Cohomology of affine "formal" schemes

by

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Let A be any commutativ ring with unit and let M be an A-module.

Then M defines a sheaf of O_X-Modules $\tilde{M}$ on the affine scheme X = Spec (A) , and by a well known theorem of Serre

$$H^i(X,\tilde{M}) = 0 \quad \text{for all} \quad i \geq 1$$

$$H^o(X,\tilde{M}) \simeq M \; .$$

Suppose now that A is an O-algebra with O a complete valuation ring, then the topology of O induces a topology on A and one may consider the topological subspace Sp(A) of Spec(A) defined by

$$Sp(A) = \{ \underline{p} \in Spec(A) / \underline{p} \text{ open} \}$$

It is easy to see that Sp(A) is the closed fiber of the morphism

$$Spec(A) \rightarrow Spec(O) \; .$$

Now the completion A^1 of A in the topology defined by O defines a sheaf $\tilde{A}^1$ on Sp(A) and the pair $(Sp(A),\tilde{A}^1)$ is a formal scheme.

Given any A-module M the completion M^1 defines a sheaf of $\tilde{A}^1$-Modules , $\tilde{M}^1$, and one easily proves the following

$$H^i(Sp(A),\tilde{M}^1) = 0 \quad \text{for all} \quad i \geq 1$$

$$H^o(Sp(A),\tilde{M}^1) = M^1 \; .$$

Washnitzer and Monsky [2], have introduced another completion of A - the Washnitzer-Monsky completion - A^+ (shortened to W.M. completion) defined as follows:

$$A^+ = \{\alpha = \sum_{\underline{m} \geq o} a_{\underline{m}} t_i^{\underline{m}} \in A^1 \mid a_{\underline{m}} \in O \quad t_1,\dots,t_n \in A \text{ and}$$
$$\text{ord } a_{\underline{m}} \geq c_\alpha(\|\underline{m}\|) \cdot \|m\|\}$$

where c_α is a function $Z^+ \to Z^+$, depending on α and constant outside a finite set.

One may prove (see Lubkin [1])

<u>Lemma 1</u>. Let $\psi: A \to B$ be a surjectiv homomorphism of O-algebras and let M be a finitely generated B-module, then

$$M \underset{A}{\otimes} A^+ \simeq M \underset{B}{\otimes} B^+ .$$

<u>Lemma 2</u>. A^+ is a flat A-module.

Let B be an A-algebra and let for every $a \in A$ the open subset $D(a) = \{ \underline{p} \in \text{Spec}(A) \mid a \notin \underline{p} \}$ of $\text{Spec}(A)$ correspond to the O module $B^+_{\{a\}}$.
This defines a presheaf $\widetilde{B}^+$ on $\text{Spec}(A)$ and Lubkin proves the following,

<u>Theorem</u> Let B be an A-algebra and M a B-module then the presheaf $\widetilde{B}^+_B \otimes M$ is a sheaf concentrated on $\text{Sp}(A)$ and

$$H^i(\text{Sp}(A), \widetilde{B}^+ \underset{B}{\otimes} M) = 0 \quad \text{for } i \geq 1$$
$$H^o(\text{Sp}(A), \widetilde{B}^+ \underset{B}{\otimes} M) = B^+ \underset{B}{\otimes} M$$

The purpose of this note is to give another proof of this theorem which also applies to more general completions.

From now on we shall therefore assume that A^+ is any odd "completion" of A defined by:

$$A^+ = \{\alpha = \sum_{\underline{m}} a_{\underline{m}} \, t_{\underline{i}}^{\underline{m}} \mid a_{\underline{m}} \in O \, , \; t_1, \ldots, t_n \in A$$

$$\operatorname{ord} a_{\underline{m}} \geq c_\alpha(\|\underline{m}\|) \cdot \|\underline{m}\| \}$$

where the (order) maps $c_\alpha : \mathbb{Z}^+ \to \mathbb{Z}^+$ are given in such a way that $A \to A^+$ is a functor <u>Rings</u> $\to$ <u>Rings</u> .

