 1$.

__Lemma (2.1.1)__ The complex $C. = \{C(\mathbb{Z},c_p),\partial_p\}_{p\geq o}$ is a resolution of $C(\mathbb{Z},c)$ in $\underline{Ab}^{\underline{M}^o}$.

__Proof.__ See M. Artin [1] p. 18.

Let $F^\bullet$ be an injective resolution of F in $\underline{Ab}^{\underline{M}^o}$ and consider the double complex

$$\mathrm{Mor}(C.,F^\bullet)$$

We shall compute the two associated spectral sequences. But first we have to establish the following lemma.

__Lemma (2.1.2)__ Let $f\colon \underline{M}/c \to \underline{M}$ be the canonical forgetful functor and let F be injective in $\underline{Ab}^{\underline{M}^o}$, then the composed functor $f \circ F\colon (\underline{M}/c)^o \to \underline{Ab}$ is injective as an object of $\underline{Ab}^{(\underline{M}/c)^o}$.

__Proof.__ The functor f induces a functor

$$f_*\colon \underline{Ab}^{\underline{M}^o} \to \underline{Ab}^{(\underline{M}/c)^o}$$

We want to prove that f_* takes injectives into injectives. To prove this we construct a left adjoint

$$\rho\colon \underline{Ab}^{(\underline{M}/c)^o} \to \underline{Ab}^{\underline{M}^o}$$

Let G be an object of $\underline{Ab}^{(\underline{M}/c)^o}$ and put

$$\rho(G)(m) = \coprod_{\varphi \in \mathrm{Mor}(m,c)} G(m \xrightarrow{\varphi} c)$$

so that $\rho(G)$ is an object of $\underline{\mathrm{Ab}}^{\underline{M}^{\circ}}$.

One easily checks that there is a canonical isomorphism

$$\mathrm{Mor}(\rho(G),F) \simeq \mathrm{Mor}(G,f_*(F))$$

proving that ρ is left adjoint to f_* . Since ρ is exact we know that f_* takes injectives into injectives.

QED.

Going back to the double complex $\mathrm{Mor}(C.,F^{\bullet})$ we find the E_2 terms of the two associated spectral sequences:

$$'E_2^{p,q} = H^p(H^q(\mathrm{Mor}(C.,F^{\bullet})))$$

$$''E_2^{p,q} = H^p(\mathrm{Mor}(H_q(C.),F^{\bullet}))$$

We know already that

$$''E_2^{p,q} = 0 \quad \text{for} \quad q \neq 0$$

$$''E_2^{n,o} = H^n(\varprojlim_{(\underline{M}/c)^{\circ}} (F^{\bullet}))$$

and by Lemma (2.1.2) we deduce that

$$''E_2^{n,o} = \varprojlim_{(\underline{M}/c)^{\circ}} {}^{(n)}F \ .$$

Since

$$\mathrm{Mor}(C_p,F^{\bullet}) = \varprojlim_{\underline{M}/c_p} F^{\bullet} \ ,$$

we find, using Lemma (2.1.1) once more that

$$'E_2^{p,q} = H^p(\varprojlim_{\underline{M}/c.} {}^{(q)}F) \ .$$

We have proved the following theorem.

<u>Theorem (2.1.3)</u> Let $\underline{M} \subseteq \underline{c}$ and $\varphi\colon c_o \to c$ be given as above. Then there exists a <u>Leray spectral sequence</u> given by:

$$E_2^{p,q} = E_2^{p,q}(\underline{M}) = H^p(\varprojlim_{(\underline{M}/c.)^o}{}^{(q)}F)$$

converging to

$$\varprojlim_{(\underline{M}/c)^o}{}^{(\cdot)}F \,.$$

<u>Remark</u> 1. The spectral sequence above is nothing but the Leray spectral sequence associated to the "covering" $\varphi\colon c_o \to c$ in an appropriate Grothendieck topology.

2. Since $c_o \in \text{ob}\, \underline{M}$ the category $\underline{M}/c_o$ has a final object. Therefore $E_2^{o,q} = 0$ for all $q \geq 1$.

We deduce from this the formulas

$$\varprojlim_{(\underline{M}/c)^o} F \simeq E_2^{o,o}$$

$$\varprojlim_{(\underline{M}/c)^o}{}^{(1)}F \simeq E_2^{1,o} \,,$$

and the exact sequence

$$0 \to E_2^{2,o} \to \varprojlim_{(\underline{M}/c)^o}{}^{(2)}F \to E_2^{1,1} \to E_2^{3,o} \to \varprojlim_{(\underline{M}/c)^o}{}^{(3)}F \,.$$

<u>Corollary (2.1.4)</u> Suppose that $\varprojlim_{(\underline{M}/c_j)^o}{}^{(i)}F = 0$ for $i \geq 1$, $i+j = p$ and for $i+j = p-1$. Then

$$\varprojlim_{(\underline{M}/c)^o}{}^{(p)}F \simeq E_2^{p,o} \,.$$

Assume for a moment that there exists a functor $i: \underline{c} \to \underline{Ab}$ commuting with fibered products.

Corollary (2.1.5) Put $g = f \circ i$ and suppose

$$\lim_{\substack{\rightarrow \\ \underline{M}/c_p}} g = i(c_p) \qquad \text{for all } p \geq 0 .$$

Then

$$\lim_{\substack{\rightarrow \\ \underline{M}/c}}{}^{(1)} g = 0 .$$

Proof. Let E be an injective abelian group and consider the functor

$$F(-) = \underline{Ab}\,(g(-),E) .$$

We know that

$$\underline{Ab}\,(\lim_{\substack{\rightarrow \\ \underline{M}/c}}{}^{(1)} g,\ E) \simeq \lim_{\substack{\leftarrow \\ (\underline{M}/c)^{o}}}{}^{(1)} F$$

$$= \ker\{ \lim_{\substack{\leftarrow \\ (\underline{M}/c_1)^{o}}} F \to \lim_{\substack{\leftarrow \\ (\underline{M}/c_2)^{o}}} F\} / \operatorname{im}\{\lim_{\substack{\leftarrow \\ (\underline{M}/c_o)^{o}}} F \to \lim_{\substack{\leftarrow \\ (\underline{M}/c_1)^{o}}} F\}$$

$$= \underline{Ab}\,(\ker\{i(c_1) \to i(c_o)\}/ \operatorname{im}\{i(c_2) \to i(c_1)\},E)$$

But since $i(c_p) = \underbrace{i(c_o) \underset{i(c)}{\times} \cdots \underset{i(c)}{\times} i(c_o)}_{p+1}$

this last group is zero.

Since this holds for all injective abelian groups E we have proved that $\lim_{\substack{\rightarrow \\ \underline{M}/c}}{}^{(1)} g = 0$.

QED.

Remark   The last corollary and the next one are important in our development of the lifting theory for algebras.

Corollary (2.1.6)   Let $\underline{M}_o \subseteq \underline{M}$ be two full subcategories of $\underline{c}$. Suppose $\underline{c}$ has fibered products and let $c \in \mathrm{ob}\, \underline{c}$.

Assume that $\underline{c}$ satisfies the following conditions:

(c_1) There exists an object c_o of $\underline{M}_o$ and an $\underline{M}$-epimorphism $\varphi: c_o \to c$.

(c_2) For any $\underline{M}$-epimorphism $\psi: d_o \to d$ im $\underline{c}$ with $d_o \in \underline{M}_o$ there exist objects $e_p \in \underline{M}_o$ and $\underline{M}$-epimorphisms

$$\psi_p: e_p \to \underbrace{d_o \underset{d}{\times} \cdots \underset{d}{\times} d_o}_{p+1} \qquad p \geq 2 .$$

Then we may conclude

$$\varprojlim_{(\underline{M}/c)^o}{}^{(\cdot)} \simeq \varprojlim_{(\underline{M}_o/c)^o}{}^{(\cdot)}$$

Proof. We first observe that (c_1) and (c_2) together with (2.1.1) imply that there are canonical isomorphisms

(1) $$\varprojlim_{(\underline{M}/c_p)^o} \simeq \varprojlim_{(\underline{M}_o/c_p)^o}$$

where $c_p = \underbrace{c_o \underset{c}{\times} \cdots \underset{c}{\times} c_o}_{p+1}$.

Now the canonical morphism

$$t^n: \varprojlim_{(\underline{M}/c)^o}{}^{(n)} \to \varprojlim_{(\underline{M}_o/c)^o}{}^{(n)}$$

induces morphisms of spectral sequences

$$t_2^{p,q}: E_2^{p,q}(\underline{M}) \to E_2^{p,q}(\underline{M}_o)$$

Using (1) we find isomorphisms

$$t_2^{p,o}: E_2^{p,o}(\underline{M}) \cong E_2^{p,o}(\underline{M}_o) \qquad p \geq 0 .$$

Thereby proving that t_2^1 is an isomorphism. By an easy induction argument we may assume that $t_2^{p,q}$ are isomorphisms for all p,q with $p+q \leq n$ or $q < n$. This implies that

$$t_\infty^{p,q}: E_\infty^{p,q}(\underline{M}) \to E_\infty^{p,q}(\underline{M}_o)$$

are isomorphisms for all p,q with $p+q = n$, thereby proving that t^n is an isomorphism.

