On Banach space valued extensions from split faces.

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The aim of this note is to prove the following theorem: Suppose \( a \) is a continuous affine map from a closed split face \( F \) of a compact convex set \( K \) with values in a Banach space \( B \) enjoying the approximation property. Suppose also that \( p \) is a strictly positive lower semi-continuous concave function on \( K \) such that \( \|a(k)\| \leq p(k) \) for all \( k \) in \( F \). Then \( a \) admits a continuous affine extension \( \tilde{a} \) to \( K \) into \( B \) such that \( \|\tilde{a}(k)\| \leq p(k) \) for all \( k \) in \( K \).

We shall use the methods of tensor products of compact convex sets as developed by Semadeni [12], Lazar [9], Namioka and Phelps [10] and Behrends and Wittstock [6] to reduce the problem to the case \( B = \mathbb{R} \), and in this case the result follows from the work of Alfsen and Hirschberg [3] and the present author [4].

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We shall be concerned with compact convex sets \( K_1 \) and \( K_2 \) in locally convex spaces \( E_1 \) and \( E_2 \) respectively. By \( A(K_i) \) we shall denote the continuous real affine functions on \( K_i \) for \( i = 1, 2 \). We let \( BA(K_1 \times K_2) \) be the Banach space of continuous biaffine functions on \( K_1 \times K_2 \). We observe that \( \Pi \in BA(K_1 \times K_2) \) and that \( BA(K_1 \times K_2) \) separates points of \( K_1 \times K_2 \). As usual we define the projective tensor product of \( K_1 \) and \( K_2 \), \( K_1 \otimes K_2 \), to be the state space of \( BA(K_1 \times K_2) \) equipped with the \( w^* \)-topology. Then \( K_1 \otimes K_2 \) is a compact convex set, and we have a homeomorphic embedding \( \omega_{K_1 \times K_2} \) (called \( \omega \), when no confusion can arise) from \( K_1 \times K_2 \) into \( K_1 \otimes K_2 \) defined by the following...
We notice that \( w \) is a biaffine map. It was proved in [10; Prop. 1.3, Th. 2.3] and [6; Satz 1.1.3] that \( \partial_e(K_1 \otimes K_2) = w(\partial_e K_1 \times \partial_e K_2) \), where in general we denote the extreme points of a convex set \( K \) by \( \partial_e K \).

For \( a \) in \( A(K_1) \) and \( b \) in \( A(K_2) \) we define the continuous biaffine function \( a \otimes b \) by

\[
a \otimes b(x_1, x_2) = a(x_1)b(x_2), \quad \text{all } (x_1, x_2) \in K_1 \times K_2
\]

We let \( A(K_1) \otimes A(K_2) \) be the real vector space

\[
A(K_1) \otimes A(K_2) = \{ \sum_{i=1}^{n} a_i \otimes b_i \mid a_i \in A(K_1), b_i \in A(K_2) \}
\]

which is a copy of the algebraic tensor product of \( A(K_1) \) and \( A(K_2) \). We denote by \( A(K_1) \otimes_e A(K_2) \) the uniform closure of \( A(K_1) \otimes A(K_2) \) in \( BA(K_1 \times K_2) \).

We recall that a Banach space \( B \) is said to have the approximation property if for each compact convex subset \( C \) of \( B \) and each \( \varepsilon > 0 \) there is a continuous linear map \( T: B \to B \) such that \( T(B) \) is finite dimensional and such that \( \|Tx - x\| < \varepsilon \) for all \( x \in C \). It is proved in [10; Lem. 2.5] that if \( A(K_1) \) (or \( A(K_2) \)) has the approximation property then \( BA(K_1 \times K_2) = A(K_1) \otimes_e A(K_2) \).

Following Lazar [9] we define \( T_1 \) and \( T_2 \) as the natural embeddings of \( A(K_1) \) and \( A(K_2) \) into \( BA(K_1 \times K_2) \), i.e.

\[
T_1 a = a \otimes 1, \quad \text{all } a \in A(K_1)
\]
\[
T_2 b = 1 \otimes b, \quad \text{all } b \in A(K_2)
\]

Let \( P_i \) be the adjoint map of \( T_i \) for \( i = 1, 2 \).
Then $P_i$ is an affine and continuous map of $K_1 \otimes K_2$ onto $K_i$ (state space of $A(K_i)$), and

$$P_i \omega(k_1,k_2) = k_i, \ i = 1,2.$$ 

The first part of the following proposition was proved by Lazar in the case where $K_1$ and $K_2$ are simplexes, but the proof holds in general. The last part was proved by Lazar in the simplex case by means of the Stone-Weierstrass Theorem for simplexes.

