Killing cohomology classes by surgery.

Jon Reed

1970

1. Introduction.

1.1. We consider the following problem. If x is a cobordism class of some sort, what is a necessary and sufficient condition, in terms of characteristic numbers, that x contains a manifold for which certain given characteristic classes vanish? Solutions are well known in many special cases.

1.2. As an example we consider the case of unoriented cobordism. If M is a differentiable manifold and $u \in H^k(BO;Z_2)$, we write uM for the tangential characteristic class of M corresponding to u . Suppose $x \in \mathfrak{N}_n$ is an unoriented cobordism class, and let $u_1,\ldots,u_r \in H^*(BO;Z_2)$. A necessary condition for x to contain a manifold M such that $u_1M = \ldots = u_rM = 0$ is clearly that all Stiefel-Whitney numbers of x involving one of the classes $u_1,\ldots,u_r$ as a factor are zero. We consider therefore the problem of the sufficiency of this condition.

1.3. For the single Stiefel-Whitney class w_1 it is a result of Wall [22] that the condition of 1.2 is sufficient, giving a description of the classes x that contain oriented manifolds.

For the collection $w_1,w_3,\ldots$ of odd dimensional Stiefel-Whitney classes the sufficiency follows from Milnor [12], where the stronger result is proved that if all Stiefel-Whitney numbers of x involving an odd dimensional Stiefel-Whitney class are zero, then x contains a weakly complex manifold.

For the collection w_1,w_2 the condition is sufficient according to D.W. Anderson, E.H. Brown and F.P. Peterson [1], thus characterizing the classes x that contain spin manifolds. A few similar cases are known in low dimensions, see Stong [16].

The sufficiency of the condition for some high dimensional cases is contained in R.L.W. Brown [3] and [4]. Here results are proved of the form that if all Stiefel-Whitney numbers of x involving certain classes are zero then x contains a manifold which is a bundle over a sphere or which can be imbedded or immersed in certain Eucledian spaces.

1.4. The following example due to Stong [15] shows that the condition of 1.2 is not always sufficient. Let $x = y^{2^s}$ where $y \in \eta_5 = Z_2$ is the generator. Then all Stiefel-Whitney numbers of x involving one of the classes $w_1, \ldots, w_{2^s}$ are zero, but for $s \geq 4$ there is no manifold M in x such that $w_1 M = \ldots = w_{2^s} M = 0$.

1.5. The problem of 1.1 can be considered for different cobordism theories. A general result for complex cobordism is given in Lashof [7] as follows. Let $u \in H^k(BU; Z)$. Then a complex cobordism class $x \in \Omega_n^U$ contains a manifold M such that $uM = 0$ mod torsion if and only if all Chern numbers of x involving u are zero.

1.6. In the following we shall show that with a dimensional restriction on the characteristic classes, the problem of 1.1 can be solved under general circumstances. The assumption we have to make is that the characteristic classes lie in the upper half of the dimensions. The result is then the simplest one could expect, namely that a collection of characteristic classes of the appropriate type can be zero for a manifold in x if and only if none of the classes show up in the characteristic numbers of x .

1.7. To state the results precisely we adopt the cobordism concept of Lashof [7], where a structure on a manifold M is defined in terms of a lifting $\nu_M : M \to BG$ of a classifying map $\nu_M : M \to BO$ for the stable normal bundle of M. If π is a ring and $u \in H^*(BG;\pi)$, we define a corresponding normal characteristic class $\bar{u}M = \nu_M^* u \in H^*(M;\pi)$. If the Thom spectrum MG is π-oriented, we get a normal characteristic number $\langle \bar{u}M, [M]_\pi \rangle \in \pi$, where $[M]_\pi \in H_n(M;\pi)$ is the fundamental class. We denote by Ω_*^G the G-cobordism group. See section 2 for details. From 3.5 we get the following two theorems for coefficient rings Z_m with $m \geq 2$, and Z respectively. The proofs make use of some general assumptions on the cobordism theory, stated in 2.8. For example the theorems apply to the standard cases $G = O, SO, Spin, U, SU$ and $G = O\langle q\rangle, U\langle q\rangle$. By a separate argument they are also valid for the piecewise linear cases $G = PL, SPL$.

1.8. **Theorem.** Let $x \in \Omega_n^G$ and let $u_1, \ldots, u_r \in H^*(BG;Z_m)$ be classes in dimensions $> (n+1)/2$. Assume MG oriented if $m > 2$. Then x contains a manifold M such that $\bar{u}_1 M = \ldots = \bar{u}_r M = 0$ if and only if $\langle \bar{v}\bar{u}_i M, [M]_m \rangle = 0$ for all $v \in H^*(BG;Z_m)$, $i = 1, \ldots, r$.

1.9. **Theorem.** Let $x \in \Omega_n^G$ and let $u_1, \ldots, u_r \in H^*(BG;Z)$ be classes in dimensions $> (n+1)/2$. Assume MG oriented. Then x contains a manifold M such that

1) $\bar{u}_1 M = \ldots = \bar{u}_r M = 0$ mod torsion if and only if $\langle \bar{v}\bar{u}_i M, [M] \rangle = 0$ for all $v \in H^*(BG;Z)$, $i = 1, \ldots, r$.

2) $\bar{u}_1 M = \ldots = \bar{u}_r M = 0$ if and only if $\langle \bar{v}\bar{u}_i M, [M] \rangle = 0$ for all $v \in H^*(BG;Z)$ and $\langle \bar{w}\bar{u}_i M, [M]_m \rangle = 0$ for $w \in H^*(BG;Z_m)$, where m runs through the values for which $H^*(BG;Z)$ has torsion of order m, $i = 1, \ldots, r$.

1.10. The theorems above are formulated in terms of normal characteristic classes since the structure on the manifolds is given by liftings of the stable normal bundles. But we note that there is a homotopy involution $I: BO \to BO$ such that $I\nu_M = \tau_M$. In the case $G = 0$ we can therefore give the result in terms of tangential classes, and similarly for other standard cases.

1.11. The theorem 1.9 does not apply to $G = 0$ or $G = PL$ since the Thom spectra for these cases are not orientable. However for $G = 0$ we get the following from 3.6,

<u>Theorem.</u> Let $x \in \Pi_n$ and let $u_1, \ldots, u_r \in H^*(BO; Z)$ be classes in dimensions $> (n+1)/2$. Then x contains a manifold M such that $u_1 M = \ldots = u_r M = 0$ if and only if $<vu_i M, [M]_2> = 0$ for all $v \in H^*(BO; Z_2)$, $i = 1, \ldots, r$.

and the analogous result is true for $G = PL$.

2. Some preliminaries on bordism theories.

2.1. We recall first some standard results about spectra and homology theories. All spaces considered will be assumed to have the homotopy type of CW-complexes and the homology theories to be defined on the category $\mathcal{C}$ of pairs having the homotopy type of CW-pairs. A spectrum in the sense of G. Whitehead [23] is a sequence $E = \{E_r, \epsilon_r\}_{r \in \mathbb{Z}}$, where E_r is a space with base point and $\epsilon_r: SE_r \to E_{r+1}$ is a base point preserving map. A spectrum E defines a homology theory where the homology groups are given by

$$H_n(X,A;E) = \varinjlim_r \pi_{n+r}(X/A \wedge E_r)$$

If X is a space with base point, we have reduced homology groups $\tilde{H}_n(X;E) = H_n(X,*;E) = \varinjlim_r \pi_{n+r}(X \wedge E_r)$, and there is a short exact sequence $0 \to H_n(*;E) \to H_n(X;E) \to \tilde{H}_n(X;E) \to 0$.