QED.

(2.2.) Lifting of algebras

Let S be any commutative ring with unit. Let $\underline{S\text{-alg}}$ denote the category of S-algebras and let $\underline{S\text{-free}}$ denote the category of free S-algebras (i.e. the category of polynomial algebras, in any set of variables, over S).

Let A be any object of $\underline{S\text{-alg}}$ and consider the subcategories $\underline{M}_o$ and $\underline{M}$ of $\underline{S\text{-alg}}/_A$ given by: $\underline{M} = \underline{S\text{-free}}/_A$ and $\underline{M}_o$ is the full subcategory of $\underline{M}$ defined by the epimorphisms $F \to A$.

Thus we have $\underline{M}_o \subseteq \underline{M} \subseteq \underline{S\text{-alg}}/_A$.

We observe that we have isomorphisms of categories:

$$\underline{M}_o \simeq \underline{M}_o/(A \xrightarrow[1_A]{} A)$$

$$\underline{M} \simeq \underline{M}/(A \xrightarrow[1_A]{} A)$$

$$\underline{S\text{-alg}}/_A \simeq (\underline{S\text{-alg}}/_A)/(A \xrightarrow[1_A]{} A) .$$

By a straight forward verification we find that $\underline{M}_o \subseteq \underline{M} \subseteq S\text{-}\underline{alg}/A$ and the object $(A \xrightarrow[1_A]{} A)$ satisfy the conditions of Corollary (2.1.6).

We therefore conclude:

<u>Lemma (2.2.1)</u> $\qquad \varprojlim_{\underline{M}}^{(\cdot)} \simeq \varprojlim_{M_o}^{(\cdot)}$

Now recall that given any A-module M the cohomology $H^\cdot(S,A;M)$ is defined by

$$H^n(S,A;M) = \varprojlim_{\underline{M}^o}^{(n)} \mathrm{Der}_S(-,M)$$

(see André [A]).

Using (2.2.1) we find

$$H^n(S,A;M) = \varprojlim_{\underline{M}_o^o}^{(n)} \mathrm{Der}_S(-,M)$$

(i.e. we may compute the cohomology of A using only surjective homomorphisms of free S-algebras onto A).

Recall also (see [La]) the standard resolving complex $C^\cdot$ for $\varprojlim$ (called $\Pi^\cdot$ in [La]), for which

$$\varprojlim_{\underline{c}^o}^{(n)} \simeq H^n(C^\cdot) .$$

$C^\cdot$ is defined by

$$C^p(F) = \prod_{c_o \xrightarrow[\psi_1]{} c_1 \to \cdots \xrightarrow[\psi_p]{} c_p} F(c_o)$$

with $d^p\colon C^p(F) \to C^{p+1}(F)$ given by

$$d^p(\xi)(\psi_1,\ldots,\psi_{p+1}) = F(\psi_1)(\xi(\psi_2,\ldots,\psi_{p+1}))$$
$$+ \sum_{i=1}^{n} (-1)^i \xi(\psi_1,\ldots,\psi_i \circ \psi_{i+1},\ldots,\psi_{p+1}) + (-1)^{n+1}\xi(\psi_1,\ldots,\psi_p) .$$

Let $\pi\colon R \to S$ be a surjective homomorphism of commutative rings and let $I = \ker \pi$. Assume that $I^2 = 0$.

Consider the functor

$$\pi'\colon R\text{-}\underline{\mathrm{alg}} \to S\text{-}\underline{\mathrm{alg}}$$

defined by tensorization with S over R .

<u>Definition (2.2.2)</u> An R-algebra A' is called a lifting of the S-algebra A to R if $\pi'(A') \simeq A$ and $\mathrm{Tor}_1^R(A',S) = 0$.

Let π be the restriction of π' to R-<u>free</u> ,

$$\pi\colon R\text{-}\underline{\mathrm{free}} \to S\text{-}\underline{\mathrm{free}}$$

We have observed already in (1.1) that

$$\mathrm{Der}(\varphi) = \mathrm{Der}_S(F_1, F_2 \underset{S}{\otimes} I)$$

where $\varphi\colon F_1 \to F_2$ is a homomorphism of free S-algebras is a <u>derivation</u> <u>functor</u> for π .

There are lots of <u>quasisections</u> of π , and we pick one quasi-section σ' . Note that all <u>stems</u> are equal.

Suppose now that there exists a <u>section</u> σ for π . Given any S-algebra A , a good candidate for a lifting of A to R would be the R-algebra

$$A' = \varinjlim_{S\text{-}\underline{\mathrm{free}}/A} (f \circ \sigma) .$$

In fact we shall see later that A' is a lifting of A .

Since there are, in general, S-algebras that cannot be lifted to R we deduce that such a σ cannot always exist.

To settle the case of a single S-algebra we must therefore be a little more subtle.

Consider the restriction g_o of $g\colon \underline{M} \to$ S-$\underline{\text{free}}$ to $\underline{M}_o$ and look at the complexes

$$D^\bullet = D^\bullet(g_o \circ \mathrm{Der}\ )$$
$$C^\bullet = C^\bullet(g_o \circ \mathrm{Der}(-, A \otimes I))$$

defined above (with $\underline{c} = \underline{M}_o$), see (1.2).

Let us first show that there exists a surjective morphism

$$j\colon D^\bullet \to C^\bullet \ .$$

In fact we have that

$$D^n = \prod_{\substack{F_o \xrightarrow{\psi_1} F_1 \to \cdots \xrightarrow{\psi_n} F_n \\ \varphi_o \searrow \ \downarrow \varphi_1 \ \swarrow \varphi_n \\ A}} \mathrm{Der}_S(F_o, F_n \otimes I)$$

$$C^n = \prod_{\substack{F_o \xrightarrow{\psi_1} F_1 \to \cdots \xrightarrow{\psi_n} F_n \\ \varphi_o \searrow \ \downarrow \varphi_1 \ \swarrow \varphi_n \\ A}} \mathrm{Der}_S(F_o, A \otimes I)$$

where the indices run over all sequences of morphisms $\varphi_o \xrightarrow[\psi_n]{} \varphi_1 \to \cdots \xrightarrow[\psi_1]{} \varphi_n$ in $\underline{M}_o$.

Now for each such index, φ_n defines a homomorphism

$$\varphi_{n*}\colon \mathrm{Der}_S(F_o, F_n \otimes I) \to \mathrm{Der}_S(F_o, A \otimes I) \ .$$

Since F_o is free and φ_n is surjective we conclude that φ_{n*} is surjective. (This is in fact the only reason why we have to consider $\underline{M}_o$ instead of $\underline{M}$.)

But these φ_{n*} induce a surjective morphism

$$j_n : D^n \to C^n$$

and a trivial verification shows that these j_n's commute with the differentials in $D^{\bullet}$ and $C^{\bullet}$. Put $K^{\bullet} = \ker j$, then the sequence

$$0 \to K^{\bullet} \to D^{\bullet} \to C^{\bullet} \to 0$$

is exact.

Corresponding to the quasisection σ' we have the 2-cocycle $\mathcal{O}(\sigma') \in D^2$ given by

$$\mathcal{O}(\sigma')(\psi_1,\psi_2) = \sigma'(\psi_1 \circ \psi_2) - \sigma'(\psi_1) \circ \sigma'(\psi_2) \in \mathrm{Der}_S(F_o, F_2 \otimes I)$$

where ψ_1 and ψ_2 are morphisms in $\underline{M}_o$ such that:

$$\begin{tikzcd} F_o \arrow[r, "\psi_1"] \arrow[dr, "\varphi_o"'] & F_1 \arrow[r, "\psi_2"] \arrow[d, "\varphi_1"] & F_2 \arrow[dl, "\varphi_2"] \\ & A & \end{tikzcd}$$

Let $\mathcal{O}(A,\sigma') = j(\mathcal{O}(\sigma'))$ and consider the corresponding element

$$\sigma(A,\sigma') \in H^2(C^{\bullet}) = H^2(S,A;A \otimes I)$$

<u>Theorem (2.2.3)</u> (i) The cohomology class $\sigma(A,\pi) = \sigma(A,\sigma')$ is independent of the choice of quasisection σ'.

(ii) There exists a lifting A' of A to R if and only if $\sigma(A,\pi) = 0$.

(iii) If $\sigma(A,\pi) = 0$ then the set:

$$F_A(R) = \{A' \in \mathrm{ob}\ R\text{-}\underline{\mathrm{alg}} \mid A' \text{ lifting of } A\} \Big/ \left\{\begin{array}{l}\text{isomorphisms} \\ A' \simeq A'' \\ \text{reducing to the} \\ \text{identity on } A\end{array}\right\}$$

is a principal homogenous

space over $H^1(S,A;A \otimes I)$.

<u>Proof</u>. Suppose σ' and σ'' are two quasisections. Let ζ be the 1-cochain of $D^\bullet$ given by

$$\zeta(\psi_1) = \sigma'(\psi_1) - \sigma''(\psi_1) .$$

Then one verifies that

$$\mathcal{O}(\sigma'') - \mathcal{O}(\sigma') = d\zeta .$$

Thus $\mathcal{O}(\sigma'') = \mathcal{O}(\sigma')$, and à fortiori

$$\mathcal{O}(A,\sigma'') = \mathcal{O}(A,\sigma')$$

proving (i).