**Proposition 1.** Let $F_1$ and $F_2$ be closed faces of compact convex sets $K_1$ and $K_2$ resp. Let $F = P_1^{-1}(F_1) \cap P_2^{-1}(F_2)$

(i) Then $F$ is a closed face in $K_1 \otimes K_2$ and

$$F = \overline{\text{co}}(\omega(F_1 \times F_2)).$$

(ii) If $A(F_1)$ or $A(F_2)$ has the approximation property then $F_1 \otimes F_2$ is affinely homeomorphic to $F$.

**Proof:** Since $P_i$ is continuous and affine it is immediate that $P_i^{-1}(F_i)$ is a closed face of $K_1 \otimes K_2$, and hence $F$ is a closed face.

Now let $p = \omega(k_1,k_2) \in \omega(F_1 \times F_2)$. Then $P_1 p = k_1 \in F_1$, and hence $p \in P_1^{-1}(F_1) \cap P_2^{-1}(F_2) = F$. By the Krein Milman Theorem: $\overline{\text{co}}(\partial_e(F_1 \times F_2)) \subseteq F$.

Conversely, let $p \in \partial_e F$. Since $F$ is a closed face we get

$$p \in \partial_e F = F \cap \partial_e(K_1 \otimes K_2) = F \cap \omega(\partial_e K_1 \times \partial_e K_2).$$

Hence $p = \omega(x_1,x_2)$, $x_i \in \partial_e K_i$. Then $P_i p = x_i$ belongs to
by the definition of \( F \). Hence \( p \in \omega(F_1 \times F_2) \), and again by the Krein Milman Theorem \( F \subseteq \overline{co}(\omega(F_1 \times F_2)) \), and (i) is proved.

Now we shall prove (ii) under the assumption that \( A(F_1) \) has the approximation property. We shall define a continuous affine map \( T: F_1 \otimes F_2 \to K_1 \otimes K_2 \) by

\[
(T\varphi)(b) = \varphi(b|_{F_1 \times F_2}), \quad \varphi \in F_1 \otimes F_2, \quad b \in BA(K_1 \times K_2)
\]

Then \( T(F_1 \otimes F_2) \) is compact and convex in \( K_1 \otimes K_2 \). If \( \varphi \in \partial_e(F_1 \otimes F_2) \) then \( \varphi = \omega_{F_1 \times F_2}(x_1, x_2) \), where \( x_i \in \partial_e F_i \), \( i = 1, 2 \). But then

\[
(T\varphi)(b) = b(x_1, x_2) = \omega_{K_1 \times K_2}(x_1, x_2)(b), \quad \text{all } b \in BA(K_1 \times K_2).
\]

Hence \( T\varphi = \omega_{K_1 \times K_2}(x_1, x_2) \in \overline{co}(\omega_{K_1 \times K_2}(F_1 \times F_2)) = F \). By the Krein Milman Theorem we conclude that \( T(F_1 \otimes F_2) \subseteq F \).

Conversely, if \( \psi \in \partial_e F \) then as \( F \) is a closed face, we get by Milman's theorem

\[
\psi \in \omega_{K_1 \times K_2}(F_1 \times F_2) \cap \omega_{K_1 \times K_2}(\partial_e K_1 \times \partial_e K_2) = \omega_{K_1 \times K_2}(\partial_e F_1 \times \partial_e F_2).
\]

If \( \psi = \omega_{K_1 \times K_2}(x_1, x_2) \), \( x_i \in \partial_e F_i \), then \( \omega_{F_1 \times F_2}(x_1, x_2) \in \partial_e(F_1 \otimes F_2) \), and as above \( \psi = T(\omega_{F_1 \times F_2}(x_1, x_2)) \). By the Klein Milman Theorem we get \( F \subseteq T(F_1 \otimes F_2) \), and so \( T \) is surjective.

