HERBRAND AND SKOLEM THEOREMS
IN INFINITARY LANGUAGES

by

Herman Ruge Jervell
Oslo
I. INTRODUCTION

In this paper I will prove the Skolem and Herbrand theorems for all the classical logics $L_{Kw}$ (semantical formulation) for regular $K$. I use the method from my paper on ordinary first order logic [1]. The new idea here is to use a completeness and Maehara proof of Takeuti[3]. He gives a syntactical formulation of $L_{Kw}$ and a completeness proof of it. Since this work does not seem to be too well known, I will sketch it in enough details for the reader to fill in the rest. Else this work parallels closely [1].

Of the above the Skolem and Herbrand theorems of $L_{ww}$ and $L_{w1w}$ is known. In his thesis [2] B. Nebres proved the Herbrand theorems of $L_{w1w}$ and $L_{A}$, $A$ countable admissible. His proofs are quite different from mine.
II. THE SYNTACTICAL FORMULATION OF $L_{K^\omega}$.

In this section we give the syntactical formulation of $L_{K^\omega}$ as given by Takeuti and Maehara [3]. K is an infinite regular cardinal. $L_{K^\omega}$ is given in a sequential formulation. The differences from ordinary sequential calculus is that sequents may consist of pairs of infinite sequences of formulae and that we can use a rule an infinite number of times simultaneously.

LANGUAGE

Connectives $\wedge$ ($\alpha$-conjunction for $\alpha < K$), $\neg$
quantifier $\forall$
parameters $a_1, a_2, \ldots, a_\xi, \ldots$ $\xi < K^+$
variables $x_1, x_2, \ldots, x_\xi, \ldots$ $\xi < K^+$
constant $e$
functionsymbols of n arguments ($n < \omega$)

$$f_1^n, f_2^n, \ldots, f_\xi^n, \ldots \quad \xi < K^+$$
predicatesymbols of n arguments ($n < \omega$)

$$P_1^n, P_2^n, \ldots, P_\xi^n, \ldots \quad \xi < K^+$$

In the usual way we build up

terms $t_1, t_2, \ldots$
atomic formulae $A_1, A_2, \ldots$
formulae $F_1, F_2, \ldots$

Sequences of length $< K$ of formulae $\Gamma_1, \Gamma_2, \ldots, \Delta, \Delta, \ldots$
A sequent is a pair $\Gamma \rightarrow \Delta$ of two sequences $\Gamma, \Delta$
of formulae of length $< K$. 
THE CALCULUS $L_{K,w}$

Axioms $\Gamma_1, \Delta, \Gamma_2 \rightarrow \Delta_1, \Delta_2$ for $A$ atomic

STRUCTURAL RULES:

Permutation $\frac{\Gamma \rightarrow \Delta}{\Gamma^* \rightarrow \Delta^*}$, where $\Gamma^*$ is obtained from $\Gamma$ by a permutation of formulae and similarly $\Delta^*$ from $\Delta$.

Thinning $\frac{\Gamma \rightarrow \Delta}{\Gamma, \Gamma_1 \rightarrow \Delta, \Delta_1}$

Contraction $\frac{\Gamma, \Delta \rightarrow \Delta}{\Gamma, \Delta \rightarrow \Delta}$

Trivial rule $\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}$, where we have $<K$ number of premisses.

LOGICAL RULES:

Below we write $\{F_{\alpha}\}_{\alpha \leq \gamma}$ for a $\gamma$-sequence of formulae.

$\frac{\Gamma, \{F_{\gamma}^\beta\}_{\alpha \leq \gamma, \beta < \delta} \rightarrow \Delta}{\Gamma, \{M_{\alpha, \beta}^i F_i\}_{\alpha \leq \gamma} \rightarrow \Delta}$, where $\delta < K$ and

$\frac{\Gamma \rightarrow \{F_{\gamma}^\beta\}_{\alpha \leq \gamma} \rightarrow \Delta}{\Gamma \rightarrow \{F_{\gamma}^\beta\}_{\alpha \leq \gamma} \rightarrow \Delta}$, for all $\alpha < \gamma$ and

$\frac{\Gamma \rightarrow \{F_{\gamma}^\beta\}_{\alpha \leq \gamma} \rightarrow \Delta}{\Gamma \rightarrow \{\neg F_{\alpha}^\beta\}_{\alpha \leq \gamma} \rightarrow \Delta}$