Suppose there exists a lifting A' of A to R and consider the obvious functor

$$\pi_2 : \underline{M}_o(A') \to \underline{M}_o(A) ,$$

where $\underline{M}_o(A')$ is the full subcategory of R-$\underline{\text{free}}/_{A'}$ defined by the surjective homomorphisms.

There are lots of quasisections of π_2 , and we pick one quasisection σ'' . If

$$\begin{tikzcd} F_o \arrow[rr, "\psi_1"] \arrow[dr, "\varphi_o"'] & & F_1 \arrow[dl, "\varphi_1"] \\ & A & \end{tikzcd}$$

is a morphism ψ_1 in S-$\underline{\text{free}}/_A$, let

$$\begin{tikzcd} F'_o \arrow[rr, "\sigma''(\psi_1)"] \arrow[dr, "\sigma''(\varphi_o)"'] & & F'_1 \arrow[dl, "\sigma''(\varphi_1)"] \\ & A' & \end{tikzcd}$$

be the morphism $\sigma''(\psi_1)$ of R-$\underline{\text{free}}/_A$.

Computing $\mathcal{O}(\sigma'')$ we find $j\mathcal{O}(\sigma'') = 0$ since, on one hand, all triangles with A' as vertex in the diagram

$$\begin{tikzcd} F_0' \arrow[r, "\sigma''(\psi_1)"] \arrow[dr, "\sigma''(\varphi_0)"'] & F_1' \arrow[r, "\sigma''(\psi_2)"] \arrow[d, "\sigma''(\varphi_1)"] & F_2' \arrow[dl, "\sigma''(\varphi_2)"] \\ & A' & \end{tikzcd}$$

commute; and, on the other hand, the diagram

$$\begin{tikzcd} F_i' \otimes_R I \simeq F_i \otimes_S I \arrow[r, "\varphi_i \otimes 1_I"] \arrow[d, hook] & A' \otimes_R I \simeq A \otimes_S I \arrow[d, hook] \\ F_i' \arrow[r, "\sigma''(\varphi_i)"'] & A' \end{tikzcd}$$

commute as a result of $\mathrm{Tor}_1^R(A',S) = 0$.

Therefore $\Theta(A,\pi) = 0$, proving the "only if" part of (ii).

Suppose $\mathcal{O}(A,\pi) = 0$. Then $j\mathcal{O}(\sigma') = d\zeta$ where ζ is a 1-cochain of $C^{\bullet}$. Since j is surjective there exists a 1-cochain ζ of $D^{\bullet}$ such that $j(\xi) = \zeta$. Let σ'' be the map

$$\mathrm{Mor}\ \underline{M}_o \to \mathrm{Mor}\ R\text{-}\underline{\mathrm{free}}$$

given by:

$$\sigma''(\psi) = \sigma'(g_o(\psi)) + \xi(\psi) .$$

Let $\varphi_o \xrightarrow{\psi_1} \varphi_1 \xrightarrow{\psi_2} \varphi_2$ be two morphisms in $\underline{M}_o$, then

$$\sigma''(\psi_1 \circ \psi_2) - \sigma''(\psi_1) \circ \sigma''(\psi_2)$$

$$= \mathcal{O}(\sigma')(\psi_1,\psi_2) - d\xi(\psi_1,\psi_2) = \omega(\psi_1,\psi_2)$$

Since $j\mathcal{O}(\sigma') = j(d\xi)$ we may assume $\omega \in K^2$.

Now

$$A' = \varinjlim_{\underline{M}_o} \sigma''$$

exists as an R-module,

$$\varinjlim_{\underline{M}_o} \sigma'' = \operatorname{coker}\Big(\coprod_{\substack{\psi_1 \\ \varphi_o \to \varphi_1}} \sigma''(\varphi_o) \rightrightarrows \coprod_{\varphi} \sigma''(\varphi) \Big) \,.$$

Consider the resolving complex $C.$ (the dual of $C^{\bullet}$) of $\varinjlim_{\underline{M}_o}$.

Recall that:

$$C_p(F) = \coprod_{c_o \xrightarrow{\psi_1} c_1 \to \cdots \xrightarrow{\psi_p} c_p} F\,(c_p)$$

Since σ'' is not a functor $C.(\sigma'')$ will not necessarily be a complex, but nevertheless we may consider the diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & A \otimes_S I \arrow[d, phantom, "\wr|"] & \\
C_2(\sigma'') \otimes_R I \arrow[r, dashed] \arrow[d] & C_1(\sigma'') \otimes_R I \arrow[r] \arrow[d] & C_o(\sigma'') \otimes_R I \arrow[r, "\beta"] \arrow[d] & A' \otimes_R I \arrow[r] \arrow[d, dashed, "\alpha"] & 0 \\
C_2(\sigma'') \arrow[r, dashed, "\delta"] \arrow[d] & C_1(\sigma'') \arrow[r, "\gamma"] \arrow[d] & C_o(\sigma'') \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
C_2(\sigma'') \otimes_R S \arrow[r] \arrow[d] & C_1(\sigma'') \otimes_R S \arrow[r] \arrow[d] & C_o(\sigma'') \otimes S \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 & 0 &
\end{tikzcd}$$

In which we know that all sequences of maps marked with solid arrows are exact. The vertical sequences are exact since all $C_p(\sigma'')$ are R-flat, the lower horizontal sequences is exact since

$$C_p(\sigma'') \otimes_R S \simeq C_p(g_o)$$

and because of Corollary (2.1.5).

The solid part of the upper horizontal sequence is exact since

$$C.(\sigma'') \otimes_R I \simeq C.(g_o) \otimes_S I \;,$$

and finally, part of the middle horizontal sequence is exact by the definition of A' .

Remember that we do not know that $\delta \circ \gamma = 0$. In fact it may well be that $\delta \circ \gamma \neq 0$. However $\operatorname{im}(\delta \circ \gamma) \subseteq C_o(\sigma'') \underset{R}{\otimes} I$ and fortunately we have arranged the situation such that

$$\beta\,(\operatorname{im}(\delta \circ \gamma)) = 0 .$$

This follows by observing that the image of $\delta \circ \gamma$ consists of sums of elements of the form

$$(\sigma''(\psi_1 \circ \psi_2) - \sigma''(\psi_1) \circ \sigma''(\psi_2))(\xi) = \omega\,(\psi_1, \psi_2)(\xi)$$

for

$$\varphi_o \xrightarrow{\psi_1} \varphi_2 \xrightarrow{\psi_2} \varphi_2$$

in $\underline{M}_o$, and by recalling that $\omega \in K^2$, such that

$$\beta(\,\omega(\psi_1, \psi_2)(\xi) = 0 .$$

Using this we may easily see that α is injective.
But α is injective if and only if

$$\operatorname{Tor}^R(A', S) = 0 .$$

We have to show that A' is an R-algebra. Consider a system of homomorphisms

$$\begin{tikzcd} F_1 \arrow[r, "d"] & F_o \underset{A}{\times} F_o \arrow[r, shift left, "p_1'"] \arrow[r, shift right, "p_2'"'] & F_o \arrow[l, bend left=60, "\Delta'"] \arrow[r, "\rho"] & A \end{tikzcd}$$

in which ρ and d are surjective, p_1' and p_2' are the projections and Δ' is the diagonal. Let $\Delta\colon F_o \to F_1$ be a homomorphism such that $\Delta \circ d = \Delta'$, and put $p_i = d \circ p_i'$.

Then A is the inductive limit of the system

$$\begin{tikzcd} F_1 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & F_o \arrow[l, bend left=60, "\Delta"] \end{tikzcd}$$

Now use the quasisection σ''' on these morphisms, and get a diagram

$$\begin{tikzcd} F'_1 \arrow[r, shift left, "\sigma'''(p_1)"] \arrow[r, shift right, "\sigma'''(p_2)"'] & F'_o \arrow[l, bend left=60, "\sigma'''(\Delta)"] \arrow[r, "\rho'"] & A' \end{tikzcd} .$$

Since we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] & & \\ F_1 \otimes I \arrow[d] \arrow[r, shift left] \arrow[r, shift right] & F_o \otimes I \arrow[d] \arrow[r, "\rho \otimes 1_I"] & A \otimes I \arrow[d] \arrow[r] \arrow[dr, "\alpha"] & 0 \\ F'_1 \arrow[d] \arrow[r, shift left, "\sigma''(p_1)"] \arrow[r, shift right, "\sigma''(p_2)"'] & F'_o \arrow[d] \arrow[r] & \operatorname{coker}(\sigma''(p_1), \sigma''(p_2)) \arrow[d] \arrow[r] & A' \arrow[dl] \\ F_1 \arrow[d] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & F_o \arrow[d] \arrow[r] & A \arrow[d] \arrow[r] & 0 \\ 0 & 0 & 0 & \end{tikzcd}$$

in which α is injective and all sequences are exact we deduce that

$$A' \simeq \operatorname{coker}\,(\sigma''(p_1), \sigma''(p_2)). = F'_o/\operatorname{im}(\sigma''(p_1) - \sigma''(p_2)) .$$

If we can show that $\operatorname{im}(\sigma''(p_1) - \sigma''(p_2))$ is an ideal of F'_o we are through.

