UNIFORM APPROXIMATION ON MANIFOLDS

By

John Erik Fornæss

1. Introduction.

Suppose that M is a real C^1-manifold of dimension m, and that Φ is a family of complex-valued C^1-functions on M. Then the exceptional set, $E(\Phi)$, is the set $\{x \in M;\ df_1 \wedge \ldots \wedge df_m(x) = 0, \forall (f_1, \ldots, f_m) \in \Phi^m\}$. We fix a compact subset X of M, and we shall often write E instead of $E(\Phi) \cap X$.

Let $A \subset C(X)$ denote the closed Banach-algebra generated by the restriction to X of the elements of Φ. Assume that A separates points in X and that $M_A = X$, where M_A is the maximal ideal space of A. It is an open problem, see [1] page 348-349, if A includes all continuous functions on X which vanish identically on E. Michael Freeman proved this in [2] under the additional hypothesis that both M and the functions in Φ are real-analytic. In this work we will solve the problem if M and the functions in Φ are of class C^r, for some sufficiently large real r.

Our result will be proved via the following corollary of theorem 3.1: If Σ is a C^r-manifold in $\mathbb{C}^n$ without complex tangents (see [4]for the precise meaning of the last term) and $K = \sigma(f_1,\dots,f_n)$ is the spectrum of some members of A, then all continuous functions on K which vanish on $K - \Sigma$ operate on K.

The proof will follow by adaptation of a technique developed in the work of Hörmander and Wermer [4].

2. Fundamental Constructions.

Assume that $r \geq 1$ and that Σ is a closed, real C^r-submanifold, without complex tangents, of an open set Ω in $\mathbb{C}^n$. Let N_1 and N_2 be some open sets in $\mathbb{C}^n$, $\bar{N}_2 \subset N_1$.

The Euclidean distance between the point x and the set A will be denoted $d(x,A)$.

<u>LEMMA 2.1</u>. Suppose that $u \in C^r(\Omega \cup N_1)$ is holomorphic in N_1. Then there exists a $v \in C^r(\Omega \cup N_2)$ with $v = u$ on $\Sigma \cup N_2$ and such that:

For every compact $F \subset \Omega \cup N_2$ and every $\eta > 0$ we can find a $\delta > 0$ with the property:

If $z \in F$ and $d(z,\Sigma) < \delta$, then $|\bar{\partial} v(z)| \leq \eta \cdot d(z,\Sigma)^{r-1}$.

Since the proof of this lemma is similar to that of lemma 4.3. in [4], we omit it here.

The next result is similar to theorem 3.1. in [4]. However, since the proof is a bit different, we will carry it out in some detail.

Σ, Ω, N_1 and N_2 will be as above with $r = 1$. Suppose A is a commutative Banach-algebra with unit, and let $f_1,\dots,f_n$ be elements of A. Define K to be the joint spectrum $\sigma(f_1,\dots,f_n)$.

LEMMA 2.2. Assume $K - N_2 \subset \Sigma$. Then there exists an $\varepsilon_o > 0$, a $t \in \langle 0,1\rangle$, elements $f_{n+1},\ldots,f_m \in A$, a compact set $F \subset \Sigma$, and for every $\varepsilon \in \langle 0,\varepsilon_o\rangle$ a domain of holomorphy $\omega_\varepsilon \subset \mathbb{C}^m$ such that

(i) $\sigma(f_1,\ldots,f_m) \subset \omega_\varepsilon \subset \mathbb{C}^n \times \{|(z_{n+1},\ldots,z_m)| < 1/t\}$,

(ii) if $z \in \mathbb{C}^m$ and

$d((z_1,\ldots,z_n,\ \varepsilon z_{n+1},\ldots,\varepsilon z_m),\ \sigma(f_1,\ldots,f_n,\ \varepsilon f_{n+1},\ldots,\varepsilon f_m)) < t\varepsilon$, then $z \in \omega_\varepsilon$,

(iii) if $z \in \omega_\varepsilon - (N_1 \times \mathbb{C}^{m-n})$, then $d((z_1,\ldots,z_n),F) < \frac{\varepsilon}{t}$.

