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AN HERBRAND THEOREM FOR A MODAL LOGIC

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I. INTRODUCTION

This is one of a sequence of papers on Skolem and Herbrand theorems for various logics (1,2,3,4). Here I will treat modal logics. My approach will be similar to the one for LI (4). It seems reasonable that one can develop Skolem and Herbrand theories for logics with proof procedures à la the one for LI. Schütte has given such procedures for the modal logics M and S4(5). As in LI I can show that the Herbrand theory works and also one way in the Skolem theory. The counterexamples to the other way are the sequents

$$\forall x\ N\ Ax \longrightarrow N\ \forall x Ax$$

$$N\ \exists x Ax \longrightarrow \exists x\ N\ Ax$$

Neither of the sequents are M-provable (S4-provable) but their Skolem transforms are. The situation seems to be rather hopeless. Especially the second sequent should not be true in a modal logic.

Below I will show one way out. I give a sequential formulation of a modal logic LB. LB is weaker than both M and S4. In fact LB is complete for Kripke-modals where the binary relation between the worlds is serial ($\forall x \exists y\ xRy$). To get an axiomatization of the logic one usually adds the schema $NA \rightarrow MA$ or $NA, N \rightarrow A \rightarrow$

I give an Herbrand theorem for LB and also show how to imbed M and S4 into LB. In this way one gets almost Herbrand-theorems for M and S4. The theorems give connections between M or S4 and the propositional part of LB.

II. THE FORMAL SYSTEM LB.

LANGUAGE

Connectives $\bigwedge$ (finite conjunction), $\neg$, N

Quantifier $\forall$

Functionsymblos and predicatesymbols.

Fixed constant c.

As usual we define:

Terms, atomic formulae, formulae

Sequents (Pair of finite sequences of formulae.)

AXIOMS

$$\Gamma_1, A, \Gamma_2 \longrightarrow \Delta_1, A, \Delta_2 \quad \text{for } A \text{ atomic}$$

STRUCTURAL RULES

Permutation, thinning, contraction, trivial rule as in LK (1).

LOGICAL RULES

$\bigwedge\longrightarrow$, $\longrightarrow\bigwedge$, $\neg\longrightarrow$, $\longrightarrow\neg$, $\forall\longrightarrow$, $\longrightarrow\forall$ as in LK (1).

In addition we have the following two

N_1 $\dfrac{\Gamma \longrightarrow}{N\Gamma \longrightarrow}$ where $N\Gamma$ = the sequence of NF's where F is in Γ .

N_2 $\dfrac{\Gamma \longrightarrow F}{N\Gamma \longrightarrow NF}$

This completes the description of LB.

It is well known that we get a sequential formulation of M if we exchange N_1 with

$N\longrightarrow$ $\dfrac{\Gamma , F \longrightarrow \Delta}{\Gamma , NF \longrightarrow \Delta}$

and we get S4 if we exchange N_1, and N_2 with $N \longrightarrow$ and

$$\longrightarrow N \qquad \frac{N\Gamma \longrightarrow A}{N\Gamma \longrightarrow NA}$$

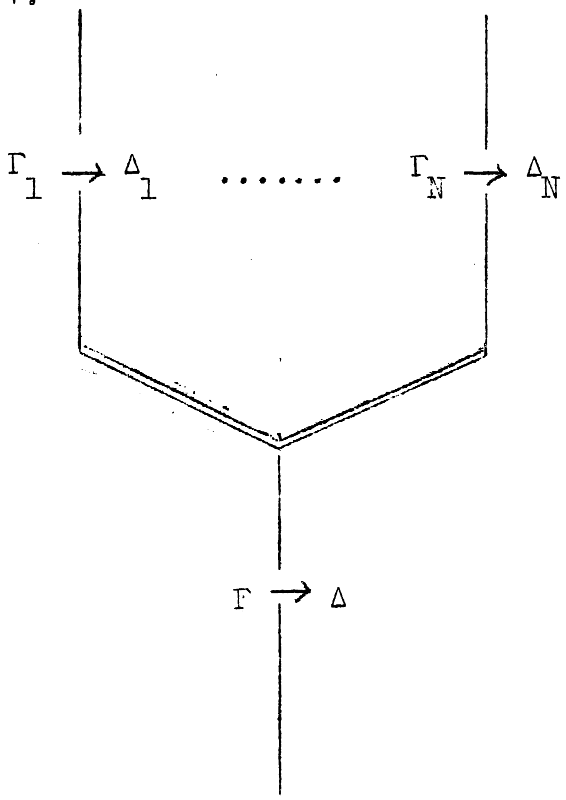
III. COMPLETENESS OF LB.