Suppose $\sigma''(1_{F_o}) = 1_{F'_o} + \xi_{F_o}$, then since

$$\sigma''(1_{F_o}) \circ \sigma''(1_{F_o}) = \sigma''(1_{F_o}) - \omega(1_{F_o}, 1_{F_o})$$

we find that $\xi_{F_o} = -\omega(1_{F_o}, 1_{F_o})$

and that

$$\rho'(\xi_{F_o}(x)) = 0$$

for all $x \in F_o'$,so that we have

$$\xi_{F_o}(x) \in \mathrm{im}\,(\sigma''(p_1) - \sigma''(p_2)) .$$

Obviously $\sigma''(1_{F_o})$ is an automorphism, such that given any $x \in F_o'$ we may find a $y \in F_o'$ such that

$$x = \sigma''(1_{F_o})(y) .$$

Let $z = (\sigma''(p_1) - \sigma''(p_2))(u)$ and look at

$$x \cdot z = \sigma''(1_{F_o})(y) \cdot (\sigma''(p_1)(u) - \sigma''(p_2)(u) .$$

We recall that

$$\begin{aligned} \sigma''(1_{F_o})(y) &= \sigma''(p_1)(\sigma''(\Delta)(y)) + \omega(\Delta,p_1)(y) \\ &= \sigma''(p_2)(\sigma''(\Delta)(y)) + \omega(\Delta,p_2)(y) . \end{aligned}$$

Therefore we get:

$$\begin{aligned} x \cdot z = {} & \sigma''(p_1)(u \cdot \sigma''(\Delta)(y)) - \sigma''(p_2)(u \cdot \sigma''(\Delta)(y)) \\ & + \omega(\Delta,p_1)(y) \cdot \sigma''(p_1)(u) \\ & - \omega(\Delta,p_2)(y) \cdot \sigma''(p_2)(u) . \end{aligned}$$

Now for $i = 1,2$, the element

$$\omega(\Delta,p_i)(y) \cdot \sigma''(p_i)(u) \in F_o \underset{S}{\otimes} I \simeq F_o' \underset{R}{\otimes} I$$

is equal to

$$\omega(\Delta,p_1)(y) \cdot p_i(\bar{u})$$

where $\bar{u}$ is the image of u in F_1 .

Since $\rho \otimes 1_I$ is a homomorphism of F_o-modules we find:

$$\begin{aligned} & (\rho \otimes 1_I)(\omega(\Delta,p_i)(y) \cdot p_i(\bar{u})) \\ & = \rho(p_i(\bar{u})) \cdot (\rho \otimes 1_I)(\omega(\Delta,p_i)(y)) = 0 \qquad i = 1,2 \end{aligned}$$

In particular we have proved

$$x \cdot z \in \ker \rho' ,$$

thus $\ker \rho'$ is an ideal of F'_o and therefore A' is an algebra, which proves (ii).

Suppose now that $\mathcal{O}(A) = 0$ and consider the quasisection σ'' defined in the beginning of the proof (p. 24).
Let c be any element of $H^1(S,A;A \underset{S}{\otimes} I)$ and let $\xi \in C^1$ represent c .
Then $\sigma'' + \xi$ is another quasisection with the property that

$$\mathcal{O}(\sigma'' + \xi) = \mathcal{O}(\sigma'') .$$

Therefore

$$\varinjlim_{\underline{M}_o} (\sigma'' + \xi) = A''$$

is a lifting of A .

Suppose on the other hand that we have two liftings A', A'' .
We may, as we claimed above, construct quasisections σ', σ'' of

$$\pi_{A'}: \underline{M}_o(A') \to \underline{M}_o$$

and

$$\pi_{A''}: \underline{M}_o(A'') \to \underline{M}_o , \qquad \text{respectively.}$$

Let

$$\xi(\psi) = \sigma'(\psi) - \sigma''(\psi) .$$

Then $\xi \in D^1$ and we know that $j(d\xi) = 0$. Therefore $j(\xi)$ defines an element

$$\lambda(A',A'') \in H^1(S,A;A \underset{S}{\otimes} I) .$$

If there exists an isomorphism

$$\mu: A' \to A''$$

reducing to the identity on A, one checks that $j(\xi)$ is a cocycle such that $\lambda(A',A'') = 0$.

Conversely if $\lambda(A',A'') = 0$ one easily shows that the 0-cochain $\zeta \in D^0$ for which

$$j(\xi - d\zeta) = 0$$

defines an isomorphism

$$\mu \colon A' \to A''$$

reducing to the identity on A.

From this we deduce (iii), thereby proving the theorem.

<u>Example 2.2.4</u>. Let $R = \mathbb{Z}/(p^2)[X]/(X^2-p)$, $S = \mathbb{Z}/(p)[X]/(X^2-p)$ and let $\pi\colon R \to S$ be the obvious homomorphism. Let $A = \mathbb{Z}/(p)$ and consider A as an S-algebra by the homomorphism $S \to A$ mapping X to 0. Suppose there exists a lifting A' of A to R, and consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & I \arrow[d] & R \otimes_V I \arrow[d] & A' \otimes_R (R \otimes_V I) \simeq A' \otimes_V I \arrow[d] \\
V \arrow[r, phantom, "="] \arrow[d] & \mathbb{Z}/(p^2) \arrow[r] \arrow[d] & \mathbb{Z}/(p^2)[X]/(X^2-p) \arrow[r] \arrow[d] & A' \arrow[d] \\
\mathbb{F}_p \arrow[r, phantom, "="] & \mathbb{Z}/(p) \arrow[r] \arrow[d] & \mathbb{Z}/(p)[X]/(X^2) \arrow[r] \arrow[d] & \mathbb{Z}/(p) \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

Since $A' \otimes_V \mathbb{F}_p \simeq A' \otimes_R (R \otimes_V \mathbb{F}_p) \simeq A' \otimes_R S \simeq A$

we find by inspecting the diagram that A' is a lifting of the $\mathbb{F}_p$-algebra A to V. But then $A' \simeq V$ and this is impossible since p is not a square in V.

This example shows that $o_\pi(A)$ is nonzero in

$$H^2(S,A;A)$$

We shall end this section by proving a result which will be used in section (3.2).

<u>Theorem (2.2.5)</u> Let $\psi: A \to B$ be a morphsm of S-algebras, then

$$\psi_*(\,o(A,\pi)) = \psi^*(\,o(B,\pi))$$

<u>Proof</u>. Let σ' be a quasisection of π: <u>R-free</u> $\to$ <u>S-free</u> .
Let $\psi_1: \varphi_0 \to \varphi_1$, $\psi_2: \varphi_1 \to \varphi_2$ be two morphisms of $\underline{M}o(A)$, then $\psi_*(\,o(A,\pi))$ is represented by the 2-cocycle O_1 given by:

$$O_1(\psi_1,\psi_2) = ([\sigma'(\psi_1 \circ \psi_2) - \sigma'(\psi_1) \circ \sigma'(\psi_2)]\varphi_2)\psi$$

and $\psi^*(\,o(B,\pi))$ is represented by the 2-cocycle O_2

$$O_2(\psi_1,\psi_2) = [\sigma'(\psi_1 \circ \psi_2) - \sigma'(\psi_1) \circ \sigma'(\psi_2)](\varphi_2 \circ \psi)$$

Obviously $O_1 = O_2$ which proves the theorem.

QED.

Chapter 3. Lifting morphisms.

(3.1) Obstructions for lifting morphisms of algebras

Let $\pi: R \to S$ be a surjective homomorphism of rings with $(\ker\pi)^2 = o$. Let $\psi: A \to B$ be a morphism of S-algebras, and suppose that A and B can be lifted to R. If A' is a lifting of A to R, and B' is a lifting of B to R and $\psi': A' \to B'$ is a morphism of R-algebras, then we shall call ψ' a lifting of ψ to R with respect to A', B', provided $\psi' \underset{R}{\otimes} 1_S \simeq \psi$.

Theorem (3.1.1) Given liftings A' and B' of A and B respectively there exists an obstruction

$$o(\psi) = o_\pi(\psi, A', B') \in H^1(S, A; B \otimes \ker\pi)$$

such that $o(\psi) = o$ if and only if there exists a lifting ψ' of ψ to R with respect to A', B'. The set of such liftings is a principal homogenous space over $\mathrm{Der}_S(A, B \otimes \ker\pi)$.