We proceed to show that \( T \) is injective. This is the case if \( BA(K_1 \times K_2)|_{F_1 \times F_2} \) is dense in \( BA(F_1 \times F_2) \). We show that \( A(K_1) \otimes A(K_2)|_{F_1 \times F_2} \) is dense in \( BA(F_1 \times F_2) \). Hence let \( \epsilon > 0 \). Since \( A(F_1) \) has the approximation property, we have that \( A(F_1) \otimes\epsilon A(F_2) = BA(F_1 \otimes F_2) \), so there exist \( a_1, \ldots, a_n \in A(F_1), b_1, \ldots, b_n \in A(F_2) \) such that

\[
\|c - \sum_{i=1}^{n} a_i \otimes b_i\|_{F_1 \times F_2} < \frac{\epsilon}{2}.
\]
Now $A(K_1)|_{F_1}$ is dense in $A(F_1)$, so we can choose $a_i^1 \in A(K_1)$, $b_i^1 \in A(K_2)$, $i = 1, \ldots, n$, such that

$$\| \sum_{i=1}^{n} a_i^1 \otimes b_i^1 - \sum_{i=1}^{n} a_i^1 \otimes b_i^1 \|_{F_1 \times F_2} < \frac{\varepsilon}{2}.$$ 

Then $\| c - \sum_{i=1}^{n} a_i^1 \otimes b_i^1 \|_{F_1 \times F_2} < \varepsilon$, and the claim follows.

The next step is to prove that $\overline{\text{co}(w(F_1 \times F_2))}$ is a closed split face of $K_1 \otimes K_2$ provided $F_1$ is a closed split face of $K_i$ for $i = 1, 2$, and f.ex. $A(F_1)$ has the approximation property.

We shall remind the reader of the following definitions and facts: If $F$ is a closed face of a compact convex $K$, then the complementary $\sigma$-face $F'$ is the union of all faces disjoint from $F$. It is always true that $K = \text{co}(F \cup F')$. $F$ is called a split face if $F'$ is a face and each point in $K \setminus (F \cup F')$ can be decomposed uniquely as convex combination of a point in $F$ and a point in $F'$. It follows from a slight modification of the proof of [2; Th. 3.5] that a closed face is a split face if and only if each non-negative u.s.c. affine function on $F$ admits an u.s.c. affine extension to $K$, which is equal to 0 on $F'$. This characterization is sometimes inconvenient because of the "non-symmetric" properties of the affine functions involved. Using the above characterization we shall give a new one involving the space $A_s(K)$ which is the smallest uniformly closed subspace of the affine bounded functions on $K$ containing the bounded u.s.c./functions. This space has been used f.ex. by Krause [8] and Behrends and Wittstock [6] in simplex theory and by Combes [7] in C*-algebra theory. We shall state some of the known properties of $A_s(K)$. 

Lemma 2.

(i) If \( a \in A_\delta(K) \) and \( a \geq 0 \) on \( \partial_e K \) then \( a \geq 0 \) on \( K \).

(ii) If \( a \in A_\delta(K) \) then \( \|a\|_K = \|a\|_{\partial_e K} \).

(iii) If \( a \in A_\delta(K) \) then \( a \) satisfies the barycentric calculus.

Sketch of proof: If \( s \) and \( t \) are u.s.c. affine functions on \( K \) and \( s \leq t \) on \( \partial_e K \) it follows by [5; Lem. 1] that \( s \leq t \) on \( K \). Hence (i) follows by a limit argument. Now (ii) follows by (i), since on \( \partial_e K \):
\[
\|a\|_{\partial_e K} \leq a \leq \|a\|_{\partial_e K}.
\]
Hence the same inequality holds on \( K \), and so \( \|a\|_K \leq \|a\|_{\partial_e K} \). The converse inequality is trivial. Finally (iii) follows from Lebesgue's theorem on dominated convergence, since the barycentric calculus holds for (differences of) u.s.c. bounded affine functions, cf. [1; Cor. I.1.4].

Proposition 3. Let \( F \) be a closed face of a compact convex set \( K \). Then \( F \) is a split face if and only if each \( a \in A_\delta(F) \) (or \( A_\delta(F)^+, A(F), A(F)^+, A(F;K), A(F;K)^+ \)) has an extension \( \tilde{a} \in A_\delta(K) \) such that \( \tilde{a} = 0 \) on \( F' \). If such an extension exists then it is unique.