$\frac{\Gamma \rightarrow \{F_{\alpha}^\beta\}_{\alpha \leq \gamma} \rightarrow \Delta}{\Gamma \rightarrow \{\neg F_{\alpha}^\beta\}_{\alpha \leq \gamma} \rightarrow \Delta}$
\[ \forall \alpha \vdash F \alpha \alpha \alpha \alpha < \gamma, \delta \rightarrow \Delta \]

\[ \forall \alpha \vdash F \alpha \alpha \alpha \alpha < \gamma, \Delta \]

where \( \delta < K \)

where the \( \alpha \) are distinct parameters not occurring in

\[ \forall \alpha \vdash F \alpha \alpha \alpha \alpha < \gamma, \Delta \]

This completes the description of the system \( L_{Kw} \). The semantics of \( L_{Kw} \) is well known. In the next section we prove completeness of the system.
III. COMPLETENESS OF $L_{Kw}$

Given the syntactical formulation of $L_{Kw}$, the completeness is proved as in LK. As in [1] we define:

1. Precedes, succeeds, strand, analysis.

2. Positive and negative occurrences.

3. General and restricted quantifiers.

4. $L_{Kw}$-tree. An $L_{Kw}$-tree over a sequent $\Gamma \rightarrow \Delta$ is a tree of sequents with $\Gamma \rightarrow \Delta$ at the downmost node and such that 
   i) a sequent at any node and the sequents at its successor-nodes are related as one of the rules of $L_{Kw}$;
   ii) the terms introduced at a node by $V \rightarrow$ are built up from symbols in $\Gamma \rightarrow \Delta$, the constant $e$, and from parameters introduced by $\rightarrow V$ somewhere in the tree;
   iii) parameters introduced in $\rightarrow V$ are distinct if we analyze quantifiers not in the same strand or with distinct analyses; and
   iv) there is a well-order of the parameters called less than or equal such that for any parameter $a$ introduced at a node $\nu$ by $\rightarrow V$, all parameters occurring in nodes below $\nu$ are strictly less than $a$.

5. Secured nodes, branches, $L_{Kw}$-trees.

6. PROVABILITY THEOREM If we have a secured $L_{Kw}$-tree over $\Gamma \rightarrow \Delta$, then $\vdash_{L_{Kw}} \Gamma \rightarrow \Delta$.

7. Analyzing branch and analyzing $L_{Kw}$-tree.
8. ANALYZING LEMMA  To any sequent we can find an analyzing \( L_{Kw} \)-tree over it.

9. FALSIFIABILITY LEMMA  If we have a not-secured analyzing branch in an \( L_{Kw} \)-tree over \( \Gamma \rightarrow \Delta \), then we can find a falsifying model for \( \Gamma \rightarrow \Delta \).

10. SOUNDNESS LEMMA  For any sequent \( \Gamma \rightarrow \Delta \), if \( \vdash_{L_{Kw}} \Gamma \rightarrow \Delta \), then there are no falsifying models of \( \Gamma \rightarrow \Delta \).

11. COMPLETENESS THEOREM  For any sequent \( \Gamma \rightarrow \Delta \), \( \vdash_{L_{Kw}} \Gamma \rightarrow \Delta \) if and only if there are no falsifying models of \( \Gamma \rightarrow \Delta \).

12. CONSISTENCY THEOREM  For any sequent \( \Gamma \rightarrow \Delta \) we have exactly one of i and ii below:
   i) a secured \( L_{Kw} \)-tree over \( \Gamma \rightarrow \Delta \),
   ii) an \( L_{Kw} \)-tree over \( \Gamma \rightarrow \Delta \) with not-secured falsifying branch.

13. The proof of the analyzing lemma goes as in [1] with only the obvious changes. Observe that as in LK we have no difficulty in making true condition iv in the definition of \( L_{Kw} \)-tree.

   It is also easy to see how regularity of \( K \) comes in - we must prove that sequences of formulae of length \( < K \) goes over into sequences of formulae of length \( < K \) when we add new nodes in the construction of analyzing \( L_{Kw} \)-trees.

14. Strong \( L_{Kw} \)-tree.

15. STRONG ANALYZING LEMMA  To any sequent we can find a strong analyzing \( L_{Kw} \)-tree over it.
IV. THE SKOLEM THEOREM

The theory can be transferred from [1] with only the obvious changes. It is straightforward to write down the proofs in $L_{Kw}$ from those given in $LK$ [1].

$L_{Kw}$-morphisms. An $L_{Kw}$-morphism is a transformation of $L_{Kw}$-trees into $L_{Kw}$-trees preserving the tree structure. A provability-morphism is an $L_{Kw}$-morphism which transforms secured trees into secured trees. An analyzing morphism transforms analyzing trees into analyzing trees. A falsifiability morphism transforms analyzing not-secured trees into analyzing not-secured trees. An $L_{Kw}$-isomorphism is an $L_{Kw}$-morphism which is both a provability and a falsifiability morphism.