Proof. Let σ be a quasisection of

$$\pi_{A,A'} : \underline{M}_o(A') \to \underline{M}_o(A)$$

(see p. 24). Since $B' \to B$ is surjective there exists for any $\varphi \in \mathrm{ob}\, \underline{M}_o(A)$ with $\sigma(\varphi): F' \to A'$ a morphism of R-algebras

$$\nu(\varphi): F' \to B'$$

such that

$$\nu(\varphi) \underset{R}{\otimes} 1_S = \varphi \circ \psi .$$

Let

$$\begin{tikzcd} F_1 \arrow[rr, "\rho"] \arrow[dr, "\varphi_1"'] & & F_2 \arrow[dl, "\varphi_2"] \\ & A & \end{tikzcd}$$

be a morphism $\rho: \varphi_1 \to \varphi_2$ of $\underline{M}_o(A)$ and consider the diagram

$$\begin{tikzcd} F_1' \arrow[dr, "\sigma(\rho)"] \arrow[drr, "\nu(\varphi_1)"] \arrow[dd, "\sigma(\varphi_1)"'] & & \\ & F_2' \arrow[r, "\nu(\varphi_2)"'] \arrow[dl, "\sigma(\varphi_2)"] & B' \arrow[dd] \\ A' \arrow[d] & & \\ A \arrow[rr, "\psi"'] & & B \end{tikzcd}$$

Since $\varphi_1 \circ \psi = \rho \circ \varphi_2 \circ \psi$ we know that

$$C(\rho) = \sigma(\rho) \circ \nu(\varphi_2) - \nu(\varphi_1) \in \mathrm{Der}_S(F_1, B \otimes \ker \pi) .$$

When σ and ν have been fixed, this formula defines an element

$$C \in C^1(S,A;B \otimes \ker \pi) .$$

Moreover C is a 1-cocycle as for any pair of morphisms $\rho_1 : \varphi_1 \to \varphi_2$, $\rho_2 : \varphi_2 \to \varphi_3$ in $\underline{M}_o(A)$ we have:

$$dC(\rho_1,\rho_2) = \rho_1 \circ C(\rho_2) - C(\rho_1 \circ \rho_2) + C(\rho_1)$$

$$= \rho_1[\sigma(\rho_2) \circ \nu(\varphi_3) - \nu(\varphi_2)] - [\sigma(\rho_1 \circ \rho_2) \circ \nu(\varphi_3) - \nu(\varphi_1)]$$

$$+ [\sigma(\rho_1) \circ \nu(\varphi_2) - \nu(\varphi_1)]$$

$$= \sigma(\rho_1) \circ \sigma(\rho_2) \circ \nu(\varphi_3) - \sigma(\rho_1) \circ \nu(\varphi_2) - \sigma(\rho_1 \circ \rho_2) \circ \nu(\varphi_3)$$

$$+ \nu(\varphi_1) + \sigma(\rho_1) \circ \nu(\varphi_2) - \nu(\varphi_1) = 0$$

knowing, as we do, that

$$[\sigma(\rho_1) \circ \sigma(\rho_2) - \sigma(\rho_1 \circ \rho_2)]\nu(\varphi_3) = [\sigma(\rho_1) \circ \sigma(\rho_1 \circ \rho_2) - \sigma(\rho_1 \circ \rho_2)](\varphi_3 \circ \psi)$$

$$= ([\sigma(\rho_1) \circ \sigma(\rho_2) - \sigma(\rho_1 \circ \rho_2)] \circ \sigma(\varphi_3))\psi = [\sigma(\rho_1) \circ \sigma(\rho_2) \circ \sigma(\varphi_3) -$$

$$\sigma(\rho_1 \circ \rho_2) \circ \sigma(\varphi_3)]\psi = 0 .$$

Let $o(\psi) = o_\pi(\psi,A',B')$ be the corresponding cohomology class, then an easy check shows that $o(\psi)$ is independent of the choice of the quasisection σ and of the choice of the map ν .

Suppose $o(\psi) = o$, then there exists a $\xi \in C^o(S,A;B\otimes \ker\pi)$ such that $C = d\,\xi$. Put

$$\nu_1(\varphi) = \nu(\varphi) - \xi(\varphi)$$

then for any morphism $\rho\colon \varphi_1 \to \varphi_2$ in $\underline{M}_o(A)$ we have

$$\sigma(\rho)\circ\nu_1(\varphi_2) - \nu_1(\varphi_1) = \sigma(\rho)\nu(\varphi_2) - \sigma(\rho)\xi(\varphi_2)$$
$$- \nu(\varphi_1) + \xi(\varphi_1) = C(\rho) - (\rho\,\xi(\varphi_2) - \xi(\varphi_1)) = (C - d\,\xi)(\rho) = o \ .$$

This implies that for any morphism $\rho\colon \varphi_1 \to \varphi_2$ in $\underline{M}_o(A)$ the diagram

$$\begin{tikzcd}
F'_1 \arrow[rr, "\nu_1(\varphi_1)"] \arrow[dr, "\sigma(\rho)"] \arrow[dd, "\sigma(\varphi_1)"'] & & B' \\
& F'_2 \arrow[ur, "\nu_1(\varphi_2)"'] \arrow[dl, "\sigma(\varphi_2)"] & \\
A' & &
\end{tikzcd}$$

is commutative.

Consequently ν_1 defines an R-algebra morphism

$$\psi'\colon A' = \varinjlim_{\underline{M}_o(A)} \sigma \to B' \ .$$

Clearly ψ' is a lifting of ψ to R with respect to A' and B'.

The rest of the conclusion of the theorem is obvious.

QED.

Let $\pi\colon R \to S$, $\psi\colon A \to B$ be as above, and put $I = \ker\pi$.

<u>Remark (3.1.2)</u> If $S[X] \underset{j}{\to} A$ is a surjective homomorphism of S-algebras then we know that

$$H^1(S,A;B\otimes I) \simeq {}^{\mathrm{Hom}_{S[X]}(\ker j,\, B\otimes I)}\!/\!_{\mathrm{Derivations}}$$

Let $R[X] \underset{j'}{\to} A'$ be a lifting of j to R and observe that

$$\ker j' \underset{R}{\otimes} S \simeq \ker j$$

since $\mathrm{Tor}_1^R(A',S) = 0$

Let $\nu'\colon R[X] \to B'$ be a lifting of $j\circ\psi$, then ν' defines an $R[X]$-module homomorphism

$$\ker j' \to B' \underset{R}{\otimes} I$$

vanishing on $\ker j' \underset{R}{\otimes} I$.

Therefore ν' induces a homomorphism

$$\nu\colon \ker j \to B' \underset{R}{\otimes} I \simeq B \underset{S}{\otimes} I .$$

One may check that ν represents the class

$$o_\pi(\psi;A',B') .$$

Let A' and B' be liftings of A and B respectively and consider the map

$$\Psi_*\colon H^1(S,A;A\otimes I) \to H^1(S,A;A\otimes I)$$

defined by

$$\Psi_*(\lambda) = o_\pi(\psi;A',B') - o_\pi(\psi;A'',B')$$

where λ corresponds to the difference $A' - A''$.

<u>Theorem (3.1.3)</u> Ψ_* is induced by $\psi\otimes 1_I\colon A\otimes I \to B\otimes I$.

<u>Proof</u>. Let $\Lambda\psi$ be a 1-cocycle representing λ and consider a quasisection σ' of

$$\pi'\colon \underline{M}_o(A') \to \underline{M}_o(A)$$

then

$$\sigma''(\rho) = \sigma'(\rho) - \Lambda(\rho)$$

is a quasisection of

$$\pi\colon \underline{M}_o(A'') \to \underline{M}_o(A)$$

It follows that

$$C(\sigma') - C(\sigma'') = \Lambda \circ \psi .$$

(see proof of (3.1.1)) QED.

Corollary (3.1.4) Suppose A and B can be lifted to R and suppose $o_\pi(\psi\colon A',B') \in \operatorname{im}\psi_*$ for some A' and B' lifting A and B respectively. Then there exists an A'' lifting A and a $\psi''\colon A'' \to B'$ lifting ψ .

Corollary (3.1.5) Let $\zeta\colon A \to B$ be an isomorphism and suppose A and B can be lifted to R . Then there exists for every lifting B' of B a unique lifting A' of A and a morphism

$$\zeta'\colon A' \to B'$$

lifting ζ .

Consider the map

$$\Psi^*\colon H^1(S,B;B\otimes I) \to H^1(S,A;B\otimes I)$$

defined by

$$\Psi^*(\mu) = o_\pi(\psi\colon A',B') - o_\pi(\psi;A',B'')$$

where μ corresponds to the difference $B'' - B'$.

Theorem (3.1.6) Ψ^* is induced by $\psi\colon A \to B$.

Proof. We know that $\varprojlim\limits_{\underline{M}(A)^o}{}^{(i)} \simeq \varprojlim\limits_{\underline{M}_o(A)^o}{}^{(i)}$. Let $E_A^{\cdot}$ be the resolving complex for $\varprojlim\limits_{\underline{M}(A)^o}$ and $C_A^{\cdot}$ the resolving comples for $\varprojlim\limits_{\underline{M}_o(A)^o}$ (see p. 20).