PROOF. As in lemma 2.1 in [4] we can find a function $\rho \in C^2(\Omega)$ which is strictly plurisubharmonic in an open set W, with $\Sigma \subset W \subset \Omega$, and which satisfies the inequality

(2.1.) $$\tfrac{1}{2}d(z,\Sigma)^2 \leq \rho(z) \leq 2d(z,\Sigma)^2, \quad z \in W.$$

Choose an open set N_3, $\bar{N}_2 \subset N_3 \subset \bar{N}_3 \subset N_1$. Define $K_1 = K \cap (\bar{N}_3 - N_2)$. K_1 will evidently be a compact subset of $N_1 \cap W$. We can now select $\psi \in C_o^\infty(\mathbb{C}^n)$ with $0 \leq \psi \leq 1$, supp $\psi \subset N_1 \cap W$ and so that $\psi \equiv 1$ in an open neighbourhood of K_1. By taking a sufficiently small $\delta > 0$, we can assume that $\rho - \delta^2\psi$ is strictly plurisubharmonic in W.

Let V_o be an open set which contains K_1, and where $\psi \equiv 1$ and $\rho < \frac{1}{2}\delta^2$. Define

(2.2.) $$V = V_o \cup (\mathbb{C}^n - \bar{N}_3) \cup N_1$$

Since $K \subset V$, we can find a finite number of elements $f_{n+1},\ldots,f_m \in A$, and a relatively compact domain of holomorphy U in C^m, such that

(2.3.) $$\sigma(f_1,\ldots,f_m) \subset U \subset V \times \mathbb{C}^{m-n}.$$

Furthermore, we define $K_2 = K - N_3 \subset \Sigma \subset W$ and find a $\varphi \in C_0^\infty(W)$ with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in an open neighbourhood of K_2. Now we choose $\varepsilon_1 > 0$ with the property that $p_\varepsilon = \rho - \varepsilon^2\varphi - \delta^2\psi$ is strictly plurisubharmonic in $W \; \forall \varepsilon \in \langle 0, \varepsilon_1]$.

There exists a compact set $F \subset \Sigma$ and an $\varepsilon_2 \in \langle 0, \varepsilon_1]$ such that we have, for all $\varepsilon \in \langle 0, \varepsilon_2\rangle$:

(i) If $z \in \mathbb{C}^n$ and $d(z, K - N_2) < \varepsilon/2$, then

(2.4.) $\quad z \in W$ and $p_\varepsilon(z) < 0$.

(ii) When $z \in W - N_1$ and $p_\varepsilon(z) < 0$, then $d(z,F) < 2\varepsilon$.

In fact, (i) follows easily from (2.1.) and the choices of ψ and φ by remarking that $K - N_2 = K_1 \cup K_2$.

As for (ii) we can find an ε_2 so small that $\{z \in \mathbb{C}^n; d(z, \operatorname{supp}\varphi) \leq 2\varepsilon_2\} \subset W$. Define $F = \{z \in \mathbb{C}^n; d(z, \operatorname{supp}\varphi) \leq 2\varepsilon_2\} \cap \Sigma$. Then F is a compact subset of Σ. Suppose that $z \in W - N_1$ and that $p_\varepsilon(z) < 0$. It follows that $\psi(z) = 0$, and so $\rho(z) < \varepsilon^2\varphi(z) \leq \varepsilon^2$. By (2.1.) there exists an $\eta \in \Sigma$ so that $d(z,\eta) < 2\varepsilon$. Consequently $\eta \in F$ and $d(z,F) < 2\varepsilon$.

In the rest of the proof we will use the convention that $z \in \mathbb{C}^m$ is written (z_1, z_2) where $z_1 \in \mathbb{C}^n$ and $z_2 \in \mathbb{C}^{m-n}$.

