Cohomology of affine "formal" schemes

by

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Let $A$ be any commutative ring with unit and let $M$ be an $A$-module. Then $M$ defines a sheaf of $O_X$-Modules $\tilde{M}$ on the affine scheme $X = \text{Spec}(A)$, and by a well known theorem of Serre

$$
\begin{align*}
H^i(X,\tilde{M}) &= 0 \quad \text{for all } i \geq 1 \\
H^0(X,\tilde{M}) &= M.
\end{align*}
$$

Suppose now that $A$ is an $O$-algebra with $O$ a complete valuation ring, then the topology of $O$ induces a topology on $A$ and one may consider the topological subspace $\text{Sp}(A)$ of $\text{Spec}(A)$ defined by

$$
\text{Sp}(A) = \{ \mathfrak{p} \in \text{Spec}(A)/ \mathfrak{p} \text{ open}\}.
$$

It is easy to see that $\text{Sp}(A)$ is the closed fiber of the morphism

$$
\text{Spec}(A) \rightarrow \text{Spec}(O).
$$

Now the completion $A^1$ of $A$ in the topology defined by $O$ defines a sheaf $\tilde{A}^1$ on $\text{Sp}(A)$ and the pair $(\text{Sp}(A),\tilde{A}^1)$ is a formal scheme. Given any $A$-module $M$ the completion $M^1$ defines a sheaf of $\tilde{A}^1$-Modules, $\tilde{M}^1$, and one easily proves the following

$$
\begin{align*}
H^i(\text{Sp}(A),\tilde{M}^1) &= 0 \quad \text{for all } i \geq 1 \\
H^0(\text{Sp}(A),\tilde{M}^1) &= M^1.
\end{align*}
$$

Washnitzer and Monsky [2], have introduced another completion of $A$ — the Washnitzer-Monsky completion $A^+$ (shortened to W.M. completion) defined as follows.
\[ A^+ = \{ \alpha = \sum_{m \geq 0} a_m t^m \in A^1 \mid a_m \in \mathbb{Z}, t_1, \ldots, t_n \in A \text{ and } \] \[ \text{ord } a_m \geq c_\alpha(||m||) \cdot ||m|| \} \]

where \( c_\alpha \) is a function \( \mathbb{Z}^+ \to \mathbb{Z}^+ \), depending on \( \alpha \) and constant outside a finite set.

One may prove (see Lubkin [1])

**Lemma 1.** Let \( \psi: A \to B \) be a surjective homomorphism of \( \mathbb{Z} \)-algebras and let \( M \) be a finitely generated \( B \)-module, then

\[ M \otimes A^+ \cong M \otimes B^+ \]

**Lemma 2.** \( A^+ \) is a flat \( A \)-module.

Let \( B \) be an \( A \)-algebra and let for every \( a \in A \) the open subset \( D(a) = \{ \mathfrak{p} \in \text{Spec}(A) \mid a \not\in \mathfrak{p} \} \) of \( \text{Spec}(A) \) correspond to the \( 0 \)-module \( B^+_{\{a\}} \).

This defines a presheaf \( \widehat{B}^+ \) on \( \text{Spec}(A) \) and Lubkin proves the following,

**Theorem** Let \( B \) be an \( A \)-algebra and \( M \) a \( B \)-module then the presheaf \( \widehat{B}^+ \otimes M \) is a sheaf concentrated on \( \text{Sp}(A) \) and

\[ H^i(\text{Sp}(A), \widehat{B}^+ \otimes M) = 0 \text{ for } i > 1 \]

\[ H^0(\text{Sp}(A), \widehat{B}^+ \otimes M) = B^+ \otimes M \]

The purpose of this note is to give another proof of this theorem which also applies to more general completions.

From now on we shall therefore assume that \( A^+ \) is any odd "completion" of \( A \) defined by:
\[ A^+ = \{ a = \sum_{m} a_m t^m \mid a_m \in \mathcal{O}, t_1, \ldots, t_n \in A \} \]

\[ \text{ord } a_m \geq c_\alpha(||m||) \cdot ||m|| \]

where the (order) maps \( c_\alpha : \mathbb{Z}^+ \to \mathbb{Z}^+ \) are given in such a way that \( A \to A^+ \) is a functor \( \text{Rings} \to \text{Rings} \).