The canonical homomorphism

$$E_A^{\cdot}(\mathrm{Der}_S(-,B\otimes I)) \to C_A^{\cdot}(\mathrm{Der}_S(-,B\otimes I))$$

therefore induces isomorphisms in cohomology.
Let σ be a quasisection of π: R-free $\to$ S-free, let ν_1 be a section of the map $\pi\colon \mathrm{ob}\,\underline{M}(A') \to \mathrm{ob}\,\underline{M}(A)$ and let ν_2 be a section of the map $\pi\colon \mathrm{ob}\,\underline{M}(B') \to \mathrm{ob}\,\underline{M}(B)$. Let $\rho\colon \varphi_o \to \varphi_1$ be a morphism of $\underline{M}(A)$ and put

$$E(\rho) = (\sigma(\rho)\nu_2(\varphi_1\psi) - \nu_2(\varphi_o\psi)) - (\sigma(\rho)\nu_1(\varphi_1) - \nu_1(\varphi_o))\psi$$

Then $E \in E_A^1(\mathrm{Der}_S(-,B\otimes I))$ and one checks that E is a cocycle in $E_A^{\cdot}$. Moreover it is easily seen that the corresponding cohomology class is independent of the choice of σ, ν_1 and ν_2.
Let C be the image of E in $C_A^1(\mathrm{Der}_S(-,B\otimes I))$, and let σ' be a quasisection of $\pi\colon \underline{M}_o(A') \to \underline{M}_o(A)$. Since the cohomology class c of C is independent of the choice of σ and ν_1 we find that c is represented by the 1-cocycle C' defined by:

$$\begin{aligned} C'(\rho) &= (\sigma'(\rho)\nu_2(\varphi_1\psi) - \nu_2(\varphi_o\psi)) - (\sigma'(\rho)\sigma'(\varphi_1) - \sigma'(\varphi_o))\psi \\ &= \sigma'(\rho)\,\nu_2(\varphi_1\psi) - \nu_2(\varphi_o\psi) \end{aligned}$$

This shows that $c = o_\pi(\psi,A',B')$. Thus E represents $o_\pi(\psi,A',B') \in \varprojlim\limits_{\underline{M}(A)^o}{}^{(1)}\mathrm{Der}_S(-,B\otimes I) = H^1(S,A;B\otimes I)$.

Let ν_3 be a section of $\pi\colon \mathrm{ob}\,\underline{M}(B'') \to \mathrm{ob}\,\underline{M}(B)$, and let for any morphism $\rho\colon \varphi_o \to \varphi_1$ of $\underline{M}(A)$

$$F(\rho) = (\sigma(\rho)\nu_3(\varphi_1\psi) - \nu_3(\varphi_o\psi)) - (\sigma(\rho)\nu_1(\varphi_1) - \nu_1(\varphi_o))\psi$$

then F is a 1-cocycle of $E^{\bullet}_A(\mathrm{Der}_S(-,B\otimes I))$ and we know that F represents $o_\pi(\psi,A',B'') \in H^1(S,A;B\otimes I)$.

Let $\alpha\colon \tau_o \to \tau_1$ be a morphism of $\underline{M}(B)$ and put

$$G(\alpha) = (\sigma(\alpha)\nu_2(\tau_1) - \nu_2(\tau_o)) - (\sigma(\alpha)\nu_3(\tau_1) - \nu_3(\tau_o)) \ ,$$

then $G \in E^1_B(\mathrm{Der}_S(-,B\otimes I))$ is a cocycle and a moments refexion will convince the reader that G represents the cohomology class μ corresponding to the difference $B'' - B$ (i.e. $\lambda(B'',B')$ see p. 30). In fact, consider the image H of G in $C^1_B(\mathrm{Der}_S(-,B\otimes I))$, let σ be a quasisection of $\pi\colon \underline{M}_o(B') \to \underline{M}_o(B)$ and let σ'' be a quasisection of $\pi\colon \underline{M}_o(B'') \to \underline{M}_o(B)$. Then H represents the same cohomology class as H' defined by:

$$\begin{aligned} H'(\alpha) &= (\sigma''(\alpha)\sigma'(\tau_1) - \sigma'(\tau_o)) - (\sigma''(\alpha)\sigma''(\tau_1) - \sigma''(\tau_o)) \\ &= \sigma''(\alpha)\sigma'(\tau_1) - \sigma'(\tau_o) \\ &= (\sigma''(\alpha) - \sigma'(\alpha))\sigma'(\tau_1) = (\sigma''(\alpha) - \sigma'(\alpha))\tau_1 \end{aligned}$$

By definition the cohomology class of H' is $\lambda(B'',B')$ (see p.30). Now let ψ also denote the functor

$$\underline{M}(A) \to \underline{M}(B)$$

defined by $\psi(\varphi) = \varphi \circ \psi$. Then

$$E - F = \psi\, G$$

which implies

$$o_\pi(\psi,A',B') - o_\pi(\psi,A',B'') = \psi^*(B'' - B')$$

QED.

Corollary (3.1.7) A and B can be lifted to R and suppose $o_\pi(\psi;A',B') \in \operatorname{im}\psi^*$ for some A',B' lifting A and B respectively. Then there exists an B'' lifting B and a $\psi'': A' \to B''$ lifting ψ.

Corollary (3.1.8) Let $\psi: A \to B$ be an isomorphism and suppose A and B can be lifted to R. Then there exists for every lifting A' of A a unique lifting B' of B and a morphism $\psi': A' \to B'$ lifting ψ.

Corollary (3.1.9) Let $\mu \in H'(S,A;A\otimes I)$ correspond to $A' - A''$ where A' and A'' are two liftings of A to R. Then $\mu = o_\pi(1_A,A',A'')$.

Proof. By (3.1.6) $o_\pi(1_A,A',A') - o_\pi(1_A,A',A'') = -1_{A*}(\mu) = -\mu$. Since $o_\pi(1_A,A',A') = o$ the Corollary follows immediately.

QED.

Theorem (3.1.10) Let $\psi_1: A \to B$ and $\psi_2: B \to C$ be two S-algebra homomorphisms and let A',B' and C' be liftings of A,B and C respectively, then

$$o_\pi(\psi_1\circ\psi_2,A',C') = \psi_1\, o_\pi(\psi_2,B',C') + o_\pi(\psi_1,A',B')\psi_2$$

Proof. Let σ be a quasisection of π: R-free $\to$ S-free. Let ν_1 be a section of $\pi: \operatorname{ob}\underline{M}(A') \to \operatorname{ob}\underline{M}(A)$, ν_2 a section of $\pi: \operatorname{ob}\underline{M}(B') \to \operatorname{ob}\underline{M}(B)$ and ν_3 a section of $\pi: \operatorname{ob}\underline{M}(C') \to \operatorname{ob}\underline{M}(\underline{C})$. Let $\rho: \varphi_o \to \varphi_1$ be a morphism of $\underline{M}(A)$ then the 1-cocycle C_1 given by

$$C_1(\rho) = (\sigma(\rho)\nu_2(\varphi_1\psi_1) - \nu_2(\varphi_o\psi_1)) - (\sigma(\rho)\nu_1(\varphi_1) - \nu_1(\varphi_o))\psi_1$$

represents $o_\pi(\psi_1, A', B')$, and the 1-cocycle C_3 given by:

$$C_3(\rho) = (\sigma(\rho)\nu_3(\varphi_1\psi_1\psi_2) - \nu_3(\varphi_o\psi_1\psi_2)) - (\sigma(\rho)\nu_1(\varphi_1) - \nu_1(\varphi_o))\psi_1\psi_2$$

represents $o_\pi(\psi_1\psi_2, A', C')$.

Let $\alpha: \tau_o \to \tau_1$ be a morphism of $\underline{M}(B)$, then the 1-cocycle C_2 given by

$$C_2(\alpha) = (\sigma(\alpha)\nu_3(\tau_1\psi_2) - \nu_3(\tau_o\psi_2)) - (\sigma(\alpha)\nu_2(\tau_1) - \nu_2(\tau_o))\psi_2$$

represents $o_\pi(\psi_2, B', C')$.

Considering the canonical functors

$$\underline{M}(A) \xrightarrow{\psi_1} \underline{M}(B) \xrightarrow{\psi_2} \underline{M}(C)$$

defined by the morphisms ψ_1 and ψ_2 we find

$$C_3 = \psi_1 C_2 + C_1 \psi_2$$

proving the theorem.

QED.

Remark (3.1.11) Consider the morphisms

$$T \xrightarrow[\rho]{} R \xrightarrow[\pi]{} S .$$

Assume ρ and π both surjective and put

$$I_o = \ker(\rho \circ \pi) , \quad I_1 = \ker \pi , \quad I_2 = \ker \rho .$$

Suppose $I_o^2 = o$ then $I_1^2 = I_2^2 = o$, and we have an exact sequence of S-modules

1) $$o \to I_2 \xrightarrow{i} I_o \xrightarrow{j} I_1 \to o .$$

Let A' be an R-algebra lifting the S-algebra A , and consider the exact sequence

2) $$\to H^1(R,A';A' \otimes I_1) \xrightarrow{\delta^1} H^2(R,A';A' \otimes I_2) \xrightarrow{i_*} H^2(R,A';A' \otimes I_o) \to$$

induced by 1).

We know (see [A]) that there are canonical isomorphisms

$$H^i(R,A';A' \underset{R}{\otimes} I_k) \simeq H^i(S,A;A \underset{S}{\otimes} I_k) \qquad k = 0,1,2, \quad i \geq 0$$

and we may verify that by these isomorphisms

(i) $$\delta^1(\lambda) = o_\rho(A') - o_\rho(A'')$$

if $\lambda \in H^1(S,A;A \underset{S}{\otimes} I_1)$ corresponds to the difference between two liftings A' and A'' of A to R .

(ii) $$i_*(o_\rho(A')) = o_{\rho \circ \pi}(A) .$$

Suppose given a lifting $\psi' : A' \to B'$ of the morphism of S-algebras

$\psi\colon A \to B$ to R with respect to the liftings A' and B' of A respectively B.