We are now able to construct the required domains of holomorphy. Define

$$O_1 = \{(z_1,z_2) \in \mathbb{C}^m; z_1 \in N_3\}$$
$$O_2 = \{(z_1,z_2) \in \mathbb{C}^m; z_1 \in \mathbb{C}^n - \bar{N}_2\}$$
$$O_3 = \{(z_1,z_2) \in \mathbb{C}^m; z_1 \in W \text{ and } p_\varepsilon(z_1) < 0\}$$

Then $O_1 \cup O_2 = \mathbb{C}^m$. Suppose $z \in O_1 \cap O_2 \cap U$. From (2.3.) we get that $z_1 \in V$, and since $z_1 \in N_3 - \bar{N}_2 \subset \bar{N}_3 - N_2$, (2.2.) implies that $z_1 \in V_0$. It follows from the definition of V_0 that

$\psi(z_1) = 1$ and $\rho(z_1) < \frac{1}{2}\delta^2$. Therefore $p_\varepsilon(z_1) < 0$, which proves that $O_1 \cap O_2 \cap U \subset O_3$. As a trivial consequence there exist unique domains of holomorphy ω_ε $\forall \varepsilon \in <0,\varepsilon_2>$ with

$$(2.5.) \quad \begin{array}{ll} \text{(i)} & \omega_\varepsilon \cap O_1 = U \cap O_1 \\ \text{(ii)} & \omega_\varepsilon \cap O_2 = U \cap O_2 \cap O_3 \end{array}$$

Since $\sigma(f_1,\dots,f_m) \subset U$ by (2.3.), we can find $t_1 > 0$ so that $\{z \in C^m; d(z,\sigma(f_1,\dots,f_m)) < t_1\} \subset U$. In addition, $K \cap \bar{N}_2 \subset N_3$. This implies the existence of $\varepsilon_o \in <0,\min\{t_1,\varepsilon_2\}>$ with the property that

$$\{z \in \mathbb{C}^n; d(z,K \cap \bar{N}_2) < \tfrac{1}{2}\varepsilon_o\} \subset N_3 .$$

Assume $\varepsilon \in <0,\varepsilon_o>$. Then:

$$(2.6.) \quad \begin{array}{ll} \text{(i)} & \text{If } z \in \mathbb{C}^m \text{ and } \eta \in \sigma(f_1,\dots,f_m) \text{ are such that} \\ & |z_1 - \eta_1| < \tfrac{1}{2}\varepsilon \text{ and } |z_2 - \eta_2| < \tfrac{1}{2}t_1 \text{ , then } z \in \omega_\varepsilon . \\ \text{(ii)} & \text{If } z \in \omega_\varepsilon - (N_1 \times \mathbb{C}^{m-n}) \text{ , then } d(z_1,F) < 2\varepsilon . \end{array}$$

We prove (i) first. Since $d(z,\sigma(f_1,\dots,f_m)) \leq \frac{1}{2}\varepsilon + \frac{1}{2}t_1 < t_1$, it is obvious that $z \in U$. Suppose that $\eta_1 \in N_2$. By the definition of ε_o, $z_1 \in N_3$. It follows from (2.5.) that $z \in \omega_\varepsilon$. Next, consider the situation when $\eta_1 \in K - N_2$. Then $z_1 \in N_3$ or $z_1 \in \mathbb{C}^n - \bar{N}_2$. If $z_1 \in N_3$, then $z \in \omega_\varepsilon$ as before. But if $z_1 \in \mathbb{C}^n - \bar{N}_2$, then $p_\varepsilon(z_1) < 0$ by (2.4.), and so (2.5.) shows that $z \in \omega_\varepsilon$.

At last we remark that (ii) follows from (2.4.).

The lemma is now an immediate consequence.

Q.E.D.

By applying a technique introduced by M. Freeman in [2], we shall now determine a class of mappings of a manifold into $\mathbb{C}^n$.

More precisely, let M be a k-dimensional real C^r-manifold, $r \geq 1$. Suppose X is a compact subset of M. Assume further that $\Phi \subset C^r_{\mathbb{C}}(M)$ and separates points in X. Let E denote the exceptional set $E(\Phi) \cap X$.