We follow LI (4) in proving that LB is complete for Kripke-models with no restriction on the binary relation between the worlds.

FOREST. A forest of trees over $\Gamma \longrightarrow \Delta$, is a tree of trees of sequents, starting with the one-sequent tree consisting of $\Gamma \longrightarrow \Delta$ alone, and between any tree in the forest and its successor-trees we have one of the following 4 possibilities:

We write single lines for branches in trees and double lines for branches in the forest:

1.



The trees above are the same as the one below except that we have $\Gamma_1 \longrightarrow \Delta_1, \ldots, \Gamma_N \longrightarrow \Delta_N$ for $\Gamma \longrightarrow \Delta$. $\Gamma_1 \longrightarrow \Delta_1, \ldots, \Gamma_N \longrightarrow \Delta_N$ and $\Gamma \longrightarrow \Delta$ are connected as one of the rules of LB except N_1 and N_2.

2.

$\Gamma_2 \rightarrow$

$\Gamma_1, N\Gamma_2 \rightarrow \Delta$

$\Gamma_1, N\Gamma_2 \rightarrow \Delta$

The tree above is the same as the one below except that we have added a new node with with sequent $\Gamma_2 \rightarrow$ immediately above $\Gamma_1, N\Gamma_2 \rightarrow \Delta$

3.

$\Gamma_2 \rightarrow F$

$\Gamma_1, N\Gamma_2 \rightarrow NF, \Delta$

$\Gamma_1, N\Gamma_2 \rightarrow NF, \Delta$

The tree above is the same as the one below except that we have added a new node with sequent $\Gamma_2 \rightarrow F$ immediately above $\Gamma_1, N\Gamma_2 \rightarrow NF, \Delta$

4.

$\Gamma_1, F \longrightarrow \Delta_1$

$\Gamma_2, NF \longrightarrow \Delta_2$

The sequents indicated are at nodes, the one immediately above the other.
The trees are the same except that we have exchanged $\Gamma_1 \longrightarrow \Delta_1$ with $\Gamma_1, F \longrightarrow \Delta_1$

$\Gamma_1 \text{ — } \Delta_1$

$\Gamma_2, NF \longrightarrow \Delta_2$

This concludes the definition of forests.

SUCCEEDS, PRECEDES As in LK (1) and LI (4) we define 'succeeds as formula', 'succeeds as formula part', 'precedes as formula', 'precedes as formula part', 'in the same strand. 'analysis'.

POSITIVE AND NEGATIVE Positive and negative occurrences in $\Gamma \longrightarrow \Delta$ are defined inductively as in (1) with the extra clause: If NF occurs positively (negatively) in $\Gamma \longrightarrow \Delta$, then F occurs positively (negatively) in $\Gamma \longrightarrow \Delta$.

GENERAL AND RESTRICTED As in LK.

DEGREE This is a new concept for LB. The degree of a subformula of $\Gamma \rightarrow \Delta$ is defined inductively by:

1. Formulae in Γ and Δ have degree 1.
2. If $\bigwedge F_i, \neg F, \forall x Gx$ have degree N, then also F_i, F, Gx.
3. If NF has degree N, then F has degree N + 1.

NODES We have a system of notations for nodes in a tree. We write $\nu \lesssim \mu$ for the usual tree-ordering between nodes.

TERM BELONGING TO A NODE. Given a forest over $\Gamma \rightarrow \Delta$. Let ν be a node occurring in one of the trees of the forest. A term t belongs to ν if it is built up from:

i) Symbols from $\Gamma \rightarrow \Delta$

ii) The constant e

iii) Parameters introduced by $\rightarrow \forall$ at nodes $\mu \lesssim \nu$ somewhere in the forest.