We shall moreover assume that <u>lemma 1</u> and <u>lemma 2</u> holds for this completion, and that there exist a sequence $\{c_n\}_{n \geq 1}$ of order maps $\mathbb{Z}^+ \to \mathbb{Z}^+$ such that for any order map c_α there exist an $n \geq 1$ with $c_\alpha \geq c_n$.

<u>Lemma 3</u>. (Cohomology of projective space)
Let $S = O[T_0, \ldots, T_N]$, and let B be any S-algebra. Denote by $\mathbb{U}$ the covering of $\operatorname{Spec}(S) - V(T_0, \ldots, T_N) \supseteq \operatorname{Sp}(S) - \{m\}$ given by all intersections of $D(T_1), \ldots, D(T_m)$. Let $B(i_1, i_2, \ldots, i_k)$ denote $B_{\{T_{i1} \cdot T_{i2} \cdot \cdot T_{ik}\}}$ and assume:

(1) For all $1 \leq i_1 < i_2 < \cdot\cdot < i_s \leq N$ the homomorphism

$$B(i_1, \ldots, i_s)^+ \to B(1, 2, \ldots, N)^+$$

is one-to-one.

(We shall therefore identify $B(i_1, \ldots, i_s)^+$ with its image in $B(1, 2, \ldots, N)^+$).

(2) If $\{i_1, \ldots, i_s\} \cap \{j_1, \ldots, j_t\} = \{k_1, \ldots, k_n\}$ and if $f \in B(i_1, \ldots, i_s)^+$, $g \in B(j_1, \ldots, j_t)^+$ with $f = g$ then $f = g \in B(k_1, \ldots, k_n)^+$.

(3) If $f \in B(i_1, \ldots, \hat{i}_k, \ldots, i_n)^+ \cap \sum_{r=k+1}^{n} B(i_1, \ldots, \hat{i}_r, \ldots, i_n)^+$ then $f \in \sum_{r=k+1}^{n} B(i_1, \ldots, \hat{i}_k, \ldots, \hat{i}_r, \ldots, i_n)^+$,

then, for $N \geq 1$

$$H^i(\mathfrak{U},\widetilde{B}^+) = \begin{cases} B^+ & \text{if } i = 0 \\ 0 & \text{if } i \neq 0,N \\ B(0,1,\ldots,N)^+ \Big/ \sum_{k=0}^{N} B(0,\ldots,\hat{k},\ldots,N)^+ & \text{if } i = N \end{cases}$$

<u>Proof.</u> If $N = 0$ then

$$H^i(\mathfrak{U},\widetilde{B}^+) = \begin{cases} B_{T_0}^+ & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

If $N = 1$ then

$$H^i(\mathfrak{U},\widetilde{B}^+) = \begin{cases} B^+ & \text{for } i = 0 \\ B(0,1)^+ / B(0)^+ + B(1)^+ & \text{for } i = 1 \\ 0 & \text{for } i \neq 0,1 \end{cases}$$

Therefore the lemma is true for $N = 1$, and we may try induction on N . Suppose $N \geq 2$. Let $\mathfrak{U}'$ be the subcovering of $\mathfrak{U}$ given by all intersections of $D(T_1),\ldots,D(T_N)$. Let $\mathfrak{U}^2$ be the subcovering of $\mathfrak{U}$ given by all intersections of $D(T_0T_1)$, $D(T_0T_2),\ldots,D(T_0T_N)$ and let $\mathfrak{U}^3$ be the subcovering given by $D(T_0)$ alone.

By a spectral sequence-argument or by a Mayer-Vietoris sequence we find an exact sequence

$$0 \to H^{i-1}(\mathfrak{U}^2,\widetilde{B}^+) \Big/ H^{i-1}(\mathfrak{U}^3,\widetilde{B}^+) + H^{i-1}(\mathfrak{U}^1,\widetilde{B}^+) \to H^i(\mathfrak{U},\widetilde{B}^+)$$

$$\to H^i(\mathfrak{U}^3,\widetilde{B}^+) \underset{H^i(\mathfrak{U}^2,\widetilde{B}^+)}{\times} H^i(\mathfrak{U}^1,\widetilde{B}^+) \to 0$$

The result follows from the hypotheses and by noting that if B satisfies the conditions (1), (2) and (3) then so does B_{T_0} .