LEMMA 2.3. For every compact set $X_o \subset X - E$ we can find an open neighbourhood V of X_o, a finite number of functions, $f_1,\ldots,f_n \in \Phi$, and an open $\Omega \subset \mathbb{C}^n$ such that

(i) $(f_1,\ldots,f_n)(V)$ is a closed C^r-submanifold of Ω of dimension k and without complex tangents, and

(ii) $(f_1,\ldots,f_n)(X - V) \subset \mathbb{C}^n - \Omega$.

PROOF. Choose a finite number of functions $f_1,\ldots,f_n \in \Phi$ and an open neighbourhood V of X_o with $E(\{f_1,\ldots,f_n\}) \cap V = \emptyset$. It follows from the inverse mapping theorem that the multiple function $(f_1,\ldots,f_n)\colon M \to \mathbb{C}^n$ is locally $1-1$ on V. Obviously, the set $\{(x,y) \in (X_o \times X_o) - \Delta; f_i(x) = f_i(y)\ \forall i = 1,\ldots,n\}$ is a compact subset of $X_o \times X_o$, where Δ denotes the diagonal in $X_o \times X_o$. Consequently, by adding some more functions if necessary, we may assume that $\{f_1,\ldots,f_n\}$ separate points in X_o. Shrinking V if necessary, we then get that $(f_1,\ldots,f_n)$ is $1-1$ on $\bar{V}$ and that $\bar{V}$ is compact.

Since Φ separates points in X, we can also suppose, after further modifications, that $(f_1,\ldots,f_n)(X - V)$ and $(f_1,\ldots,f_n)(V)$ are disjoint.

The choice $\Omega = \mathbb{C}^n - (f_1,\ldots,f_n)(\bar{V} - V)$ finishes the proof.

Q.E.D.

3. Approximation Theorems.

Let X be a compact Hausdorff space, and let $C(X)$ denote the Banach space under the supremum norm of continuous complex-valued functions. The notation $A \propto C(X)$ means that A is a closed linear subspace which is closed under pointwise multiplication, separates points and contains the constant functions.

If K is a compact subset of $\mathbb{C}^n$, then $A(K)$ is defined to be the class of continuous, complex-valued functions on K which can be uniformly approximated on K by functions holomorphic in a neighbourhood of K.

THEOREM 3.1. Suppose $A \propto C(X)$, where X is compact Hausdorff space. Let Σ be a closed k-dimensional submanifold of an open set $\Omega \subset \mathbb{C}^n$, without complex tangents, and of class C^r, $r = \frac{k}{2} + 1$. Choose $f_1, \ldots, f_n \in A$ and define $K = \sigma(f_1, \ldots, f_n)$, $K_0 = \overline{K - \Sigma}$.

If $u \in C(K)$ with $u_{|K_0} \in A(K_0)$, then $u \circ (f_1, \ldots, f_n) \in A$.

REMARK. We can replace the condition $r = \frac{k}{2} + 1$ with $r = \max\{\frac{k}{2}, 1\}$, but the proof will then be more involved. More specific, we need a stronger version of lemma 2.2.

PROOF OF THEOREM 3.1. Evidently, we may assume that $u \in C^r(\Omega \cup N_1)$ for some open neighbourhood N_1 of K_0, and also that u is holomorphic in N_1. We will further suppose that u has been modified as described in lemma 2.1. If N_2 is chosen as an open set with $K_0 \subset N_2 \subset\subset N_1$, we can apply lemma 2.2.

Define $v: (\Omega \cup N_1) \times \mathbb{C}^{m-n} \to \mathbb{C} : (z_1, \ldots, z_n) \to u(z_1, \ldots, z_n)$. Condition (iii) in lemma 2.2. ensures that v is defined in ω_ε for all small enough $\varepsilon > 0$.