LB-FOREST An LB-forest over $\Gamma \rightarrow \Delta$, is a forest F over $\Gamma \rightarrow \Delta$ such that

i) a term introduced at a node ν by $\forall \rightarrow$ belongs to ν

ii) parameters introduced by $\rightarrow \forall$ are distinct if we analyze quantifiers in distinct strands or with distinct analyses:, and

iii) we have a well-order of the parameters such that for any parameter a introduced by $\rightarrow \forall$ somewhere in F, all parameters occurring in the analysis of the quantifierformula we use to introduce a, they are strictly less than a (in the well-order.)

STANDARD FOREST A forest is standard if we do not use possibility 4 in giving the connection between the nodes.

SECURED A node in a tree is secured if there is an axiom at it. A tree is secured if it contains a secured node. A branch in a forest is secured if it contains a secured tree. A forest is secured if all its branches are secured.

PROVABILITY LEMMA. If there is a secured standard LB-forest over $\Gamma \longrightarrow \Delta$, then $\vdash_{LB} \Gamma \longrightarrow \Delta$.

Proof:
As in LI (4)

F FINITENESS LEMMA If there is a secured LB-forest over $\Gamma \longrightarrow \Delta$, then there is a finite secured LB-forest over $\Gamma \longrightarrow \Delta$

LEMMA If a subformula of $\Gamma \longrightarrow \Delta$ of degree N is analyzed in a forest over $\Gamma \longrightarrow \Delta$, it is analyzed at a node of height N over the bottomnode.

Proof:
Immediate from the definition of forest.

QED

This lemma holds neither in M nor in S4. (In both M and S4 we can define forests in an obvious way but the lemma will not hold.)

Now as in LI we show how to permute the applications of the possibilities in the LB-forests to make them standard.

LEMMA Given a finite secured LB-forest over $\Gamma \longrightarrow \Delta$. We can then find a finite secured LB-forest over $\Gamma \longrightarrow \Delta$ where the applications of possibility 1 at nodes of height N precedes the applications of possibilities 2,3,4 at nodes of height N.

The proof is straight forward from the hints given for the corresponding proof of LI (4). As in LI we get:

LEMMA Given a finite secured LB-forest over $\Gamma \longrightarrow \Delta$. We can then find a finite secured standard LB-forest over $\Gamma \longrightarrow \Delta$

and:

PROVAVIBITY THEOREM If we have a secured LB-forest over $\Gamma \longrightarrow \Delta$, then $\vdash_{LB} \Gamma \longrightarrow \Delta$.

ANALYZING BRANCH A branch β in a forest F is analyzing if:

i) for every $\bigwedge F_i$ occurring in an antecedent at a node ν in a tree of β , then each F_i occurs as a successor to $\bigwedge F_i$ in an antecedent at ν in a tree of β ;

ii) for every $\bigwedge F_i$ occurring in a succeedent at a node ν in a tree of β , then some F_i occurs as a successor to $\bigwedge F_i$ in a succeedent at ν in a tree of β

iii) for $\neg F$ occurring in an antecedent at a node ν in a tree of β , then F occurs as a successor to $\neg F$ in a succeedent at ν in a tree of β;

iv) for $\neg F$ occurring in a succeedent at a node ν in a tree of β , then F occurs as a successor to $\neg F$ in an antecedent at ν in a tree of β;

v) for $\forall xFx$ occurring in an antecedent at a node ν in a tree of β and for every term t belonging to ν Ft occurs in an antecedent at ν in a tree of β ;

vi) for $\forall xFx$ occurring in a succeedent at a node ν in a tree of β , there is a term t such that Ft occurs as a successor to $\forall xFx$ in a succeedent at ν in a tree of β;

vii) for NF occurring in an antecedent at a node ν in a tree of β , and for every μ immediately after ν F occurs as a successor to NF in an antecedent at μ in a tree of β ; and

viii) for NF occurring in a succeedent at a node ν in a tree of β , there is a μ immediately after ν such that F occurs as a successor to NF in a succeedent at μ in a tree of β .