Skolem transforms. Let $P$ be a set of positions. $S_P$ is the transformation which eliminates general quantifiers at positions in $P$ by introducing new functionsymbols. It is easily seen to be well defined up to the names of the new functionsymbols. For $P =$ the set of all positions we get the transformation $S$.

Skolem morphism. The morphisms $\mathcal{S}_P$ and $\mathcal{S}$ are defined from $S_P$ and $S$ as in [1].

Following [1] we get:

SKOLEM THEOREM $\mathcal{S}_P$ and $\mathcal{S}$ are $L_{Kw}$-isomorphisms.

Using the completeness theorem from the preceding section we get:

COROLLARY For any formula $F$ in $L_{Kw}$:

$F$ is valid $\iff$ $S_P(F)$ is valid
$\iff$ $S(F)$ is valid.

$S_P(F)$ and $S(F)$ above are defined in the obvious way.
V. THE HERBRAND THEOREM

As with the Skolem theorem we follow [1] with only the obvious changes.

Herbrand domain A sequence of terms $\mathfrak{D}$ is an Herbrand domain if for any sequent $\Gamma \rightarrow \Delta$, the subsequence of $\mathfrak{D}$ of terms built up from the constant $e$ and symbols of $\Gamma \rightarrow \Delta$ is of length $< K$.

Let $\mathfrak{D}$ be an Herbrand domain. The Herbrand transforms $H_{\mathfrak{D}}$ and the Herbrand morphisms $H_{\mathfrak{D}}$ are defined as in [1]. With them if we operate on $L_{Kw}$-trees over sequents without general quantifiers, we get rid of all restricted quantifiers by writing them as conjunctions.

HERBRAND THEOREM. $\mathfrak{D}$ is an Herbrand domain.

1. $H_{\mathfrak{D}}$ is a falsifiability morphism.
2. If $\mathfrak{D} \subseteq \mathfrak{C}$ and $H_{\mathfrak{D}}(\mathcal{T})$ is secured, then also $H_{\mathfrak{C}}(\mathcal{T})$.
3. If $\mathcal{T}$ is secured, then we can find $\mathfrak{C}$ with $H_{\mathfrak{C}}(\mathcal{T})$ secured.

This theorems give the Herbrand theorem for all $L_{Kw}$, $K$ regular cardinal. We can of course formulate a corollary using sequents:

COROLLARY For any formula $F$ in $L_{Kw}$:

$F$ is valid $\iff \exists \mathfrak{D} H_{\mathfrak{D}}(F)$ is valid.

We have given the Herbrand theorem for $L_{Kw}$. Now an interesting thing happens if $K > \omega$.

THEOREM. Let $K > \omega$ be a regular cardinal. Let $\mathfrak{D}$ be a se-
quence enumerating all terms of $L_{K\omega}$ without repetitions. Then $\mathcal{D}$ is an Herbrand domain.

Proof:
In any sequent $\Gamma \rightarrow \Delta$ there are $< K$ symbols. Hence we get that the subsequence of $\mathcal{D}$ of terms built up from the symbols of $\Gamma \rightarrow \Delta$ and the constant $e$ has length $< K$.

The transform and the morphism given by the Herbrand domain above is denoted by $H$ and $\mathcal{H}$.

THEOREM. For $K > \omega$:
$\mathcal{H}$ is an $L_{K\omega}$-isomorphism.

COROLLARY. In $L_{K\omega}$ ($K$ regular $> \omega$) for any formula $F$:
$F$ is valid $\iff$ $HS(F)$ is valid.

Above $HS(F)$ is a quantifierfree formula in $L_{K\omega}$.
VI. CONCLUSION

We have clearly given the Skolem and the Herbrand theorems for $L_{K\omega}$, $K$ regular, above. Since $K^+$ is regular for all $K$, we can give theorems for $L_{K\omega}$, $K$ singular, by embedding it into $L_{K^+\omega}$. In that way we get the Skolem theorem of $L_{K\omega}$, $K$ singular. In the Herbrand theorem we get into problems. We are not guaranteed to write restricted quantifiers as conjunctions of length $< K$, only of length $< K^+$.

If we work in $L_{\omega_1\omega}$ or subsystems of it, we can do with sequents of finite length, but we cannot do this in general in $L_{K\omega}$. We do not get completeness in $L_{\omega_2\omega}$ if we do not allow infinite sequents. For instance if we write $A^0$ for $A$ and $A^1$ for $\neg A$ and assume $|2^\omega| = \omega_1$ then the sequent

$$\vdash f \in 2^\omega \rightarrow \forall n \in \omega \cdot A^f(n)$$

is derivable in $L_{\omega_2\omega}$ allowing infinite sequents but not if we only allow finite sequents.
REFERENCES