Since Σ is k-dimensional, and because of (i) and (iii) in lemma 2.2., the volume of $\omega_\varepsilon - (N_1 \times \mathbb{C}^{m-n})$ is $O(\varepsilon^{2n-k})$ when $\varepsilon \to 0$. Combining lemma 2.1. and once again (iii) in lemma 2.2., we get

$$\|\bar\partial v\|^2_{L^2(\omega_\varepsilon)} = o(\varepsilon^{2n-k} \cdot \varepsilon^{2r-2})$$

and consequently

$$\|\bar\partial v\|_{L^2(\omega_\varepsilon)} = o(\varepsilon^n) \text{ when } \varepsilon \to 0 .$$

According to theorem 2.2.3. in [3] there exist functions $h_\varepsilon \in L^2(\omega_\varepsilon)$ with $\bar\partial h_\varepsilon = \bar\partial v$ in ω_ε, and

$$\|h_\varepsilon\|_{L^2(\omega_\varepsilon)} = o(\varepsilon^n) \text{ when } \varepsilon \to 0 .$$

Now we need lemma 4.4. in [4]:

Let $B_\varepsilon = \{z \in \mathbb{C}^m;\ |z| < \varepsilon\}$; let $u \in L^2(B_\varepsilon)$ and $\bar\partial u = f$ in the sense of distribution theory. If f is continuous, then u is continuous, and we have

$$|u(0)| \leq C(\varepsilon^{-m}\|u\|_{L^2(B_\varepsilon)} + \varepsilon \sup_{B_\varepsilon}|f|)$$

Here C is a constant which only depends on m, and not on ε.

Define $\omega_\varepsilon^1 = \{(z_1,\ldots,z_n, \varepsilon z_{n+1},\ldots,\varepsilon z_m)\ ;\ (z_1,\ldots,z_m) \in \omega_\varepsilon\}$, $v_\varepsilon^1 : \omega_\varepsilon^1 \to \mathbb{C} : (z_1,\ldots,z_m) \to u(z_1,\ldots,z_n)$ and $h_\varepsilon^1 : \omega_\varepsilon^1 \to \mathbb{C} : (z_1,\ldots,z_m) \to h_\varepsilon(z_1,\ldots,z_n, \frac{1}{\varepsilon}z_{n+1},\ldots,\frac{1}{\varepsilon}z_m)$.

Then $\bar\partial h_\varepsilon^1 = v_\varepsilon^1$ and $\|h_\varepsilon^1\|_{L^2(\omega_\varepsilon)} = o(\varepsilon^m)$ when $\varepsilon \to 0$. By the above lemma and (ii) in lemma 2.2., we get uniformly for all $x \in X$

$$|h_\varepsilon^1(f_1(x),\ldots,f_n(x), \varepsilon f_{n+1}(x),\ldots,\varepsilon f_m(x)| \leq$$
$$C((t\varepsilon)^{-m} \cdot o(\varepsilon^m) + \varepsilon \cdot o(\varepsilon^{r-1})) .$$

Consequently

$$\|h_\varepsilon\|_{L^\infty(\sigma(f_1,\ldots,f_m))} \to 0 \quad \text{when} \quad \varepsilon \to 0 .$$

Therefore, the functions $v - h_\varepsilon$ are holomorphic in ω_ε and converge uniformly on $\sigma(f_1,\ldots,f_m)$ to v. It is well-known that holomorphic functions operate on spectra. This concludes the proof.

Q.E.D.

We are now able to prove a generalization of a result by Freeman [2].

<u>THEOREM 3.2</u>. Let M be a k-dimensional real manifold of class C^r, $r = \frac{k}{2}+1$. Suppose that $\Phi \subset C^r(M)$ separates points on a compact subset X of M. Define $E = E(\Phi) \cap X$ and A = the supnormalgebra in $C(X)$ generated by Φ. If $M_A = X$, then

$$A \supset \{g \in C(X);\ g_{|E} \equiv 0\} .$$

<u>PROOF</u>. Choose any compact subset $X_o \subset X - E$ and use lemma 2.3. It follows from theorem 3.1. that the family

$$\{g \in A \cap C_{\mathbb{R}}(X);\ g_{|E} \equiv 0 \text{ and } 0 \notin g(X_o)\}$$

is nonempty and separates points in X_o. The theorem now is a consequence of Stone-Weierstrass.

Q.E.D.

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