Proof: The uniqueness statement follows from Lemma 2 (ii), since \( \partial_e K \subseteq F \cup F' \).

Assume \( F \) is a split face and let \( a \in A_\delta(F) \). If \( a \) is u.s.c. affine and non-negative \( a \) has as noted above an u.s.c. affine extension \( \tilde{a} \) with \( \tilde{a} = 0 \) on \( F' \). Hence the result follows if \( a \) is the difference of two non-negative u.s.c. affine functions on \( K \). In general there are \( b_n, c_n \) u.s.c. affine and non-negative, \( a_n = b_n - c_n \), such that \( \|a_n - a\|_F \to \infty \). We
use Lemma 2 (ii) and the fact that $\partial_e K \subseteq F \cup F'$ to conclude that

$$\|\tilde{a}_n - \tilde{a}_m\| = \|\tilde{a}_n - \tilde{a}_m\|_{\partial_e K} = \|a_n - a_m\|_{\partial_e F} = \|a_n - a_m\|_F$$

Hence $\{\tilde{a}_n\}_{n=1}^{\infty}$ is Cauchy in $A_s(K)$. Then $\tilde{a} = \lim \tilde{a}_n \in A_s(K)$ will be an extension of $a$ with $\tilde{a} = 0$ on $F'$.

Conversely, assume that each $a \in A(F;K)^+$ has an extension $\tilde{a} \in A_s(K)$ such that $\tilde{a} = 0$ on $F'$. Let $x \in K \setminus (F \cup F')$, $x = \lambda y + (1 - \lambda)z$, where $y \in F$, $z \in F'$ and $0 < \lambda < 1$. Then $\lambda = \tilde{w}(x)$, and since $\lambda$ is uniquely determined, $\tilde{x}_F'$ is affine, and hence $F' = \tilde{x}_F'^{-1}(0)$ is a face, cf. [2; Prop. 1.1, Cor. 1.2]. Now the uniqueness of $F,F'$ components is easy, since $A(F;K)^+$ separates points of $F$.

The following lemma can be derived from [6; Formula (1), p.263, Satz 2.1.3]. For the readers convenience we shall give a proof.

**Lemma 4.** Let $K_1$ and $K_2$ be compact convex sets and $a \in A_s(K_1)$, $b \in A_s(K_2)$. Then there is a function $c \in A_s(K_1 \otimes K_2)$, denoted by $a \otimes b$, such that

$$c(\omega(x_1,x_2)) = a(x_1)b(x_2), \text{ all } (x_1,x_2) \in K_1 \times K_2.$$

**Proof:** First we shall consider the case where $a$ and $b$ are non-negative u.s.c. and affine. Then there exist nets $\{a_\alpha\} \subseteq A(K_1)^+$, $\{b_\beta\} \subseteq A(K_2)^+$ such that $a_\alpha \wedge a$, $b_\beta \wedge b$, pointwise. Then $\{a_\alpha \otimes b_\beta\}$ is a decreasing net in $BA(K_1 \times K_2)^+$, and therefore there is an u.s.c. affine function $c$ on $K_1 \otimes K_2$ such that

$$c(\varphi) = \inf_{\alpha,\beta} \varphi(a_\alpha \otimes b_\beta), \text{ all } \varphi \in K_1 \otimes K_2.$$
Especially, for all \((x_1, x_2) \in K_1 \times K_2\)

\[ c(w(x_1, x_2)) = \inf a_\alpha(x_1)b_\beta(x_2) = a(x_1)b(x_2). \]

If

\((*)\) \quad a = a_1 - a_2, \quad b = b_1 - b_2

where \(a_1\) is u.s.c. non-negative and affine on \(K_1\), \(b_1\) is u.s.c. non-negative and affine on \(K_2\), then \((x_1, x_2) - a(x_1)b(x_2)\) is linear combination of four terms of the kind considered in the first part of the proof, and we can choose \(c\) as the corresponding linear combination of elements from \(A_s(K_1 \otimes K_2)\).