Consider the exact sequence

$$\to \mathrm{Der}_R(A',B' \underset{R}{\otimes} I_1) \xrightarrow{\delta^0} H^1(R,A';B' \underset{R}{\otimes} I_2) \xrightarrow{i_*} H^1(R,A';B' \otimes I_0) \to$$

induced by 1).

As above we have canonical isomorphisms

$$H^i(R,A';B' \underset{R}{\otimes} I_k) \simeq H^i(S,A;B \underset{S}{\otimes} I_k) \qquad k = 0,1,2, \quad i \geq 0$$

and we may verify that by these isomorphisms

(iii) $\qquad \delta^0(\xi) = o_\rho(\psi';A'',B'') - o_\rho(\psi'';A'',B'')$

if $\xi \in \mathrm{Der}_S(A,B \underset{S}{\otimes} I_1)$ corresponds to the difference between two liftings ψ' and ψ'' of ψ to R with respect to A', B', and if A'', B'' are liftings of A', B' respectively, to T.

(iv) $\qquad i_*(o_\rho(\psi';A'',B'')) = o_{\rho\circ\pi}(\psi;A'',B'')$.

(3.2) Lifting diagrams of morphisms of algebras

Let $\underline{c}$ be a small subcategory of $\underline{S\text{-alg}}$, and let $\pi\colon R \to S$ be as before.

Let $\psi\colon A \to B$ be a morphism of $\underline{c}$, consider B as an A module and put

$$H^i(\psi) = H^i(S,A;B \underset{S}{\otimes} \ker \pi) \qquad\qquad i \geq 0 .$$

Let $(\lambda,\mu)\colon \varphi \to \psi$ be a morphism of $\underline{\mathrm{Mor}}\ \underline{c}$ so that the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\lambda} & A_2 \\ \varphi\downarrow & & \downarrow\psi \\ B_1 & \xleftarrow[\mu]{} & B_2 \end{array}$$

commutes, and define the homomorphism

$$H^i(\lambda,\mu)\colon H^i(\psi) \to H^i(\varphi)$$

by

$$H^i(\lambda,\mu)(\alpha) = \mu_*^1\ \lambda^*(\alpha) = \lambda_1^*\ \mu_*(\alpha)$$

where

$$\lambda^*\colon H^i(\psi) \to H^i(\lambda\circ\psi)\ ,\ \lambda_1^*\colon H^i(\psi\circ\mu) \to H^i(\varphi)$$

$$\mu_*^1\colon H^i(\lambda\circ\psi) \to H^i(\varphi)\ ,\ \mu_*\colon H^i(\psi) \to H^i(\psi\circ\mu)$$

are induced by λ, λ, μ and μ respectively.

Then $\psi \longmapsto H^i(\psi)$ and $(\lambda,\mu) \longmapsto H^i(\lambda,\mu)$ define a functor

$$H^i\colon \underline{\mathrm{Mor}}\ \underline{c} \to \underline{\mathrm{Ab}}\ .$$

As in (1.1) we shall use the notations

$$\lambda\,\alpha = H^i(\lambda,1)(\alpha)\ ,\ \alpha\,\mu = H^i(1,\mu)(\alpha)\ .$$

Now, consider the complex $D^\bullet(H^i)$ as defined in the proof of (1.2.1). Recall that $D^\bullet$ looks like

$$\begin{array}{rcl} D^0(H^i) & = & \prod\limits_{A\in \mathrm{ob}\,\underline{c}} H^i(1_A) = \prod\limits_{A\in \mathrm{ob}\,\underline{c}} H^i(S,A;A\otimes \ker\pi) \\ d^0 \downarrow & & \\ D^1(H^i) & = & \prod\limits_{\psi\in \mathrm{Mor}\,\underline{c}} H^i(\psi) \\ d^1 \downarrow & & \\ D^2(H^i) & = & \prod\limits_{\psi_1,\psi_2\in \mathrm{Mor}\,\underline{c}} H^i(\psi_1\circ\psi_2) \end{array}$$

with $d^0(\xi)(\psi) = \psi\,\xi_B - \xi_A\,\psi$ for $\psi\colon A \to B$,

$$d^1(C)(\psi_1,\psi_2) = \psi_1 C(\psi_2) - C(\psi_1\circ\psi_2) + C(\psi_1)\psi_2$$

Consider the o-cochain o_o of $D^\bullet(H^2)$ defined by:

$$o_o(A) = o(A,\pi) \in H^2(S,A;A\otimes \ker\pi)$$

By (2.2.5) o_o is a o-cocycle. Let $\underline{o}_o$ be the corresponding cohomology class, then we have proved,

Theorem (3.2.1) There exists an obstruction

$$\underline{o}_o \in \varprojlim_{\underline{\mathrm{Mor}}\,\underline{c}} H^2$$

such that $\underline{o}_o = o$ if and only if every object of $\underline{c}$ can be lifted to R .

Consider the 1-cochain o_1 of $D^{\cdot}(H^1)$ defined by

$$o_1(\psi) = o_\pi(\psi, A'.B') \in H^1(\psi)$$

supposing of course that $\psi: A \to B$ and that A and B admit liftings A', B' respectively.

By (3.1.10) o_1 is a 1-cocycle. Let $\underline{o}_1$ be the corresponding cohomology class.

Let $\underline{C}$ be the subcategory of $\underline{R\text{-alg}}$ defined by:

$$\mathrm{ob}\,\underline{C} = \{A' \in \mathrm{ob}\,\underline{R\text{-alg}} \mid \pi(A') \in \mathrm{ob}\,\underline{c}\ ,\ \mathrm{Tor}_1^R(A'.S) = o\}$$

$$\mathrm{mor}\,\underline{C} = \{\psi' \in \mathrm{mor}\,\underline{R\text{-alg}} \mid \pi(\psi') \in \mathrm{mor}\,\underline{c}\ \}$$

and let

$$\pi: \underline{C} \to \underline{c}$$

denote the functor tensorization with S over R .

Then we have,

Theorem (3.2.2) Suppose $\underline{o}_o = o$, then there exists an obstruction

$$\underline{o}_1 \in \varprojlim_{\underline{\mathrm{Mor}}\,\underline{c}}{}^{(1)} H^1$$

such that $\underline{o}_1 = o$ if and only if there exists a quasisection of $\pi: \underline{C} \to \underline{c}$.

The set of stems of such quasisections is a principal homogenous space over $\varprojlim_{\underline{\mathrm{Mor}\ c}}^{(o)} H^1$.

Proof. An easy check shows that $\underline{o}_1$ is independent of the choice of liftings A', B' used to define o_1 .
Suppose $\underline{o}_1 = o$, then there exists a o-cochain $\xi \in D^o(H^1)$ such that for $\psi: A \to B$ in mor $\underline{c}$

$$o_\pi(\psi, A', B') = d\,\xi(\psi) = \psi\,\xi_B - \xi_A\,\psi$$

By (3.1.9) $\xi_A = -o_\pi(1_A, A', A'')$ for some lifting A" of A and $\xi_B = -o_\pi(1_B, B', B'')$ for some lifting B" of B .
Apply (3.1.10) to the compositions $1_A \circ \psi = \psi \circ 1_B$ and the liftings A',A",B" and A',B',B" respectively, then we find

$$o_\pi(\psi, A'.B'') = 1_A\, o_\pi(\psi, A'', B'') + o_\pi(1_A, A', A'')\psi$$

$$= \psi\, o_\pi(1_B, B', B'') + o_\pi(\psi, A', B')1_B$$

or equivalentely

$$o_\pi(\psi, A'', B'') = o_\pi(\psi, A', B') + \psi\, o_\pi(1_B, B', B'')$$

$$- o_\pi(1_A, A', A'')\psi = o_\pi(\psi, A', B') - \psi\,\xi_B + \xi_A\,\psi = o .$$

The rest will be left to the reader as an exercise.

QED.

Combining (1.2.1), (3.2.1) and (3.2.2) we have proved.

Theorem (3.2.3) Suppose $\underline{o}_o = o$, $\underline{o}_1 = o$ then there exists a set of obstructions

$$O(\pi) \subseteq \varprojlim_{\underline{\mathrm{Mor}\ c}} H^o$$

such that π has a section if and only if

$$o \in O(\pi)$$

The set of sections with a fixed stem is a principal homogenous space over $\varprojlim\limits_{\underline{\mathrm{Mor}}\ \underline{c}}{}^{(1)} H^{o}$.