ANALYZING FOREST An LB-forest is analyzing if every branch in it is.

ANALYZING LEMMA To any sequent we can construct an analyzing LB-forest over it.

Proof:

The construction is as usual by stages. (see (1)).

STAGE 3N-2 Make conditions i - ii - iii - iv - vii - viii true in the definition of analyzing branch tree for all branches in the forest constructed so far.

STAGE 3N-1 Make condition v true for all terms of length $\leqslant N$ in all branches so far.

STAGE 3N Make condition vi true for all branches so far.

The details in the construction should be clear with this sketch and the detailed construction in (1).

QED

SOUNDNESS THEOREM If $\vdash_{LB} \Gamma \longrightarrow \Delta$, then we can find no Kripke-model which falsifies $\Gamma \longrightarrow \Delta$. As long as the binary relation between the worlds in the Kripke model is serial.

Proof:
By induction over the proof of $\Gamma \longrightarrow \Delta$ in LB.

QED

FALSIFIABILITY THEOREM If we have an LB-forest over $\Gamma \longrightarrow \Delta$ with not-secured analyzing branch, then there is a falsifying Kripke-model of $\Gamma \longrightarrow \Delta$. (with serial relations between the worlds.)

COMPLETENESS THEOREM For any sequent $\Gamma \longrightarrow \Delta$, $_{LB}\ \Gamma \longrightarrow \Delta$ if and only if there are no falsifying Kripkemodels (with arbitrary binary relation) of $\Gamma \longrightarrow \Delta$

CONSISTENCT THEOREM For any sequent $\Gamma \longrightarrow \Delta$, exactly one of the following two possibilities are true:

i) there is a secured LB-forest over $\Gamma \longrightarrow \Delta$

ii) there is a not-secured LB-forest over $\Gamma \longrightarrow \Delta$ with analyzing branch.

Now as in LK (1) we go to the strong analyzing lemma.

STRONG LB-FOREST An LB-forest is strong if parameters introduced by $\longrightarrow\forall$ are equal if and only if they arise from formulae in the same strand and with the same analysis.

Now we check the construction in the analyzing lemma and as in LK (1) we get:

STRONG ANALYZING LEMMA. To any sequent we can find a strong analyzing LB-forest over it.

IV. THE HERBRAND THEOREM

We have now done all the preliminaries necessary to proceed to the Skolem and Herbrand theories. The development here is parallell to the previous papers (1, 2, 3, 4)

MORPHISM An LB-morphism is a transformation of LB-forests into Lb-forests preserving the forest structure. A provability morphism transforms secured LB-forests into secured LB-forests. An analyzing morphism transforms analyzing LB-forests into analyzing LB-forests. A falsifiability morphism transform not-secured LB-forests into analyzing not-secured LB-forests. An LB-isomorphism is an LB-morphism which is both a provability and a falsifiability morphism.

SKOLEM TRANSFORMS AND MORPHISMS The Skolem transforms S, S_{π}^{x} and the Skolem morphisms $\mathfrak{S}$, $\mathfrak{S}_{\pi}$ are defined in the obvious way. (See 1,2,3,4)

Given an LB-forest $\mathcal{F}$ then as usual $\mathfrak{S}$ induces a transformation of parameters in $\mathcal{F}$ into terms in $\mathfrak{S}(\mathcal{F}.)$ If $\mathcal{F}$ is a strong LB-forest then the induced transformation is injective.

THEOREM $\mathfrak{S}$ and $\mathfrak{S}_{\pi}$ are provability LB-morphisms.

This is parallel to LI(4). Also the next follows LI ; namely $\mathfrak{S}$ is not in general an analyzing morphism. To get a counterexample apply $\mathfrak{S}$ to a strong analyzing LB-forest over

$$\forall x\ N\ Ax \longrightarrow N\ \forall x Ax$$

or

$$N\neg \forall x \neg Ax \longrightarrow \neg \forall x \neg N Ax$$

DEGREE OF QUANTIFIER A quantifier $\forall x$ occurring as $\forall x Fx$ in $\Gamma \rightarrow \Delta$ has degree N if $\forall x Fx$ has degree N in $\Gamma \rightarrow \Delta$.