If \(a \in A_s(K_1), b \in A_s(K_2)\) are arbitrary then we can find \(a'_n, b'_n\) of the type \((*)\), such that \(\|b - b'_n\|_{K_2} < \frac{1}{n}\), \(\|a - a'_n\|_{K_1} < \frac{1}{n}\) and \(c_n \in A_s(K_1 \otimes K_2)\) such that

\((**)\) \quad \(c_n(w(x_1, x_2)) = a'_n(x_1)b'_n(x_2),\) all \((x_1, x_2) \in K_1 \times K_2.\)

Then for all \((x_1, x_2) \in \partial e K_2\)

\[ |a(x_1)b(x_2) - c_n(w(x_1, x_2))| < \frac{1}{n^2} + \frac{1}{n}(\|a\|_{K_1} + \|b\|_{K_2}) .\]

From this it follows that \(\{c_n \mid \partial e(K_1 \otimes K_2)\}\) is Cauchy, and hence \(\{c_n\}\) is Cauchy on \(K_1 \otimes K_2\) by Lemma 2 (ii). Let \(c = \lim c_n \in A_s(K_1 \otimes K_2)\). Then it is obvious from \((**)\) that \(c\) satisfies the requirement.

**Theorem 5.** Let \(K_1\) and \(K_2\) be compact convex sets, and \(F_1\) and \(F_2\) closed faces of \(K_1\) and \(K_2\) respectively. Let \(F\) be the face \(\text{co}(w(F_1 \times F_2))\) in \(K_1 \otimes K_2\). Then the following holds

(i) If \(F\) is a split face of \(K_1 \otimes K_2\) then \(F_1\) and \(F_2\) are split faces of \(K_1\) and \(K_2\).

(ii) If either \(A(F_1)\) or \(A(F_2)\) has the approximation property,
and $F_1$ and $F_2$ are split faces of $K_1$ and $K_2$, then $F$ is a split face of $K_1 \otimes K_2$.

Proof: To prove (i) we assume that $F$ is a split face. As noted before $\partial_e F = w(\partial_e F_1 \times \partial_e F_2)$. Let $a \in A(K_1)$ such that $a \geq 0$ on $F_1$, i.e. $a|_{F_1} \in A(F_1; K_1)^+$. By Proposition 3 it will suffice to show that $(a \cdot \chi_{F_1})^\wedge$ is affine on $K_1$. We know that $((a \otimes 1) \cdot \chi_F)^\wedge$ is u.s.c. and affine on $K_1 \otimes K_2$, since $a \otimes 1$ is non-negative on $w(F_1 \times F_2)$ and hence on $F$. Now we fix $x_2 \in \partial_e F_2$. Then the function $g(x_2) : x \mapsto ((a \otimes 1) \cdot \chi_F)^\wedge(w(x, x_2))$ is u.s.c. and affine on $K_1$. On $F_1$ $g(x_2)$ agrees with $a$, and since $w(\partial_e F_1' \times \partial_e F_2) \subseteq F'$, we have that $g(x_2) = 0$ on $\partial_e F_1'$.

Since $g(x_2)$ and $(a \cdot \chi_{F_1})^\wedge$ agree on $\partial_e K_1$, and $g(x_2)$ is u.s.c. affine, while $(a \cdot \chi_{F_1})^\wedge$ is u.s.c. concave it follows from Bauers principle [5; Lem.1] that $g(x_2) \leq (a \cdot \chi_{F_1})^\wedge$. Moreover $g(x_2) \geq a \cdot \chi_{F_1}$, and since $(a \cdot \chi_{F_1})^\wedge$ is the smallest u.s.c. concave majorant of $a \cdot \chi_{F_1}$, we have $g(x_2) = (a \cdot \chi_{F_1})^\wedge$, and (i) follows.

To prove (ii) we shall assume that $F_1$ and $F_2$ are split faces, and that $A(F_1)$ has the approximation property. By Proposition 3 we have to show that if $a \in A(F)^+$ then $a$ admits an extension $\tilde{a} \in A_s(K_1 \otimes K_2)$ such that $\tilde{a} = 0$ on $F'$. Now $a \circ (w_{K_1 \times K_2} | F_1 \times F_2)$ belongs to $BA(F_1 \times F_2) = A(F_1) \otimes_e A(F_2)$. If $\epsilon > 0$ is arbitrary we can choose $a_1, \ldots, a_n \in A(F_1)$ and $b_1, \ldots, b_n \in A(F_2)$ such that

$$\|a \circ w_{K_1 \times K_2} - \sum_{i=1}^n a_i \otimes b_i\|_{F_1 \times F_2} < \epsilon.$$ 

By Proposition 3 we can choose $\tilde{a}_i \in A_s(K_1)$, $\tilde{b}_i \in A_s(K_2)$ such that $\tilde{a}_i = a_i$ on $F_1$ and $\tilde{a}_i = 0$ on $F'_1$, while $\tilde{b}_i = b_i$. 

on $F_2$ and $\tilde{b}_i = 0$ on $F'_2$.