2.2. The homomorphisms in the direct system above are by definition the compositions

$$\pi_{n+r}(X/A \wedge E_r) \xrightarrow{\ S\ } \pi_{n+r+1}(X/A \wedge SE_r) \xrightarrow{(1 \wedge \epsilon_r)_*} \pi_{n+r+1}(X/A \wedge E_{r+1})$$

where S is the suspension homomorphism. Assume that E_r is $(r-1)$-connected and that $\epsilon_r: SE_r \to E_{r+1}$ is a $2r$-equivalence for all r . Let X/A be $(k-1)$-connected. Then $X/A \wedge E_r$ is $(k+r-1)$-connected, and by the suspension theorem, Spanier [14], S is an isomorphism for $r \geq n-2k+2$ Furthermore $(1 \wedge \epsilon_r)_*$ is an isomorphism for $r \geq n-k+2$. Hence the direct system is stable for $r \geq n-k+2$.

2.3. Theorem (switching spectra). Let E and F be spectra satisfying the connectivity condition of 2.2. Then there is an

isomorphism $\sigma\colon \widetilde{H}_n(F_k;E) \to \widetilde{H}_n(E_k;F)$ for $n \leq 2k-2$.

Proof. By 2.2 we have

$$\widetilde{H}_n(F_k;E) = \pi_{n+r}(F_k \wedge E_r)$$

$$\widetilde{H}_n(E_k;F) = \pi_{n+r}(E_k \wedge F_r)$$

for $r \geq n-k+2$. Taking $k = r$ we get the isomorphism σ from the canonical homeomorphism $F_k \wedge E_k \to E_k \wedge F_k$.

2.4. If π is an abelian group, the Eilenberg-MacLane spectrum $K(\pi,) = \{K(\pi,k)\}$ satisfies the connectivity condition of 2.2. The corresponding homology theory coincides with the singular homology theory with coefficients π on the category $\mathfrak{C}$.

2.5. We recall the general cobordism concept introduced by Lashof [7]. Suppose given a sequence of fibrations $p\colon BG_k \to BO_k$ together with fiber maps $i\colon BG_k \to BG_{k+1}$ over the natural maps $i\colon BO_k \to BO_{k+1}$, so that we have a commutative diagram

$$\begin{array}{ccc} \cdots \to & BG_k & \overset{i}{\to} BG_{k+1} \to \cdots \\ & p\downarrow & \downarrow p \\ \cdots \to & BO_k & \underset{i}{\to} BO_{k+1} \to \cdots \end{array}$$

Passing to the direct limit we get a space $BG = \varinjlim_k BG_k$ and a fibration $p\colon BG \to BO$. (We use the telescope construction.)

If M is a compact differentiable n-manifold, we define a G-<u>structure</u> on M to be a vertical homotopy class of liftings in the diagram

$$\begin{array}{ccc} & & BG \\ & \nearrow & \downarrow p \\ M & \underset{\nu_M}{\to} & BO \end{array}$$

where ν_M is a classifying map for the stable normal bundle of M . We call M together with a G-structure a G-manifold. There is a difficulty with this definition due to the fact that ν_M is determined only up to homotopy, and there is in general no canonical bijection between the vertical homotopy classes of liftings of two homotopic maps. To get around this we choose an immersion of M in $\mathbb{R}^{n+k}$ and use as a classifying map ν_M the composition $M \overset{g}{\to} G_k(\mathbb{R}^{n+k}) \hookrightarrow BO_n \overset{i}{\to} BO$, where g is the normal Gauss map. ν_M depends then only on the choice of immersion, and is therefore unique up to a homotopy which is itself unique up to a homotopy with stationary ends. The vertical homotopy classes of liftings of such maps ν_M correspond now bijectively in a unique way. A G-structure on M is precisely a set of corresponding classes of liftings. It follows that if M is a G-manifold, there is a lifted map ν_M: M → BG unique up to homotopy. A G-structure on M induces one on ∂M , once we make a choice between inward or outward normal along the boundary.

2.6. Let Ω_*^G denote the G-bordism theory defined on the category $\mathcal{b}$. The elements of $\Omega_n^G(X,A)$ are G-bordism classes [M,f] represented by a G-manifold M of dimension n and a map f: (M,∂M) → (X,A) .

Let γ_k be the k-vector bundle over BG_k classified by p: BG_k → BO_k , and let $MG_k = D\gamma_k/S\gamma_k$ be the Thom space of γ_k . The natural bundle maps $\gamma_k + 1 \to \gamma_{k+1}$ over i: $BG_k \to BG_{k+1}$ induce maps $SMG_k \to MG_{k+1}$ defining the Thom spectrum $MG = \{MG_k\}$. The Pontrjagin-Thom construction gives a natural equivalence $\Omega_*^G = H_*(\ ;MG)$ of homology theories.

2.7. Let π be a ring and $v \in H^*(BG;\pi)$. If M is a G-manifold with lifted map $\nu_M\colon M \to BG$, we define the <u>normal characteristic class</u> $\bar{v}M = \nu_M^* v \in H^*(M;\pi)$. Assume the Thom spectrum MG is π-oriented. Let $f\colon M \to X$ be a map of a closed G-manifold M of dimension n and $c \in H^*(X;\pi)$. Let $[M]_\pi \in H_n(M;\pi)$ denote the fundamental class. Then we get a <u>normal characteristic number</u> $\langle\bar{v}M\; f^*c,[M]_\pi\rangle \in \pi$. By Pontrjagins theorem it depends only on the G-bordism class $[M,f] \in \Omega_n^G X$. Stong [16] .

2.8. For all applications we make the assumptions that BG_k has the homotopy type of a CW-complex, that $i\colon BG_k \to BG_{k+1}$ is a k-equivalence and that $H_*(BG;\mathbb{Z})$ is of finite type. These assumptions are for example satisfied for the standard cases $G = O,SO,$ Spin,U,SU and $G = O\langle q\rangle, U\langle q\rangle$, mentioned in the introduction.

It follows that the Thom spectrum MG satisfies the connectivity condition of 2.2. $MG_k = M\gamma_k$ is (k-1)-connected as the Thom complex of a k-vector bundle. Furthermore we have

$$
\begin{array}{ccccc}
SMG_k & \to & MG_{k+1} & \to & (MG_{k+1}, SMG_k) \\
\| & & \| & & \| \\
M(\gamma_k+1) & \to & M\gamma_{k+1} & \to & (M\gamma_{k+1}, M(\gamma_k+1))
\end{array}
$$

The term $(M\gamma_{k+1}, M(\gamma_k+1))$ is the relative Thom complex of γ_{k+1} over the k-connected pair (BG_{k+1}, BG_k) . Hence $(M\gamma_{k+1}, M(\gamma_k+1))$ is (2k+1)-connected, which means that $SMG_k \to MG_{k+1}$ is a (2k+1)-equivalence.

2.9. Let E be a spectrum with the connectivity property of 2.2 and suppose the Thom spectrum MG is E-oriented, Stong [16]. We define for $n \le 2k-2$ an isomorphism

$$
\psi\colon \widetilde{\Omega}_n^G\; E_k \;\tilde{\rightleftarrows}\; H_{n-k}(BG;E)
$$

by switching spectra and using the Thom isomorphism. In other words ψ is the composition

$$\widetilde{\Omega}_n^G E_k \overset{\sigma}{\underset{\approx}{\to}} \widetilde{H}_n(MG_k;E) \overset{\varphi}{\underset{\approx}{\to}} H_{n-k}(BG_k;E) \overset{i^*}{\underset{\approx}{\to}} H_{n-k}(BG;E)$$

where φ is the Thom isomorphism. This isomorphism ψ is very useful. We describe it more explicitly in the cases where E is $K(\pi,)$ or MO .