Let $\underline{c}$ consist of the two objects A and B and the three morphisms 1_A, $\psi: A \to B$ and 1_B . Then $\underline{\mathrm{Mor}}\ \underline{c}$ consists of 3 objects and 5 morphisms ilustrated in the following diagram

$$\begin{tikzcd}
1_A \arrow[loop above, "(1_A,1_A)"] & & 1_B \arrow[loop above, "(1_B,1_B)"] \\
& \psi \arrow[ul, "(1_A,\psi)"] \arrow[ur, "(\psi,1_B)"'] \arrow[loop below, "(1_A,1_B)"] &
\end{tikzcd}$$

Then (see [L]) we have:

$$\varprojlim_{\underline{\mathrm{Mor}}\ \underline{c}}{}^{(o)} H^{1} = H^{1}(S,A;A \otimes \ker \pi) \times_{H^{1}(S,A;B \otimes \ker \pi)} H^{1}(S,B;B \otimes \ker \pi)$$

where the fibered product is taken with respect to the homomorphisms ψ_* and ψ^* respectively, and

$$\varprojlim_{\underline{\mathrm{Mor}}\ \underline{c}}{}^{(1)} H^{1} = H^{1}(S,A;B \otimes \ker \pi) / \operatorname{im} \psi_* + \operatorname{im} \psi^*$$

This proves the following result,

<u>Corollary (3.2.4)</u> Given a morphism $\psi: A \to B$ of S-algebras. Suppose A and B can be lifted to R , then ψ admits a lifting if and only if

$$o_{\pi}(\psi, A', B') \in \operatorname{im} \psi^* + \operatorname{im} \psi_*$$

for some liftings A', B' of A and B respectively.

((3.2.4) is, of course, a trivial consequence of (3.1.9) and (3.1.10).)

Lemma (3.2.5) Let $\underline{c}$ be any small category and assume that every morphism of $\underline{c}$ is an isomorphism. Then there is a full equivalence of categories

$$\underline{Ab}^{\underline{c}} \simeq \underline{Ab}^{\underline{Mor}\,\underline{c}}$$

inducing an isomorphism of functors

$$\varprojlim_{\underline{c}}{}^{(i)} \simeq \varprojlim_{\underline{Mor}\,\underline{c}}{}^{(i)}$$

Proof. If $\psi: c \to d$ is a morphism of $\underline{c}$, put $S(\psi) = c$, $B(\psi) = d$.

Let F be an object of $\underline{Ab}^{\underline{c}}$ and define the object $\nu(F)$ of $\underline{Ab}^{\underline{Mor}\,\underline{c}}$ by

$$\nu(F)(\psi) = F(S_\psi)$$

$$\nu(F)(\lambda,\mu) = F(\lambda)$$

for $(\lambda,\mu): \varphi \to \psi$ im $\underline{Mor}\,\underline{c}$.

Let G be an object of $\underline{Ab}^{\underline{Mor}\,\underline{c}}$ and define the object $\varkappa(G)$ by

$$\varkappa(G)(c) = G(1c)$$

$$\varkappa(G)(\psi) = G(\psi,\psi^{-1})$$

Obviously $\nu\circ\varkappa = 1$ and if

$$\begin{tikzcd} c_1 \arrow[r, "\lambda"] \arrow[d, "\varphi"'] & c_2 \arrow[d, "\psi"] \\ d_1 & d_2 \arrow[l, "\mu"] \end{tikzcd}$$

is a morphism in <u>Mor c</u> we find that

$$\begin{array}{ccccccc} (\varkappa\circ\nu)(G)(\varphi) & = & \nu(\varkappa(G))(\varphi) & = & G(1_{c_1}) & \overset{G(1,\varphi)}{\underset{\simeq}{\leftarrow}} & G(\varphi) \\ {\scriptstyle(\varkappa\circ\nu)(G)(\lambda,\mu)}\downarrow & & \downarrow & & \downarrow{\scriptstyle G(\lambda,\lambda^{-1})} & & \downarrow{\scriptstyle G(\lambda,\mu)} \\ (\varkappa\circ\nu)(G)(\psi) & = & \nu(\varkappa(G))(\psi) & = & G(1_{c_2}) & \underset{G(1,\psi)}{\overset{\simeq}{\leftarrow}} & G(\psi) \end{array}$$

commutes since

$$(1,\varphi)\circ(\lambda,\lambda^{-1})\circ(1,\psi^{-1}) = (\lambda,\psi^{-1}\circ\lambda^{-1}\circ\varphi) = (\lambda,\mu) \ .$$

But this proves that there exists an isomorphism of functors

$$\varkappa\circ\nu \simeq 1 \ .$$

The rest is clear.

QED.

<u>Corollary (3.2.6)</u> Let G be a group acting on the S-algebra A. Then there exists an obstruction

$$\underline{o}_0 \in H^0(G,H^2(S,A;A\otimes\ker\pi))$$

such that $\underline{o}_0 = 0$ if and only if A can be lifted.

If $\underline{o}_0 = o$ there exists an obstruction

$$\underline{o}_1 \in H^1(G,H^1(S,A;A\otimes\ker\pi))$$

such that $\underline{o}_1 = o$ if and only if for every $g \in G$ the action g can be lifted to a common lifting A' of A.

If $\underline{o}_0 = o$, $\underline{o}_1 = o$ there exists a set of obstructions

$$O(\pi) \subseteq H^2(G,H^0(S,A;A\otimes\ker\pi))$$

such that $o \in O(\pi)$ if and only if the action of G can be lifted to a lifting A' of A .

Proof. This follows from (3.2.1), (3.2.2), (3.2.3) and (3.2.5). In fact, by (3.2.5), if $\underline{c}$ is the category consisting of one object A and the morphisms corresponding to the elements in G, then

$$\varprojlim_{\underline{\mathrm{Mor}\,c}}{}^{(i)} \simeq \varprojlim_{\underline{c}}{}^{(i)} \simeq H^i(G,-)$$

(see [La]).

QED.

Example (3.2.7) If one wants to lift affine group-schemes, or equivalently, bialgebras, the main problem is the following: Let A be an S-bialgebra with coalgebra structure defined by

$$m: A \to A \underset{S}{\otimes} A .$$

Find a lifting A' of the S-algebra A to R, and a lifting m' of m with

$$m': A' \to A' \underset{R}{\otimes} A' \;!$$

I claim that this can be done if and only if we can lift the diagram $\underline{c}$;

$$\begin{tikzcd} A \arrow[r, shift left=4, "1\otimes\epsilon"] \arrow[r, "m"] \arrow[r, shift right=4, "\epsilon\otimes 1"'] & A \underset{S}{\otimes} A \end{tikzcd}$$

where $\epsilon: S \to A$ is the structure morphism.

In fact, suppose we can lift this diagram to the diagram

$$\begin{tikzcd} A' \arrow[r, shift left=4, "(1\otimes\epsilon)'"] \arrow[r, "m''"] \arrow[r, shift right=4, "(\epsilon\otimes 1)'"'] & B' \end{tikzcd}$$

Then the morphism of R-algebras $\alpha: A' \underset{R}{\otimes} A' \to B'$ defined by $(1\otimes\epsilon)'$ and $(\epsilon\otimes 1)'$ is a lifting of $1_{A\otimes A}$. In particular α

is an isomorphism. Put $m' = m'' \circ \alpha^{-1}$, then $m' : A' \to A' \underset{R}{\otimes} A'$ is a lifting of m .

Next we notice that $\underline{\mathrm{Mor}}\ \underline{c}$ is the ordered set with 5 objects and 6 non-trivial relations illustrated by the diagram

$$\begin{tikzcd}
1_A & & 1_{A\otimes A} \\
1\otimes\varepsilon \arrow[u] \arrow[urr] & m \arrow[ul] \arrow[ur] & \varepsilon\otimes 1 \arrow[ull] \arrow[u]
\end{tikzcd}$$

An easy calculation (see [L]) then shows that

$$\varprojlim_{\underline{\mathrm{Mor}\ c}} H^1 = \ker \psi$$

$$\varprojlim_{\underline{\mathrm{Mor}\ c}}{}^{(i)} H^1 = \operatorname{coker} \psi$$

where

$$\psi : H^1(S,A;A\otimes\ker\pi) \times H^1(S,A\otimes A;A\otimes A\otimes\ker\pi)$$
$$\to H^1(1\otimes\varepsilon) \times H^1(m) \times H^1(\varepsilon\otimes 1)$$

is defined by

$$\psi(\alpha,\beta) = ((1\otimes\varepsilon)_*(\alpha) - (1\otimes\varepsilon)^*(\beta), m_*(\alpha) - m^*(\beta),$$
$$(\varepsilon\otimes 1)_*(\alpha) - (\varepsilon\otimes 1)^*(\beta)) .$$

Having this, we obtain the following result,

<u>Corollary (3.2.8)</u> In the situation above m can be lifted to an $m' : A' \to A' \underset{R}{\otimes} A'$ if and only if

$$(o_\pi(1\otimes\varepsilon, A'', A''\underset{R}{\otimes}A''), o_\pi(m, A'', A''\underset{R}{\otimes}A''), o_\pi(\varepsilon\otimes 1, A'', A''\underset{R}{\otimes}A'')) \in \operatorname{im}\psi$$

for some lifting A'' of A .

We shall, hopefully, return to this problem in a later paper.

THE READER SHOULD ALSO CONSULT

Luc. Illusie: Complexe Cotangent et Déformations I.
Lecture Notes in Mathematics, Vol. 239.
Springer-Verlag 1971.

Among other things Illusie's paper, which appeared while this paper was in print, contains some of the material covered in this repport. Exactely how much I do not know yet.

Anyway, our methods seem to be quite different.

Encl. to:

PREPRINT SERIES - Mathematics. No 12. O.A. Laudal: Sections of functors and the problem of lifting algebraic structures.
1971.

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