By Lemma 4 $\sum_{i=1}^{n} a_i \otimes \tilde{b}_i \in A_s(K_1 \otimes K_2)$ and on $w(F_1 \times F_2)$ it equals $\sum_{i=1}^{n} a_i \otimes \tilde{b}_i$, while $\sum_{i=1}^{n} \tilde{a}_i \otimes \tilde{b}_i = 0$ on $\partial_e(K_1 \otimes K_2) \setminus \partial_e F$.

As $A_s(K_1 \otimes K_2)$ is complete in $\| \|_{\partial e(K_1 \otimes K_2)}$ and the norm of $\sum_{i=1}^{n} \tilde{a}_i \otimes \tilde{b}_i$ is obtained at $w(F_1 \times F_2)$, this argument leads to the existence of $\tilde{a} \in A_s(K_1 \otimes K_2)$ such that $\tilde{a} = a$ on $w(F_1 \times F_2)$, and $\tilde{a} = 0$ on $\partial_e F' = \partial_e(K_1 \otimes K_2) \setminus F$. It remains to show that $\tilde{a} = a$ on $F$ and $\tilde{a} = 0$ on $F'$.

Now let $x \in F$ and represent $x$ by a probability measure on $w(F_1 \times F_2)$. Since $\tilde{a}$ satisfies the barycentric calculus we get

$$\tilde{a}(x) = \int_{K_1 \otimes K_2} \tilde{a} \, d\mu = \int_{w(F_1 \times F_2)} \tilde{a} \, d\mu = \int_{F} a \, d\mu = a(x)$$

and so $\tilde{a} = a$ on $F$.

To show that $\tilde{a} = 0$ on $F'$ we let $b \in A(K_1 \otimes K_2)$ with $b > 0$ on $K_1 \otimes K_2$ and $b > a$ on $F$. Then $b \geq \tilde{a}$ on $\partial_e(K \otimes K_2)$, and by Lemma 2 (i), $b \geq \tilde{a}$ on $K_1 \otimes K_2$. For $\rho \in K_1 \otimes K_2$ we have

$$(a \cdot \chi_F) \wedge (\rho) = \inf \{ b(\rho) \mid b \in A(K_1 \otimes K_2), b > a \cdot \chi_F \} \geq \tilde{a}(\rho) \geq 0.$$ 

Since $(a \cdot \chi_F) \wedge = 0$ on $F'$, we get $\tilde{a} = 0$ on $F'$, and the proof is complete.

Remark: It is easy to see from Lemma 4 that the embedding of the product of two parallel faces $F_1$ and $F_2$ in the sense of [11] gives rise to a parallel face $F$ without the assumption of the presence of the approximation property in $A(F_1)$. In fact, $\chi_F = \chi_{F_1} \otimes \chi_{F_2}$ is affine.
Theorem 6. Let \( F \) be a closed split face of a compact convex set \( K \). Let \( B \) be a real Banach space having the approximation property. Let \( p \) be a concave l.s.c. strictly positive real function on \( K \). Let \( a : F \to B \) be an affine continuous map such that
\[
\|a(k)\| \leq p(k), \text{ all } k \in F.
\]

Then \( a \) has an extension to a continuous affine map \( \tilde{a} : K \to B \) such that
\[
\|\tilde{a}(k)\| \leq p(k), \text{ all } k \in K.
\]

Proof: Let \( C \) be the unit ball of \( B^* \) with \( w^* \)-topology. \( B \times \mathbb{R} \) is normed by
\[
\|(x, r)\| = \|x\| + |r|.
\]
It was observed in [10] that \((x, r) \mapsto \cdot(x) + r\) is an isometric isomorphism of \( B \times \mathbb{R} \) onto \( A(C) \). Hence if \( B \) has the approximation property then \( A(C) \) has.