2.10. Let π be a ring as before and assume MG is π-oriented. We consider the isomorphism $\psi: \widetilde{\Omega}_n^G K(\pi,k) \underset{\approx}{\to} H_{n-k}(BG;\pi)$ in the stable range $n \leq 2k-2$. Elements of $\widetilde{\Omega}_n^G K(\pi,k)$ are of the form [M,c] , where M is a closed G-manifold of dimension n and $c \in H^k(M;\pi)$. Then we have

<u>Theorem.</u> $\langle v,\psi[M,c]\rangle = \pm \langle \overline{v}Mc,[M]_\pi\rangle$ for all $v \in H^*(BG;\pi)$.

Proof: There is a commutative diagram with $K_k = K(\pi,k)$ and $\rho : E \to M$ the projection

$$E/\partial E \xrightarrow{(c\rho,\nu_M)} K_k \times D\gamma_r / K_k \times S\gamma_r \cup * \times D\gamma_r$$

where E is a tubular neighborhood of M and q is the collapsing map. Let $v \in H^{n-k}(BG;\pi) = H^{n-k}(BG_k;\pi)$. If we pull back the class $\varphi v \wedge \iota_r$ and evaluate on $[S^{n+r}]$, we get $\langle \varphi v,\sigma[M,c]\rangle = \langle v,\varphi\sigma[M,c]\rangle = \langle v,\psi[M,c]\rangle$ on one hand, and on the other hand

$<\bar{v}Mc,[M]_\pi>$ up to sign.

2.11. Since we assume $H_*(BG;\mathbb{Z})$ is of finite type the natural homomorphism

$$H_i(BG;\mathbb{Z}_m) \rightarrow \mathrm{Hom}(H^i(BG;\mathbb{Z}_m),\mathbb{Z}_m)$$

is an isomorphism (m not necessarily prime!). Therefore, applying 2.10 with $\pi = \mathbb{Z}_m$, we get, assuming MG oriented if $m > 2$,

Corollary. $[M,c] \in \widetilde{\Omega}_n^G K(\mathbb{Z}_m,k)$ is zero if and only if $<\bar{v}Mc,[M]_m> = 0$ for all $v \in H^*(BG;\mathbb{Z}_m)$, $n \leq 2k-2$.

2.12. We consider next the case $\pi = \mathbb{Z}$. There is a short exact sequence

$$0 \rightarrow \mathrm{Ext}(H^{n-k+1}(BG;\mathbb{Z}),\mathbb{Z}) \rightarrow H_{n-k}(BG;\mathbb{Z}) \rightarrow \mathrm{Hom}(H^{n-k}(BG;\mathbb{Z}),\mathbb{Z}) \rightarrow 0$$

using again that $H_*(BG;\mathbb{Z})$ is of finite type, Spanier [14]. The Ext term is equal to the torsion subgroup of $H^{n-k+1}(BG;\mathbb{Z})$. Hence we get, assuming MG oriented,

Corollary. $[M,c] \in \widetilde{\Omega}_n^G K(\mathbb{Z},k)$ is a torsion element if and only if $<\bar{v}Mc,[M]> = 0$ for all $v \in H^*(BG;\mathbb{Z})$, $n \leq 2k-2$.

2.13. There is a commutative diagram

$$
\begin{array}{ccc}
\widetilde{\Omega}_n^G K(\mathbb{Z},k) & \overset{\psi}{\underset{\approx}{\rightarrow}} & H_{n-k}(BG;\mathbb{Z}) \\
\downarrow & & \downarrow \\
\widetilde{\Omega}_n^G K(\mathbb{Z}_m,k) & \overset{\approx}{\underset{\psi}{\rightarrow}} & H_{n-k}(BG;\mathbb{Z}_m)
\end{array}
$$

where the vertical maps are reduction mod m . Therefore we have also, again assuming MG oriented

Corollary. $[M,c] \in \widetilde{\Omega}_n^G K(\mathbb{Z},k)$ is a multiple of m if and only if $\langle \bar{v}Mc, [M]_m \rangle = 0$ for all $v \in H^*(BG;\mathbb{Z}_m)$, $n \leq 2k-2$.

2.14. From 2.12 and 2.13, with the assumption MG oriented, we get

Corollary. $[M,c] \in \widetilde{\Omega}_n^G K(\mathbb{Z},k)$ is zero if and only if $\langle \bar{v}Mc, [M] \rangle = 0$ for all $v \in H^*(BG;\mathbb{Z})$ and $\langle \bar{w}Mc, [M]_m \rangle = 0$ for all $w \in H^*(BG;\mathbb{Z}_m)$, where m runs through the values for which $H^*(BG;\mathbb{Z})$ has torsion of order m , $n \leq 2k-2$.

2.15. The previous results for detecting bordism classes by characteristic numbers came from the method of switching spectra. For the case $G = 0$ one can use a different argument which is valid without any dimensional restriction. We have

Theorem. $[M,c] \in \widetilde{\Pi}_n K(\mathbb{Z}_2,k)$ is zero if and only if $\langle vMc, [M]_2 \rangle = 0$ for all $v \in H^*(BO;\mathbb{Z}_2)$.

Proof. From the assumption $\langle vMc, [M]_2 \rangle = 0$ for all v one proves inductively that $\langle vMSq^i c, [M]_2 \rangle = 0$ for all v . In fact, if we have the last equation for $i < j$, we get

$$\langle vMSq^j c, [M]_2 \rangle = \langle Sq^j(vMc), [M]_2 \rangle = \langle v_j vMc, [M]_2 \rangle = 0$$

where $v_j \in H^j(BO;\mathbb{Z}_2)$ is the j-te Wu class. Since the Sq^j's generate the Steenrod algebra $\mathcal{O}_2$ we get $\langle vM\alpha c, [M]_2 \rangle = 0$ for all $\alpha \in \mathcal{O}_2$. Hence the Stiefel-Whitney numbers of $[M,c]$, except those of M itself, are zero. Therefore $[M,c] = 0$ in $\widetilde{\Pi}_n K(\mathbb{Z}_2,k)$ by Conner and Floyd [5].

2.16. <u>Corollary.</u> $[M,c] \in \tilde{n}_n K(\mathbb{Z},k)$ is zero if and only if

$\langle vMc,[M]_2 \rangle = 0$ for all $v \in H^*(BO;\mathbb{Z}_2)$

Proof. Mod 2 reduction $\tilde{n}_n K(\mathbb{Z},k) \to \tilde{n}_n K(\mathbb{Z}_2,k)$ is a monomorphism.

2.17. We give a geometrice description of the isomorphism

$\psi: \tilde{\Omega}_n^G MO_k \overset{\approx}{\to} n_{n-k} BG$ defined for $n \leq 2k-2$. An element of $\tilde{\Omega}_n MO_k$ is of the form $[M,\mu]$, where M is a closed G-manifold of dimension n and $\mu: M \to MO_k$ a map. We may take μ to be transversal to the zero section BO_k so that $N = \mu^{-1} BO_k$ is a submanifold of M . The lifted map $\nu_M: M \to BG$ induces $\nu_M/N: N \to BG$. Then $\psi[M,\mu] = [N,\nu_M/N]$. This can be proved by comparing Stiefel-Whitney numbers of both sides. We do not need the result in the following. For our application in section 4 we can just use the geometric description above as a definitinn. We get then a well defined homomorphism ψ by the relative transversality theorem.

3. Killing cohomology classes.

3.1. In this section we introduce the concept of killing a cohomology class, not necessarily a characteristic class, by a cobordism, and prove the theorems of the introduction in a more general form.