We shall moreover assume that \textbf{lemma 1} and \textbf{lemma 2} holds for this completion, and that there exist a sequence \( \{c_n\}_{n \geq 1} \) of order maps \( \mathbb{Z}^+ \to \mathbb{Z}^+ \) such that for any order map \( c_\alpha \) there exist an \( n \geq 1 \) with \( c_\alpha \geq c_n \).

\textbf{Lemma 3.} (Cohomology of projective space)

Let \( S = 0[T_0, \ldots, T_N] \), and let \( B \) be any \( S \)-algebra. Denote by \( \mathcal{U} \) the covering of \( \text{Spec}(S) - V(T_0, \ldots, T_N) \supset \text{Sp}(S) - \{m\} \) given by all intersections of \( D(T_1), \ldots, D(T_m) \). Let \( B(i_1, i_2, \ldots, i_k) \) denote \( B(T_{i_1}, T_{i_2}, \ldots, T_{i_k}) \) and assume:

1. For all \( 1 \leq i_1 < i_2 < \cdots < i_s \leq N \) the homomorphism

\[ B(1, 2, \ldots, N)^+ \to B(i_1, i_2, \ldots, i_s)^+ \]

is one-to-one.

(We shall therefore identify \( B(i_1, \ldots, i_s)^+ \) with its image in \( B(1, 2, \ldots, N)^+ \).)

2. If \( \{i_1, \ldots, i_s\} \cap \{j_1, \ldots, j_r\} = \{k_1, \ldots, k_n\} \) and if \( f \in B(i_1, \ldots, i_s)^+ \), \( g \in B(j_1, \ldots, j_r)^+ \) with \( f = g \) then \( f = g \in B(k_1, \ldots, k_n)^+ \).

3. If \( f \in B(i_1, \ldots, i_k, \ldots, i_n)^+ \cap \sum_{r=k+1}^n B(i_1, \ldots, i_k, \ldots, \hat{i}_r, \ldots, i_n)^+ \) then \( f \in \sum_{r=k+1}^n B(i_1, \ldots, i_k, \ldots, \hat{i}_r, \ldots, i_n)^+ \).
then, for $N \geq 1$

$$H^i(\mathcal{U}, \mathcal{B}^+) = \begin{cases} 
B^+ & \text{if } i = 0 \\
0 & \text{if } i \neq 0, N \\
\frac{B(0,1,\ldots,N)^+}{\sum_{k=0}^N B(0,\ldots,k,\ldots,N)^+} & \text{if } i = N
\end{cases}$$

Proof. If $N = 0$ then

$$H^i(\mathcal{U}, \mathcal{B}^+) = \begin{cases} 
\mathcal{B}_0^+ & \text{for } i = 0 \\
0 & \text{for } i \neq 0
\end{cases}$$

If $N = 1$ then

$$H^i(\mathcal{U}, \mathcal{B}^+) = \begin{cases} 
B^+ & \text{for } i = 0 \\
B(0,1)^+ / B(0)^+ + B(1)^+ & \text{for } i = 1 \\
0 & \text{for } i \neq 0, 1
\end{cases}$$

Therefore the lemma is true for $N = 1$, and we may try induction on $N$. Suppose $N \geq 2$. Let $\mathcal{W}'$ be the subcovering of $\mathcal{W}$ given by all intersections of $D(T_1)$, $\ldots$, $D(T_N)$. Let $\mathcal{W}^2$ be the subcovering of $\mathcal{W}$ given by all intersections of $D(T_0 T_1)$, $D(T_0 T_2)$, $\ldots$, $D(T_0 T_N)$ and let $\mathcal{W}^3$ be the subcovering given by $D(T_0)$ alone.