Lemma 4. Let R be any commutativ ring and consider $S = R[X_0,\dots,X_N]$. Let C be an S-module such that for all $c \in C$ there exists an $n \geq 1$ for which $X_i^n \in \mathrm{Ann}(C)$ for $i = 0,\dots,N$. Let M be an S-module and suppose there exists an element $Q = \sum_{i=0}^{N} Q_i X_i$ in S operating on M as the identity. Then

$$\mathrm{Tor}_p^S(M,C) = 0 \quad \text{for all} \quad p = 0,1,\dots,N .$$

Proof. Q acts as the identity on $\mathrm{Tor}_p^S(M,C)$ since Q acts as the identity on M. On the other hand, taking a free resolution of M, we find that every element in $\mathrm{Tor}_p^S(M,C)$ is annihilated by some power of Q, it follows that $\mathrm{Tor}_p^S(M,C) = 0$ for all $p \geq 0$.

QED.

Lemma 5. Let $U = O[T_0,T_1,\dots,T_N,T_{N+1},\dots,T_{N+M}]$

Then

$$D = U(0,1,\dots,N)^+ \Big/ \sum_{k=0}^{N} U(0,\dots,\hat{k},\dots,N)^+$$

$$\simeq \{\alpha = \sum_{\substack{m_i<0 \text{ for } 0\leq i\leq N \\ m_i>0 \text{ for } i\geq N+1}} a_{\underline{m}} T_{\underline{i}}^{\underline{m}} \mid a_{\underline{m}} \in O \,,\ \mathrm{ord}\, a_{\underline{m}} \geq c_\alpha(\underline{m})\|m\|\}$$

Let $D_n = \{x \in D \mid x = \sum a_{\underline{m}} T_{\underline{i}}^{\underline{m}},\ \mathrm{ord}\, a_{\underline{m}} \geq c_n(\underline{m})\cdot\|\underline{m}\| + d_x\}$ where d_x is a constant depending on x. Then $D_n = \hat{D}_n$ and $D = \varinjlim_{n>1} D_n$, as S-modules.

Proof. Clearly D_n is an S-module, the rest is equally obvious.

QED.

Let $D_{n,k} = D_n/t^k D_n$, then as we have seen

$$D_n = \varprojlim_{k\geq 1} D_{n,k}$$

and, for all $i = 0,\ldots,N$ there exists an n_k such that

$$T_i^{n_k} \in \operatorname{Ann}(D_{n,k}) \; .$$

<u>Lemma 6</u>. Suppose M is a U-module such that there is an element $A = \sum_{i=o}^{N} Q_i T_i$ in U operating on M as the identity, then

$$\operatorname{Tor}_p^U(D,M) = 0 \quad \text{for all } p \; .$$

<u>Proof</u>. By <u>lemma 4</u> $\operatorname{Tor}_p^U(D_{n,k},M) = 0$ for all p . By a standard argument

$$\operatorname{Tor}_p^U(D_n,M') = 0 \quad \text{for all } p$$

and all finitely generated U-modules M' . Since Tor_p^U commutes with $\varinjlim_{\mathbb{Z}}$ we find

$$\operatorname{Tor}_p^U(D,M) = 0 \quad \text{for all } p \; .$$

QED.

Let A be any finitely generated O-algebra. Let B be a finitely generated A-algebra. Let $x_1,\ldots,x_n$ be generators for A as O-algebra, and let $y_1,\ldots,y_M$ be generators for B as A-algebra.