Let M be a closed manifold of dimension n and $c \in H^k(M;\pi)$. If V is a cobordism from M to M' , we get a diagram

$$H^k(M;\pi) \overset{i^*}{\leftarrow} H^k(V;\pi) \overset{i'^*}{\to} H^k(M';\pi)$$

induced by inclusion maps, and we have a corresponding additive relation $\rho_V = i'^* i^{*-1}$, where actually ρ_V depends on the pair (V,M) . If $\rho_V c = 0$ with zero indeterminacy, we say that c is <u>killed</u> <u>by</u> the cobordism V .

If $H^k(V,M;\pi) = H^{k+1}(V,M;\pi) = 0$, then by exactness of the cohomology sequence of the pair (V,M) , we have a homomorphism $\rho_V \colon H^k(M;\pi) \to H^k(M',\pi)$.

In general, if V is a cobordism from M to M' and V' a cobordism from M' to M'' , we get a composite cobordism VV' from M to M' and the equation $\rho_{VV'} = \rho_{V'}\rho_V$ between additive relations.

3.2. Suppose M is a G-manifold, $c = \bar{u}M$ a characteristic class and V a G-cobordism from M to M' . Then $\rho_V(\bar{u}M) = \bar{u}M'$. Thus $\bar{u}M' = 0$ if $\bar{u}M$ is killed by V .

3.3. It is clear that if $c \in H^k(M;\pi)$ can be killed by a G-cobordism, then $[M,c] = 0$ in $\widetilde{\Omega}_n^G K(\pi,k)$. We prove the converse with a dimensional restriction.

<u>Theorem</u>. Let M be a closed G-manifold of dimension n and $c \in H^k(M;\pi)$, where $k > (n+1)/2$. Then c can be killed by a

G-cobordism if and only if $[M,c] = 0$ in $\widetilde{\Omega}_n^G K(\pi,k)$

Proof. Assume $[M,c] = 0$ in $\widetilde{\Omega}_n^G K(\pi,k)$. Geometrically this means that there exists a G-cobordism V from M to an M' and a class $d \in H^k(V;\pi)$ such that $d/M = c$ and $d/M' = 0$. We consider V as a handlebody on M and we can assume handles attached in order of increasing dimension, Smale [13]. Let V' be the part of the handlebody obtained by attaching the handles through dimension $n-k+1$, V'' the rest and $M'' = V' \cap V''$.
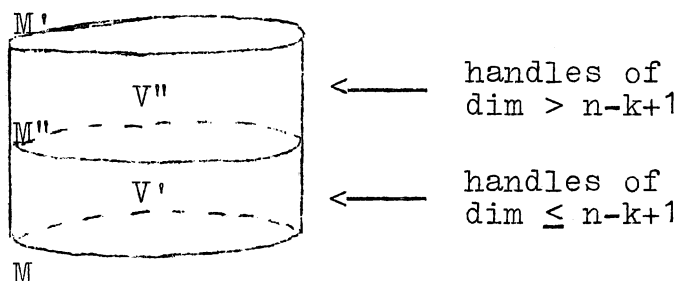
We put $d' = d/V'$ and $d'' = d/V''$. By duality V'' is a handlebody on M' with handles of dimensions $< k$. Therefore $H^k(V'',M';\pi) = 0$. Since $d''/M' = d/M' = 0$ we get $d'' = 0$ and hence $d'/M'' = d''/M'' = 0$. Because $H^p(V',M;\pi) = 0$ for $p > n-k+1$ the additive relation $\rho_{V'} : H^p(M;\pi)$ is a homomorphism for $p > n-k+1$. Thus c is killed by V' for $k > (n+1)/2$, and V' is a G-cobordism.

3.4. <u>Corollary.</u> A collection of classes $c_1, \ldots, c_r \in H^*(M;\pi)$ in dimensions $> (n+1)/2$ can be simultaneously killed by a G-cobordism if and only if each class can be separately killed by a G-cobordism.

Proof. We can kill one class at a time by a G-cobordism for which the corresponding additive relation is a homomorphism, and then take the composite G-cobordism.

3.5. We obtain now the following main result as an immediate consequence. The assumptions of 2.8 are understood, and except for $m = 2$ in case 1) the Thom spectrum MG is supposed to be oriented.

<u>Theorem.</u> Let M be a closed G-manifold of dimension n and suppose $k > (n+1)/2$. Then

1) $c \in H^k(M;\mathbb{Z}_m)$ can be killed by a G-cobordism if and only if $\langle \bar{v}Mc, [M]_m \rangle = 0$ for all $v \in H^*(BG;\mathbb{Z}_m)$.

2) $c \in H^k(M;\mathbb{Z})$ can be killed mod torsion by a G-cobordism if and only if $\langle \bar{v}Mc, [M] \rangle = 0$ for all $v \in H^*(BG;\mathbb{Z})$.

3) $c \in H^k(M;\mathbb{Z})$ can be killed by a G-cobordism if and only if $\langle \bar{v}Mc, [M] \rangle = 0$ for all $v \in H^*(BG;\mathbb{Z})$ and $\langle \bar{w}Mc, [M]_m \rangle = 0$ for all $w \in H^*(BG;\mathbb{Z}_m)$, where m runs through the values for which $H^*(BG;\mathbb{Z})$ has torsion of order m .

Proof. The result follows from 3.3 together with 2.11, 2.12 and 2.13.

3.6. If we want to kill an integral class by an unoriented cobordism, the previous theorem does not apply since MO is not oriented. However from 2.16 we get

<u>Theorem.</u> Let M be a closed n-manifold and $c \in H^k(M;\mathbb{Z})$, where $k > (n+1)/2$. Then c can be killed by an unoriented cobordism if and only if $\langle vMc, [M]_2 \rangle = 0$ for all $v \in H^*(BO;\mathbb{Z}_2)$.

4. Twisted surgery.

4.1. In this section we shall describe a general surgery method which gives an altermative way of killing cohomology classes. Let π be a ring and M a closed differentiable n-manifold. If π has elements not of order 2 , we assume M oriented. We can then consider the following condition on a cohomology class $c \in H^k(M;\pi)$.

C. There exist a dual submanifold N to c in M and a compact manifold W such that $\partial W = N$ and such that the normal bundle ν_{NM} extends to a bundle ω over W . If π has elements not of order 2 , the manifolds and bundles are assumed oriented.

Algebraically the condition means that the bordism class $[N, \nu_{NM}]$ in $\mathfrak{N}_{n-k}BO_k$ or $\Omega_{n-k}BSO_k$ is zero.

4.2. If condition C is satisfied, we can perform a surgery on M as follows. Let E be the disk bundle over W associated to ω , which we may take to be a differentiable bundle. By the tubular neighborhood theorem there exists an imbedding $\alpha: E/N \to M$ onto a tubular neighborhood of N . In the oriented case we take α to be orientation reversing. We define a new manifold

$$M' = (M - \text{int } \alpha(E/N)) \cup_\alpha \partial E$$

where we straighten the angle to obtain a differentiable structure on M' . The construction is called a twisted surgery on M .

If the normal bundle ν_{NM} is trivial, we can take the extension ω over W to be trivial. The surgery is then of the type used by Lashof [7]. If N is a sphere with trivial normal bundle in M , we can take W to be a disk, and the surgery is the standard one introduced in Milnor [10].