By a spectral sequence-argument or by a Mayer-Vietoris sequence we find an exact sequence

$$0 \to H^{i-1}(\mathcal{W}^2, \mathcal{B}^+) \to H^i(\mathcal{W}, \mathcal{B}^+) \to H^{i-1}(\mathcal{W}^3, \mathcal{B}^+) + H^{i-1}(\mathcal{W}^1, \mathcal{B}^+)$$

The result follows from the hypotheses and by noting that if $\mathcal{B}$ satisfies the conditions (1), (2) and (3) then so does $\mathcal{B}_0$.
Lemma 4. Let $R$ be any commutative ring and consider $S = R[X_0, \ldots, X_N]$. Let $C$ be an $S$-module such that for all $c \in C$ there exists an $n \geq 1$ for which $X_i^c \in \text{Ann}(C)$ for $i = 0, \ldots, N$. Let $M$ be an $S$-module and suppose there exists an element $Q = \sum_{i=0}^{N} Q_i X_i$ in $S$ operating on $M$ as the identity. Then

$$\text{Tor}_p^S(M, C) = 0 \text{ for all } p = 0, 1, \ldots, N.$$ 

Proof. $Q$ acts as the identity on $\text{Tor}_p^S(M, C)$ since $Q$ acts as the identity on $M$. On the other hand, taking a free resolution of $M$, we find that every element in $\text{Tor}_p^S(M, C)$ is annihilated by some power of $Q$, it follows that $\text{Tor}_p^S(M, C) = 0$ for all $p \geq 0$.

QED.

Lemma 5. Let $U = 0[T_0, T_1, \ldots, T_N, T_{N+1}, \ldots, T_{N+M}]$ Then

$$D = \frac{U(0,1,\ldots,N)^+}{\sum_{k=0}^{N} U(0,\ldots,k,\ldots,N)^+}$$

$$\cong \{ \alpha = \sum_{m_i < 0 \text{ for } 0 \leq i \leq N} a_m T_i^m \mid a_m \in O, \text{ord } a_m \geq c_a(m) \|m\| \} \quad \text{for } m_i > 0 \text{ for } i > N+1$$

Let $D_n = \{ x \in D \mid x = \sum a_m T_i^m, \text{ord } a_m \geq c_n(m) \|m\| + d_x \}$ where $d_x$ is a constant depending on $x$. Then $D_n = \hat{D}_n$ and $D = \lim_{n \to \infty} D_n$, as $S$-modules.

Proof. Clearly $D_n$ is an $S$-module, the rest is equally obvious.

QED.
Let \( D_n,k = D_n/k D_n \), then as we have seen

\[
D_n = \lim_{k \to \infty} D_n,k
\]

and, for all \( i = 0, \ldots, N \) there exists an \( n_k \) such that

\[
T^*_{i} \in \text{Ann}(D_n,k).
\]

**Lemma 6.** Suppose \( M \) is a \( U \)-module such that there is an element \( A = \sum_{i=0}^{N} Q_i T_i \) in \( U \) operating on \( M \) as the identity, then

\[
\text{Tor}_p^U(D,M) = 0 \quad \text{for all } p.
\]

**Proof.** By **Lemma 4** \( \text{Tor}_p^U(D_n,k,M) = 0 \) for all \( p \). By a standard argument

\[
\text{Tor}_p^U(D_n,M') = 0 \quad \text{for all } p
\]

and all finitely generated \( U \)-modules \( M' \). Since \( \text{Tor}_p^U \) commutes with \( \lim \), we find

\[
\text{Tor}_p^U(D,M) = 0 \quad \text{for all } p.
\]

QED.

Let \( A \) be any finitely generated \( \mathbb{O} \)-algebra. Let \( B \) be a finitely generated \( A \)-algebra. Let \( x_1, \ldots, x_n \) be generators for \( A \) as \( \mathbb{O} \)-algebra, and let \( y_1, \ldots, y_M \) be generators for \( B \) as \( A \)-algebra.