We define a biaffine continuous function \( b \) on \( F \times C \) by
\[
b(x, x^*) = x^*(a(x)), \text{ all } x \in F, x^* \in C
\]

By Proposition 1 (ii) there is an affine homeomorphism between \( F \otimes C \) and \( \overline{co}(w_{K \times C}(F \times C)) \) defined by
\[
T(p)(d) = \rho(d|_{F \times C}) \quad \text{for } d \in BA(K \times C).
\]

Since \( b \) is naturally a continuous affine function on \( F \otimes C \) there is a continuous affine function \( b_1 \) on \( \overline{co}(w_{K \times C}(F \times C)) \) such that
\[
b_1(T \circ w_{F \times C}(x, x^*)) = x^*(a(x)), \text{ all } (x, x^*) \in F \times C.
\]

Moreover \( \rho = \rho(P_1(\rho)) \) is concave, strictly positive and l.s.c. on \( K \otimes C \). For \( \rho \in \partial_e(\overline{co}(w_{K \times C}(F \times C))) = w_{K \times C}(\partial_e F \times \partial_e C) \) we have \( \rho = w_{K \times C}(x, x^*) \) with \((x, x^*) \in \partial_e F \times \partial_e C\) and hence
\[ |b_1(\rho)| = |x^*(a(x))| \leq \|a(x)\| \leq p(x) = p(P_1(\rho)) . \]

Since \( \rho - |b_1(\rho)| \) is convex and continuous and \( \rho \rightarrow p(P_1(\rho)) \) is concave and 1.s.c., it follows from Baur’s principle [5; Lem.1] that \( |b_1| \leq p \circ P_1 \) on \( \overline{co}(w_{K} \times C) \).

Now it follows from Theorem 5 that \( \overline{co}(w_{K} \times C) \) is a split face of \( K \otimes C \). By [1; Th.II.6.12] and [3; Th.2.2 and Th.4.5] it follows that there is a function \( c \in A(K \otimes C) \) such that \( c \) extends \( b_1 \) and

\[ |c(\rho)| \leq p(P_1(\rho)) , \text{ all } \rho \in K \otimes C . \]

(Actually, it follows from [1; Cor.I.5.2] that a concave 1.s.c. function on a compact convex set is \( A(K) \)-superharmonic in the sense of [3]. Moreover it should be remarked that the theorems 2.2 and 4.5 of [3] are stated for complex spaces, but the proofs hold almost unchanged for the real case.)

Now we can define a continuous affine map \( c_1 : K \rightarrow A(C) \) by

\[ c_1(k)(\cdot) = c(w(k, \cdot)) \]

Then for \( k \in K \)

\[ \|c_1(k)\| = \sup_{x^* \in C} \|c(w(k, x^*))\| \leq \sup_{x^* \in C} p(P_1(w(k, x^*))) = p(k) \]

By composing the isometry \( S \) between \( A(C) \) and \( B \times \mathbb{R} \) with the canonical projection \( Q \) from \( B \times \mathbb{R} \) to \( B \), which has norm 1, we get an affine continuous map \( \tilde{a} = Q \circ S \circ c_1 \) of \( K \) into \( B \) such that

\[ \|\tilde{a}(k)\| = \|(Q \circ S \circ c_1)(k)\| \leq \|c_1(k)\| \leq p(k) \]

for all \( k \in K \). Moreover, for \( k \in F \), \( x^* \in C \)

\[
\begin{align*}
    x^*(\tilde{a}(k)) &= x^*((Q \circ S \circ c_1)(k)) = c_1(k)(x^*) \\
    &= c(w(k, x^*)) = b_1(w(k, x^*)) = x^*(a(k))
\end{align*}
\]
Hence for \( k \in F \): \( \tilde{a}(k) = a(k) \).

Q.E.D.

**Corollary.** Let \( F \) be a closed split face of a compact convex set \( K \). Let \( B \) be a real Banach space having the approximation property. Let \( a : F \to B \) be a continuous affine map. Then \( a \) admits an extension to a continuous affine function \( \tilde{a} : K \to B \) such that \( \max_{k \in F} \|a(k)\| = \max_{k \in K} \|\tilde{a}(k)\| \).

**Remark:** Conclusions similar to those of Theorem 6 and the Corollary hold with no assumptions on \( B \), if instead we know that \( A(F) \) has the approximation property. This is f.ex. the case, if \( K \) is a simplex.
References


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