4.3. The manifold M' is cobordant to M by the cobordism

$$V = M \times I \ \cup_{(\alpha,1)} E$$

where we again straighten the angle to get a differentiable struc-
ture on V . In the oriented case we note that the orientation on
M extends to V and we get an induced orientation on M' .

disk bundle
E attached
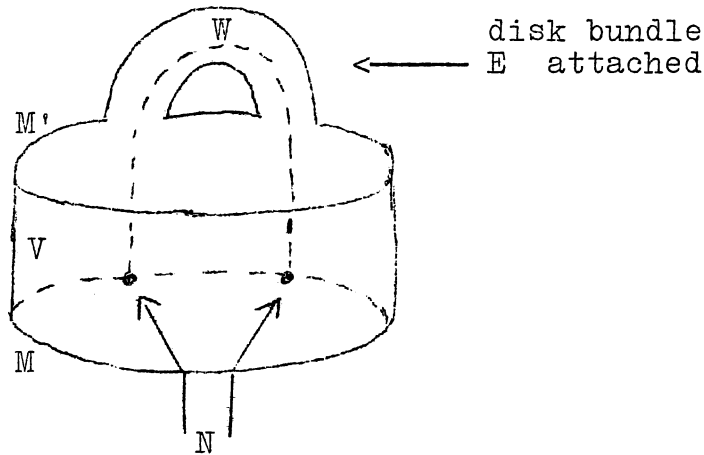
4.4. <u>Lemma</u>. If $k > (n+1)/2$, the class $c \in H^k(M;\pi)$ is killed
by the surgery (i.e. by the cobordism).

Proof. We get $H^p(V,M;\pi) = H^p(W,N;\pi) = 0$ for $p > n-k+1$. There-
fore the additive relation $\rho_V \colon H^p(M;\pi) \to H^p(M';\pi)$ is a homomor-
phism for $p > n-k+1$, in particular for $p = k$ if $k > (n+1)/2$.
Let $j \colon N \to M$, $i \colon M \to V$ and $i' \colon M' \to V$ be the inclusion maps.
Then

$$i'_*(\rho_V c \cap [M']) = i'_*(i'^* i^{*-1} c \cap [M']) = i^{*-1} c \cap i'_*[M'] =$$

$$i^{*-1} c \cap i_*[M] = i_*(c \cap [M]) = i_* j_*[N] = 0$$

Since $H_{n-k+1}(V,M';\pi) = H^k(V,M;\pi) = 0$ it follows that
$i'_* \colon H_{n-k}(M';\pi) \to H_{n-k}(V;\pi)$ is injective. Hence $\rho_V c \cap [M'] = 0$
and therefore $\rho_V c = 0$.

4.5. If M has a G-structure, we call the surgery of 4.2 a G-surgery if the G-structure on M extends to V .

Lemma. Let M be a closed G-manifold of dimension n and $c \in H^k(M;\pi)$, where $k > (n+1)/2$. If π has elements not of order 2 , assume MG oriented. Suppose c has a dual submanifold N in M such that $[N,\nu_M/N] = 0$ in $\mathfrak{N}_{n-k}BG$, or in the oriented case in $\Omega_{n-k}BG$. Then c can be killed by a twisted G-surgery on M

Proof. We consider first the unoriented case. The assumption $[N,\nu_M/N] = 0$ in $\mathfrak{N}_{n-k}BG$ gives a compact manifold W such that $\partial W = N$ together with an extension $\theta: W \to BG$ of $\nu_M/N: N \to BG$. Let $\omega = \nu_W - p\theta \in \widetilde{KO}(W)$, where $p: BG \to BO$ is the projection. Then $\omega/N = \nu_{NM}: N \to BO$. We have the diagram

$$
\begin{array}{ccc}
F & \longrightarrow & BO_k \\
 & \nearrow \nu_{NM} \nearrow & \downarrow i \\
N \hookrightarrow W & \xrightarrow[\omega]{} & BO
\end{array}
$$

where $F = O/O_k$ is (k-1)-connected. The groups $H^{i+1}(W,N;\pi_i F)$ are all zero. Hence there is no obstruction to a lifting $\omega: W \to BO_k$ extending $\nu_{NM}: N \to BO_k$.

Now we do the surgery as in 4.2 to kill c . The cobordism $V = M \times I \cup_{(\alpha,1)} E$ has a stable normal bundle classified by $\nu_V: V \to BO$. If we let $p_M: M \times I \to M$ and $\rho: E \to W$ denote the projections, we have $\nu_V/M \times I = \nu_M p_M$ and $\nu_V/E = (\nu_W - \omega)\rho = p\theta\rho$. There exists a lifting $\nu_M: M \to BG$ representing the G-structure on M , and changing the lifting by a homotopy we may assume $\nu_M \alpha = \nu_M \rho$ on E/N . Then $\nu_V/M \times I: M \times I \to BO$ lifts to

$\nu_M p_M$: $M \times I \to BG$ and ν_V/E: $E \to BO$ lifts to $\theta\rho$: $E \to BG$, where the two liftings commute with the identification map $(\alpha,1)$:$E/N \to M \times I$. Hence we get a lifting ν_V: $V \to BG$ extending ν_M: $M \to BG$. This proves the lemma in the unoriented case.

In the oriented case we have a lifting p: $BG \to BSO$ of p: $BG \to BO$. The G-structure on M induces an orientation. The proof is then as before, where we replace $\mathfrak{N}$ by Ω and O by SO everywhere.

4.6. The main result we obtain by the surgery method is the following. Compared with 3.5 the theorem gives a more explicit description of the cobordism involved, namely as the trace of a twisted surgery. On the other hand we do not get the analogs of case 1) for $m > 2$ or case 3) in 3.5. The reason is that we can not in general represent the cohomology class by a dual submanifold.

Theorem. Let M be a closed G-manifold of dimension n and suppose $k > (n+1)/2$. In case 2) assume MG oriented. Then

1) $c \in H^k(M;\mathbb{Z}_2)$ can be killed by a twisted G-surgery if and only if $\langle \bar{v}Mc, [M]_2 \rangle = 0$ for all $v \in H^*(BG;\mathbb{Z}_2)$

2) $c \in H^k(M;\mathbb{Z})$ can be killed mod torsion by a twisted G-surgery if and only if $\langle \bar{v}Mc, [M] \rangle = 0$ for all $v \in H^*(BG;\mathbb{Z})$.

Proof. By Thom [17] there is a $(2k-1)$-equivalence f: $MO_k \to K$, where $K = K(\mathbb{Z}_2,k) \times \ldots$ is a product of Eilenberg-MacLane spaces $K(\mathbb{Z}_2,)$'s . We have a diagram

$$\tilde{\Omega}_n^G K(\mathbb{Z}_2,k) \xrightarrow{i_*} \tilde{\Omega}_n^G K \xleftarrow[\approx]{f_*} \tilde{\Omega}_n^G MO_k \xrightarrow{\psi} \mathfrak{N}_{n-k} BG$$

where i: $K(\mathbb{Z}_2,k) \to K$ is the inclusion map of the first factor. There exists a lifting μ in the diagram

$$M \xrightarrow{c} K(\mathbb{Z}_2,k) \xrightarrow{i} K$$

with $\mu \to MO_k$ and $\downarrow f$.

From the geometric description 2.17 of ψ we get $\psi f_*^{-1} i_*[M,c] = \psi[M,\mu] = [N, \nu_M/N]$, where N is a dual submanifold to c in M . If we assume $\langle \bar{v}Mc, [M]_2 \rangle = 0$ for all $v \in H^*(BG;\mathbb{Z}_2)$, we get $[M,c] = 0$ in $\tilde{\Omega}_n^G K(\mathbb{Z}_2,k)$ by 2.11. Hence $[N, \nu_M/N] = 0$ in $\eta_{n-k}BG$, and by 4.5 we can kill c by a twisted G-surgery. This proves 1).

The proof of 2) is similar. We use the result of Wall [22] that there is a map $f: MSO_k \to K$, where $K = K(\mathbb{Z},k) \times \ldots$ is a product of Eilenberg-MacLane spaces $K(\mathbb{Z},)$'s and $K(\mathbb{Z}_2,)$'s , such that f is a $(2k-1)$-equivalence mod odd torsion, together with 2.12.