Let \( \mathbb{W} \) be a covering of \( \text{Spec}(A) \), \( \mathbb{W} = \{ D(a_i) \}_{i=1}^{r} \). Then the ideal of \( A \) generated by the \( a_i \)'s is equal to \( A \). Therefore there exist elements \( b_i \in A \) such that
\[ \sum_{i=1}^{r} b_i a_i = 1 \quad (7) \]

\[ x_j = \sum_{i=1}^{r} (x_j b_i) a_i \quad \text{for} \quad j = 1, \ldots, n \]

Let \( \mathcal{W}' \) be the covering consisting of all \( D(a_i) \quad i = 1, \ldots, r \) and all \( \mathcal{D}(x_j b_i a_i) \quad i = 1, \ldots, r, \quad j = 1, \ldots, n \) and of all intersections of these sets. Clearly \( \mathcal{W}' \) is finer than \( \mathcal{W}' \) and we may write

\[ \mathcal{W}' = \{D(a_i)\}_{i=0}^{N} \quad \text{for some} \quad a_1 \in A. \]

Obviously \( a_1, l = 0, \ldots, N \) generates \( A \) as \( 0 \)-algebra and \( a_1, l = 0, \ldots, N \) (strictly speaking the image of \( a_1 \) in \( B \)) and \( y_k, k = 1, \ldots, M \) generate \( B \) as \( 0 \)-algebra. Therefore there is a commutative diagram of homomorphisms of \( 0 \)-algebras

\[
\begin{cases}
S = O[T_0, \ldots, T_N] & \xrightarrow{\varphi} A \\
{
s}
U = O[T_0, \ldots, T_N, T_{N+1}, \ldots, T_{N+M}] & \xrightarrow{\psi} B
\end{cases}
\]

the horizontal homomorphisms being onto.

Let \( M \) be a \( B \)-module.

By lemma 1 we have isomorphisms of \( \check{\text{Č}} \vspace{1.5pt} \text{ech-complexes} \)

\[ C^*(\mathcal{W}, \widetilde{U}^+_* \otimes M) \cong C^*(\mathcal{W}', \widetilde{B}^+_* \otimes M) \]

\[ 
\]

\[ C^*(\mathcal{W}, \widetilde{U}^+) \otimes M \]

Let \( L \) be a free resolution of \( M \) as \( U \)-module, and look at the double complex

\[ C^*(\mathcal{W}, \widetilde{U}^+) \otimes L. \quad (8) \]

Since the completion is flat \( C^*(\mathcal{W}, \widetilde{U}^+) \) is flat over \( U \).
The first spectral sequence of the double-complex (8) degenerates, i.e.
\[ E_{2}^{p,q} = \begin{cases} 0 & \text{if } q \neq 0 \\ H^{p}(\mathbb{W}, \mathbb{B}^{+} \otimes M) & \text{if } q = 0 \end{cases} \]

The second spectral sequence has the form
\[ E_{2}^{p,q} = \text{Tor}^{B}_{p}(H^{q}(\mathbb{W}, \mathbb{B}^{+}), M) \]

Assuming $N \geq 1$ it follows from lemma 2 that
\[ E_{2}^{p,q} = 0 \quad \text{for } q \neq 0, N \]
\[ E_{2}^{p,0} = 0 \quad \text{for } p \neq 0. \]

By (7) there is a $Q = \sum_{i=0}^{N} Q_{i}T_{i}$ in $S$ operating as the identity on $A$ therefore on $B$ and consequently on $M$.

Lemma 6 implies
\[ E_{2}^{p,N} = \text{Tor}^{B}_{p}(D, M) = 0 \quad \text{for all } p. \]

Since both spectral sequences converge we have proved:

Theorem. In the situation above:
\[ H^{n}(\mathbb{W}, \mathbb{B}^{+} \otimes M) = \begin{cases} \mathbb{B}^{+} \otimes M & \text{if } n = 0 \\ B \otimes M & \text{if } n \neq 0 \end{cases} \]

In particular $\mathbb{B}^{+} \otimes M$ is a sheaf on Spec($A$).

Bibliography.