Let $\mathbb{U}'$ be a covering of $\operatorname{Spec}(A)$, $\mathbb{U}' = \{D(a_i)\}_{i=1}^{r}$. Then the ideal of A generated by the a_i's is equal to A . Therefore there exist elements $b_i \in A$ such that

$$(7)\qquad \begin{aligned} &\sum_{i=1}^{r} b_i a_i = 1 \\ &x_j = \sum_{i=1}^{r} (x_j b_i) a_i \quad \text{for } j = 1,\ldots,n \end{aligned}$$

Let $\mathbb{U}''$ be the covering consisting of all $D(a_i)$ $i = 1,\ldots,r$ and all $D(x_j b_i a_i)$ $i = 1,\ldots,r$, $j = 1,\ldots,n$ and of all intersections of these sets. Clearly $\mathbb{U}''$ is finer than $\mathbb{U}'$ and we may write

$$\mathbb{U}'' = \{D(\alpha_l)\}_{l=0}^{N} \quad \text{for some } \alpha_l \in A \,.$$

Obviously α_l , $l = 0,\ldots,N$ generates A as O-algebra and α_l , $l = 0,\ldots,N$ (strictely speaking the image of α_l in B .) and y_k , $k = 1,\ldots,M$ generate B as O-algebra.
Therefore there is a commutative diagram of homomorphisms of O-algebras

$$\begin{array}{lll} S = O[T_0,\ldots,T_N] \xrightarrow{\varphi} A & & \varphi(T_l) = \alpha_l \,,\ l = 0,\ldots,N \\ \qquad\qquad \downarrow & \searrow & \\ U = O[T_0,\ldots,T_N,T_{N+1},\ldots,T_{N+M}] \xrightarrow[\psi]{} B & & \psi(T_{N+K}) = y_k,\ k = 1,\ldots,M \end{array}$$

the horizontal homomorphisms being onto.
Let M be a B-module.
By <u>lemma 1</u> we have isomorphisms of Čech-cmmplexes

$$\begin{array}{c} C^{\bullet}(\mathbb{U},\widetilde{U}^{+}_{U} \otimes M) \simeq C^{\bullet}(\mathbb{U}''.\widetilde{B}^{+} \underset{B}{\otimes} M) \\ \wr\! | \\ C^{\bullet}(\mathbb{U},\widetilde{U}^{+}) \underset{U}{\otimes} M \end{array}$$

Let L. be a free resolution of M as U-module, and look at the double complex

$$(8)\qquad C^{\bullet}(\mathbb{U},\widetilde{U}^{+}) \underset{U}{\otimes} L. \quad .$$

Since the completion is flat $C^{\bullet}(\mathbb{U},\widetilde{U}^{+})$ is flat over U .

The first spectral sequence of the double-complex (8) degenerates, i.e.

$$'E_2^{p,q} = \begin{cases} 0 & \text{if} \quad q \neq 0 \\ H^p(\mathbb{W}'', \widetilde{B}^+ \otimes_B M) & \text{if} \quad q = 0 \end{cases}$$

The second spectral sequence has the form

$$''E_2^{p,q} = \mathrm{Tor}^U_{-p}(H^q(\mathbb{W}, \widetilde{U}^+), M)$$

Assumeing $N \geq 1$ it follows from <u>lemma 2</u> that

$$''E_2^{p,q} = 0 \quad \text{for} \quad q \neq 0, N$$

$$''E_2^{p,o} = 0 \quad \text{for} \quad p \neq 0 \; .$$

By (7) there is a $Q = \sum_{i=o}^{N} Q_i T_i$ in S operating as the identity on A therefore on B and consequently on M .

Lemma 6 implies

$$''E_2^{p,N} = \mathrm{Tor}_{-p}(D, M) = 0 \quad \text{for all} \quad p \; .$$

Since both spectral sequences converge we have proved:

<u>Theorem</u>. In the situation above:

$$H^n(\mathbb{W}'', \widetilde{B}^+ \otimes_B M) = \begin{cases} B^+ \otimes_B M & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n \neq 0 \end{cases}$$

In particular $\widetilde{B}^+ \otimes_B M$ is a sheaf on $\mathrm{Spec}(A)$.

<u>Bibliography</u>.

[1] Saul Lubkin: A p-adic proof of Weil's conjectures: Annals of Math. Vol 87 (1968) pp. 105-255.

[2] G. Washnitzer and P. Monsky: The construction of formal cohomology sheaves. Proc. Natl. Acad. Sci. USA 52 (1964) pp. 1511-1514.