5.   The   PL   case.

5.1. The methods of sections 2-4 can be carried over to the   PL
category without difficulties. We let Ω_*^{PL} denote the   PL   bor-
dism theory defined geometrically on the category $\mathcal{C}$.   Furthermore
let BPL_k be the classifying space for   PL   microbundles of dimen-
sion   k   as defined by Milnor [11], and   MPL = $\{MPL_k\}$ the corres-
ponding Thom spectrum.   The Pontrjagin-Thom construction gives a
natural equivalence Ω_*^{PL} = $H_*($;MPL)   of homology theories on the
category $\mathcal{C}$.   This follows since we get an isomorphism on the co-
efficient groups by Williamson [24], and the two homology theories
commute with direct limits.   The natural map   i: $BPL_k \to BPL_{k+1}$ is
a   k-equivalence by Haefliger and Wall [6].   It follows therefore
from 2.8 that the spectrum   MPL   satisfies the connectivity condi-
tion of 2.2.   With   BPL = $\lim_{\overrightarrow{k}} BPL_k$ it is well known that $H_*(BPL;\mathbb{Z})$
is of finite type.

Similarly we have in the oriented case classifying spaces $BSPL_k$
and an oriented Thom spectrum   MSPL = $\{MSPL_k\}$.   There is a natural
equivalence Ω_*^{SPL} = $H_*($;MSPL)   of homology theories, and the spec-
trum   MSPL   satisfies the conditions of 2.8, hence the connectivity
condition of 2.2.

If   M   is a   PL   manifold, there exists by Milnor [11] a stable
normal microbundle classified by ν_M: M $\to$ BPL , or in the oriented
case by ν_M: M $\to$ BSPL , where ν_M is unique up to homotopy. 2.10-
2.14 is then valid for   G = PL,SPL .   For the   PL   analog of 2.15
we use the fact that   PL   bordism classes are determined by charac-
teristic numbers from $H^*(BPL;\mathbb{Z}_2)$.   The proof of this is similar
to that of the differentiable case given in Conner and Floyd [5].
One uses the result of Browder, Liulevicius and Peterson [2] that
the elements of the coefficient group Ω_*^{PL} are determined by the

characteristic numbers from $H^*(BPL;\mathbb{Z}_2)$, and the fact that the Steenrod representation $\mu: \Omega_*^{PL}X \to H_*(X;\mathbb{Z}_2)$ is surjective so that the spectral sequence of $\Omega_*^{PL}X$ collapses and we have $\Omega_*^{PL}X = H_*(X;\mathbb{Z}_2) \otimes \Omega_*^{PL}$.

If M is a closed PL manifold of dimension n , then $Sq^j: H^{n-j}(M;\mathbb{Z}_2) \to H^n(M;\mathbb{Z}_2)$ is multiplication with the Wu class v_j . Therefore we get the PL analog of 2.15 and hence 2.16.

5.2. In the proof of theorem 3.3 for $G = PL,SPL$ we make the following modification. With a triangulation of V we let V' be a closed regular neighborhood of $M \cup K$, where K is the $(n-k+1)$-skeleton of V minus an open collar of M' , and let $V'' = cl(V-V')$. Then (V',M) and (V'',M') have the homotopy types of relative CW-complexes with cells of dimension $\leq n-k+1$ and $< k$ respectively. The rest of the proof is then unchanged, and we get the results of section 3 for the PL case.

5.3. The lemma 4.5 and theorem 4.6 can be proved similarly in the PL case. The tubular neighborhood theorem we need to do the twisted PL surgery is contained in Haefliger and Wall [6]. From Browder, Liulevicius and Peterson [2] there is a $(2k-1)$-equivalence $f: MPL_k \to K$, where $K = K(\mathbb{Z}_2,k) \times \ldots$ is a product of $K(\mathbb{Z}_2,)$'s. Also there is a $(2k-1)$-equivalence mod odd torsion $f: MSPL_k \to K$, where $K = K(\mathbb{Z},k) \times \ldots$ is a product of $K(\mathbb{Z},)$'s and $K(\mathbb{Z}_{2^r},)$'s. Finally by the transversality theorem of Williamson [24] we can define geometrically, as in 2.17, a homomorphism

$\psi: \widetilde{\Omega}_n^{PL}MPL_k \to \Omega_{n-k}^{PL}BPL$ and similarly for the oriented case.

6. Applications to secondary cohomology operations.

6.1. Let ξ be a k-vector bundle over a closed differentiable n-manifold M. We consider the problem of computing a secondary characteristic class $\alpha\xi$ in $H^n(M;\mathbb{Z}_2)$. As usual $\alpha\xi$ is assumed to be given by a pull back in a universal example

$$
\begin{array}{ccc}
& E & \xrightarrow{\alpha} K(\mathbb{Z}_2,n) \\
\widetilde{\xi}\nearrow & \downarrow p & \\
M & \xrightarrow{\xi} BO_k &
\end{array}
$$

By definition $\alpha\xi = \{\widetilde{\xi}^*\alpha : p\widetilde{\xi} = \xi\}$

6.2. By means of the generating class theorem of E. Thomas [20] the problem of computing $\alpha\xi$ can be transformed under general circumstances to that of computing a corresponding secondary cohomology operation Φ of degree n on the Thom class $U_\xi \in H^k(M\xi;\mathbb{Z}_2)$.

6.3. For $\xi = \nu$ the normal bundle of some imbedding of M in S^{n+k} it follows from the Pontrjagin-Thom construction that the group $H^{n+k}(M\nu;\mathbb{Z}_2)$ is generated by the spherical class $[M]_2^* U_\nu$. Therefore any cohomology operation with values in $H^{n+k}(M\nu;\mathbb{Z}_2)$ is zero with zero indeterminacy. In perticular $\Phi U_\nu = 0$ with zero indeterminacy. This method was originally used by Mahowald and Peterson [8] to compute secondary obstructions for cross-sections of normal bundles. We shall generalize it by replacing the sphere S^{n+k} with a closed (n+k)-manifold.

6.4. Suppose Φ is a secondary cohomology operation assiciated to some relation $\sum_i \alpha_i\beta_i = 0$ in $\mathcal{O}_2$, where $\deg \alpha_i > 0$ for all i. Assume $j: M \to D$ is an imbedding in a closed (n+k)-manifold D

such that the following three properties are satisfied.

a) $j: M \to D$ has normal bundle ξ .

b) M is homologous to zero mod 2 in D .

c) $\text{Indet}^{n+k}(D;\Phi) = 0$.

Then we get the following

__Lemma__ $\Phi U_\xi = 0$ with zero indeterminacy.

Proof. Let $p: D \to M\xi$ be the Pontrjagin-Thom construction. By b)
we have $p^* U_\xi \cap [D]_2 = j_*[M]_2 = 0$, hence $p^* U_\xi = 0$. Also
$p^*([M]_2^* U_\xi) = [D]_2^*$. By naturality of Φ we get $p^* \Phi U_\xi \subseteq \Phi p^* U_\xi$.
Hence c) implies $\text{Indet}^{n+k}(M\xi;\Phi) = 0$ and then $\Phi U_\xi = 0$.

6.5. To construct an imbedding $j: M \to D$ with the properties a) -
c) of 6.4 we start with D equal to the double of the disk bundle
of ξ and j the zero-section into one of the halfs. Then a)
is satisfied, and the idea is to perform surgery on D away from
M to get also b) and c) . To get b) it is sufficient to kill
the dual class c of $j_*[M]_2 \in H_n(D;\mathbb{Z}_2)$ by a twisted surgery.
This requires a condition on the bundle ξ . Then we will make sure
that c) is satisfied by making D sufficiently perallelizable.
The first step leads to a result of some interest in its own, namely

6.6. __Theorem.__ Let ξ be a k-vector bundle over a closed n-mani-
fold M , where $k > n+1$. For the existence of a closed (n+k)-
manifold D and an imbedding $j: M \to D$ which is homologous to zero
mod 2 and has normal bundle ξ it is necessary and sufficient that
$<v(\tau_M+\xi),[M]_2> = 0$ for all $v \in H^*(BO;\mathbb{Z}_2)$.

Proof. Suppose the imbedding $j: M \to D$ with the required proper-
ties exists. Then

$$\langle v(\tau_M + \xi), [M]_2 \rangle = \langle vj^*\tau_D, [M]_2 = \langle vD, j_*[M]_2 \rangle = 0$$

by the assumption $j_*[M]_2 = 0$.

On the other hand, suppose $\langle v(\tau_M + \xi), [M]_2 \rangle = 0$ for all $v \in H^*(BO; \mathbb{Z}_2)$.
Let $j: M \to D$ be the imbedding in the double of the disk bundle
of ξ as before, and $c \in H^k(D; \mathbb{Z}_2)$ dual to M in D . Then

$$\langle vDc, [D]_2 \rangle = \langle vD, j_*[M]_2 \rangle = \langle v(\tau_M + \xi), [M]_2 \rangle = 0$$

for all $v \in H^*(BO; \mathbb{Z}_2)$. By 4.6 we can kill c by a twisted sur-
gery on D , where we use the dimensional assumption $k > n+1$.
The dual submanifold to c used for this surgery can be isotoped
off M for dimensional reasons. Therefore we can do the surgery
away from M , and M will then be imbedded in the new manifold D'
with normal bundle ξ as before. Finally M is homologous to
zero mod 2 in D' .

6.7. As usual we call a k-vector bundle ξ over a CW-complex X
$(q-1)$-parallelizable if ξ is trivial when restricted to the $(q-1)$-
skeleton of X . The property is equivalent to the existence of a
lifting in the diagram

$$
\begin{array}{ccc}
 & & BO_k\langle q\rangle \\
 & \nearrow & \downarrow \\
X & \xrightarrow[\xi]{} & BO_k
\end{array}
$$

where $BO_k\langle q\rangle$ is the $(q-1)$-connective covering of BO_k . A differ-
entiable manifold is called $(q-1)$-parallelizable if its tangent
bundle is $(q-1)$-parallelizable.

6.8. <u>Lemma.</u> Let ξ be a k-vector bundle over the n-manifold M ,
and let D be the double of the disk bundle of ξ . If $\tau_M + \xi$ is
(q-1)-parallelizable, then D is (q-1)-parallelizable, $q \leq k$.

Proof. We identify M with a submanifold of D by the imbedding
j so that $\tau_D / M = \tau_M + \xi$. Assume $\tau_M + \xi$ is (q-1)-parallelizable.
Then we have a diagram

$$
\begin{array}{ccc}
F & \longrightarrow & BO_{n+k}<q> \\
 & \overset{\tau_M+\xi}{\nearrow} \nearrow \nearrow & \downarrow \\
M & \hookrightarrow D \underset{\tau_D}{\longrightarrow} & BO_{n+k}
\end{array}
$$

where $F = O_{n+k} / O_{n+k}<q>$. The obstructions to a lifting
$\tau_D \colon D \to BO_{n+k}<q>$ extending $\tau_M + \xi$ lie in the groups $H^{i+1}(D,M;\pi_i F) =$
$\widetilde{H}^{i+1}(M\xi;\pi_i F)$ which are zero since $M\xi$ is (k-1)-connected and
$\pi_i F = 0$ for $i \geq q-1$. Hence D is (q-1)-parallelizable.

6.9. <u>Lemma.</u> Let M be a (q-1)-connected closed n-manifold and
$c \in H^k(M;\mathbb{Z}_2)$, where $k > (n+1)/2$. Suppose $<vMc,[M]_2> = 0$ for
all $v \in H^*(BO;\mathbb{Z}_2)$. Then c can be killed by a twisted $O<q>$-
surgery on M .

Proof. From 4.6 it follows that c can be killed by a twisted
surgery on M . Let $N = \partial W$ be a dual submanifold to c , We
consider W as a handlebody on N with handles attached in order
of increasing dimensions. Let W' be the part obtained by attaching
the handles in dimensions $\leq q$, W" the rest and $N' = W' \cap W"$.
Since M is (q-1)-connected there are no obstructions to an exten-
sion of j: N → M to a map h: W' → M . We assume $q \leq n-k$ since
the lemma is otherwise trivial. Then we can assume that h: W' → M
is an imbedding, and identify N' with a submanifold of M . It

follows that N' is another dual submanifold to c . We have

$N' = \partial W''$, where (W'',N') is q-connected. Hence we get

$[N',\nu_M/N'] = 0$ in $\tilde{h}_{n-k}BO<q>$, and by 4.5 we can kill c by a

twisted $O<q>$-surgery.

6.10. The main result of this section is the following.

<u>Theorem.</u> Let ξ be a k-vector bundle over a closed n-manifold

M , where $k > n+1$. Assume $\tau_M+\xi$ is $(q-1)$-parallelizable, where

$q \leq [\frac{n+k}{2}]$, and $<v(\tau_M+\xi),[M]_2> = 0$ for all $v \in H^*(BO;\mathbb{Z}_2)$. Let

Φ be a secondary cohomology operation of degree n associated to

a relation $\sum_i \alpha_i \beta_i = 0$ in $\mathcal{O}l_2$, where $0 < \deg \alpha_i < q$ for all i .

Assume Φ is defined on the Thom class $U_\xi \in H^k(M\xi;\mathbb{Z}_2)$. Then

$\Phi U_\xi = 0$ with zero indeterminacy.

Proof. Let $j: M \to D$ be the imbedding in the double of the disk

bundle of ξ as before. By 6.8 we know that D is $(q-1)$-paral-

lelizable. Now we do the surgeries on D away from M . First we

can make D a $(q-1)$-connected manifold by standard surgeries on

imbedded spheres of dimensions $< q$. If $c \in H^k(D;\mathbb{Z}_2)$ is the dual

class to M in D , we have $<vDc,[D]_2> = <v(\tau_M+\xi),[M]_2> = 0$ for

all $v \in H^*(BO;\mathbb{Z}_2)$. Hence c can be killed by a twisted $O<q>$-

surgery on D by 6.9. M is then homologous to zero mod 2 in D ,

and D is $(q-1)$-parallelizable.

We have

$$\mathrm{Indet}^{n+k}(D;\Phi) = \sum_i \alpha_i H^{n+k-a_i}(D;\mathbb{Z}_2)$$

where $a_i = \deg \alpha_i$. By Wu's theorem $\alpha_i: H^{n+k-a_i}(D;\mathbb{Z}_2) \to H^{n+k}(D;\mathbb{Z}_2)$

is multiplication by $v_{\alpha_i}D$, where $v_{\alpha_i} \in H^{a_i}(BO;\mathbb{Z}_2)$ is the Wu

class of α_i ($v_i = v_{Sq^i}$ is the ordinary Wu class). Since D is

(q-1)-parallelizable with $a_i < q$ for all i, we have $v_{\alpha_i}D = 0$ and the indeterminacy above is zero. Therefore we have an imbedding $j: M \to D$ satisfying a) - c) of 6.4. Hence $\hat{\Phi}U_\xi = 0$ with zero indeterminacy.

6.11. We remark that the conclusion of the theorem may be obtained even if the condition $\deg \alpha_i < q$ is not satisfied for all i. What we have used is only that the Wu classes $v_{\alpha_i}D$ vanish. This may be true also in other cases. For example if D is orientable, we have $v_{2i+1}D = 0$ as a consequence of the Adem relation $Sq^{2i+1} = Sq^1 Sq^{2i}$. If D is a spin manifold, then $v_{4i+2}D = 0$ from the relation $Sq^{4i+2} = Sq^2 Sq^{4i} + Sq^1 Sq^{4i} Sq^1$.

6.12. As an application of the theorem we may take $\xi = \tau_M$ to be the stable tangent bundle of an orientable M. Then $\tau_M + \xi = 2\tau_M$ is a spin bundle, i.e. 3-parallelizable. Hence we can take $q = 4$. We need also the condition $\langle v(2\tau_M), [M]_2 \rangle = 0$ for all $v \in H^*(BO; \mathbb{Z}_2)$. Since $w_{2i}(2\tau_M) = w_i^2 M$ and $w_{2i+1}(2\tau_M) = 0$ the condition is always satisfied if $n \neq 0 \mod 4$, and if $n = 4i$ it is equivalent to $\langle v_{2i}uM, [M]_2 \rangle = 0$ for all $u \in H^*(BO; \mathbb{Z}_2)$.

6.13. We consider the problem of deciding if the stable span of an orientable M^n is ≥ 2. Stable span $M \geq 2$ means that the tangent bundle of M is stably equivalent to an n-vector bundle with 2 linearly independent cross-sections, and this is equivalent to the existence of a lifting in the diagram

$$
\begin{array}{ccc}
 & & BSO_{n-2} \\
 & \nearrow & \downarrow \\
M & \xrightarrow{\tau_M} & BSO
\end{array}
$$

We assume $n \geq 5$ in the following and consider the two first stages of a Postnikov resolution for the fibration $BSO_{n-2} \to BSO$.

$n = 1 \mod 4$) The primary obstruction is $w_{u-1}M$, and there is no secondary obstruction.

$n = 2 \mod 4$)

$$i_1^* \alpha_n = Sq^2 \iota_{n-2}$$

We have $w_n M = 0$ from the Wu relations and $W_{n-1}M = 0$ by Massey [9]. Thus we get a lifting τ_1 . From the generating class theorem and the results of Mahowald, Peterson [8] we get

$(w_2 w_{n-2}M + \tau_1^* \alpha_n)U_\tau \in \tilde{\Phi} U_\tau$, where $U_\tau \in H^*(M_\tau; \mathbb{Z})$ is the Thom class of the stable tengent bundle τ of M , and $\tilde{\Phi}$ is a secondary operation associated to the relation $Sq^2(\delta Sq^{n-2}) = 0$ on integral classes. Let $\tilde{\Phi}$ be a secondary operation defined on mod 2 classes by the relation $Sq^2 Sq^{n-1} + Sq^n Sq^1 = 0$ in $\mathcal{O}_2$. Then $\tilde{\Phi}$ is defined on U_τ and can be chosen so that $\tilde{\Phi} U_\tau \subseteq \tilde{\Phi} U_\tau$. By theorem 6.10 and the remarks of 6.11 we get $\tilde{\Phi} U_\tau = 0$ with zero indeterminacy. Hence $\tau_1^* \alpha_n = w_2 w_{n-2}M = 0$ by a Wu relation. Hence there is no secondary obstruction.

$n = 3 \mod 4$)

$$i_1^* \alpha_n = Sq^2 \iota_{n-2}$$

By Massey [9] we have $w_{n-1}M = 0$ and get a lifting τ_1 . Let Φ be a secondary operation associated to the relation $Sq^2Sq^{n-1} = 0$ on integral classes. The rest of the argument is as in the previous case, and we find that there is no secondary obstruction.

n = 0 mod 4) The method does not work in this case, The assumption $\langle v_{2i}uM, [M]_2 \rangle = 0$ for all $u \in H^*(BO; \mathbb{Z}_2)$ will exclude some manifolds with stable span ≥ 2 , and furthermore the secondary operation $\tilde{\Phi}$ in this case is associated to the relation $Sq^2Sq^{n-1} + Sq^1Sq^n + Sq^nSq^1 = 0$, which is not covered by theorem 6.10 or the remarks of 6.11.

6.14. Suppose ξ and η are stably equivalent, orientable n-vector bundles over M^n . Then for n even, ξ and η are isomorphic if and only if $\chi\xi = \chi\eta$, and for n = 3 mod 4 , ξ and η have simultaneously span ≥ 2 . See E. Thomas [21] and [18]. Therefore from 6.13 we have a new proof of the following result of E. Thomas [19].

Corollary. Let M be a closed orientable n-manifold. Then M has a tangent 2-field if and only if $\chi M = 0$ in case n = 2 mod 4, and always in case n = 3 mod 4.

References.

1. D.W. Anderson, E.H Brown, F.P. Peterson. The structure of the spin cobordism ring. Ann. of Math. 86 (1967) 271-298.

2. W. Browder, A. Liulevicius, F.P. Peterson. Cobordism theories. Ann. of Math. 84 (1966) 91-101.

3. R.L.W. Brown. Cobordism and bundles over spheres. Mich. Math. J. 16 (1969) 315-320.

4. ———— Imbeddings, immersions and cobordism of differtiable manifolds. Bull. Am. Math. Soc. 76 (1970) 763-766.

5. P. Conner, E. Floyd. Differentiable periodic maps. Springer Verlag, 1964.

6. A. Haefliger, C.T.C. Wall. Piecewise linear bundles in the stable range. Topology 4 (1965) 209-214.

7. R. Lashof. Poincaré duality and cobordism. Trans. Amer. Math. Soc. 109 (1963) 257-277.

8. M. Mahowald, F. Peterson. Secondary operations on the Thom class. Topology 2 (1964) 367-377.

9. W. Massey. On the Stiefel-Whitney classes of a manifold. Amer. J. Math. 82 (1960) 92-102.

10. J. Milnor. A procedure for killing homotopy groups of differentiable manifolds. Symp. Pure. Math. A.M.S. III (1961) 39-55.

11. ———— Microbundles and differentiable structures. Mimeo. Princeton University 1961.

12. ———— On the Stiefel-Whitney numbers of complex manifolds and of spin manifolds. Topology 3 (1965) 223-230.

13. S. Smale. Generalized Poincaré's conjecture in dimensions greater than four. Ann. of Math. 74 (1961) 391-406.

14. E Spanier. Algebraic topology. Mc Graw-Hill, 1966.

15. R. Stong.Cobordism and Stiefel-Whitney numbers, Topology 4 (1965) 241-256.

16. R. Stong. Notes on cobordism theory. Princeton Univ. Press. 1968.

17. R. Thom. Quelques propriétés globales des variétés différentiables. Comment. Math. Helv. 28 (1954) 17-86.

18. E. Thomas. Cross-sections of stably equivalent vector bundles. Quart. J. Math. Oxford Ser (2) 17 (1966) 53-57.

19. ———— Postnikov invariants and higher order cohomology operations. Ann. of Math.(2) 85 (1967) 184-217.

20. ———— The index of a tangent 2-field. Comment. Math. Helv. 42 (1967) 86-110.

21. ———— Vector fields on low dimensional manifolds. Math. Z. 103 (1968) 85-93.

22. C.T.C. Wall. Determination of the cobordism ring. Ann. of Math. 72 (1960) 292-311.

23. G.W. Whitehead. Generalized homology theories. Trans. Am. Math. Soc. 102 (1962) 227-283.

24. R. Williamson. Cobordism of combinatorial manifolds. Ann. of Math. 83 (1966) 1-33.