Using this concept we introduce the degree of the Skolem functions.

DEGREE OF SKOLEM FUNCTION A Skolem function f introduced for a quantifier in $\Gamma \longrightarrow \Delta$ is assigned the degree of the quantifier.

HERBRAND DOMAIN A set $\mathcal{D}$ of term is an Herbrand domain if for any finite set of symbols, there are only a finite number built up from them which are in $\mathcal{D}$.

Now we are ready to define the Herbrand transforms and the Herbrand morphisms in LB. First observe that we obviously can define $\mathcal{H}^{x}_{\pi,\mathcal{D}}$ in LB analogously to (1) and (4).

HERBRAND-MORPHISM IN LB. We define directly the Herbrand morphisms. The Herbrand transforms come by specialization to the one-sequent one-tree forests.

Given an LB-forest $\mathcal{F}$ over $\Gamma \longrightarrow \Delta$.

i) Apply $\mathcal{S}$ to get the LB-forest $\mathcal{S}(F)$ over $S(\Gamma \longrightarrow \Delta)$ where in $S(\Gamma \longrightarrow \Delta)$ we have assigned degrees to the Skolem functions.

ii) Now for each restricted variable x of degree N applying $\mathcal{H}^{x}_{\pi,\mathcal{D}_N}$ where π is the position x occurs in and $\mathcal{D}_N$ is the subset of $\mathcal{D}$ built up from symbols in $\Gamma \longrightarrow \Delta$, the constant e, and Skolem functions from $S(\Gamma \longrightarrow \Delta)$ of degree $\leq N$.

iii) Having applied ii sufficiently many times to get rid of also all the restricted variables we are left with a sequent we denote by $HER_{\mathcal{D}}(\Gamma \longrightarrow \Delta)$.

The definition above depends on the names of the Skolem functions we use to get $S(\Gamma \longrightarrow \Delta)$.

For the important case where we let

$$\mathcal{D} = \mathcal{D}_n = \{\text{terms of length} < n\}$$

the definition above gives $HER_{\mathcal{D}n}$ well-defined up to the names of the Skolem functions.

We denote $HER_{\mathcal{D}n}$ with HER_n.

We can now pull together some important lines of arguments. Note the following points.

1: A quantifier of degree N is if it is analyzed, analyzed at a node of height N over the bottom node.

2: For general quantifiers of degree N we insert Skolem-functions of degree N.

3: Restricted quantifiers of degree N can be analyzed with terms built up from symbols in the bottomsequent, the constant e, and from parameters introduced by a general variables of degree $<$ N.

4: After applying $\mathfrak{S}$ we analyze restricted quantifiers of degree N with terms built up from the original bottomsequent, the constant e, and Skolem functions of degree N.

Using this we get:

THEOREM HER_n is an **analyzing** LB-morphism.

THEOREM HER_n is a falsifiability morphism

Proof:

Let $\mathcal{F}$ be a not-secured analyzing LB-forest over $\Gamma \to \Delta$ there is then a strong analyzing LB-forest, $\mathcal{F}^*$, over $\Gamma \to \Delta$. $\mathcal{F}^*$ must be not secured since $\mathcal{F}$ is.

By the theorem above **both** $HER_n(\mathcal{F})$ and $HER_n(\mathcal{F}^*)$ is analyzing.

Since $\mathcal{F}^*$ is strong and not-secured, both $\mathfrak{S}(\mathcal{F}^*)$ and $HER_n(\mathcal{F}^*)$ must be not-secured.

By consistency theorem $HER_n(\mathcal{F})$ is not-secured.

QED

By the points above we also get

THEOREM If $\mathcal{F}$ is a secured LB-forest, then for some n $HER_n(\mathcal{F})$ is secured.

Lastly as in LI:

THEOREM If $HER_n(\mathcal{F})$ is secured and $n \leq m$ then $HER_m(\mathcal{F})$ is also secured.

The theorems above give the Herbrand theorem for LB.

Now we can also check the examples of the introduction. All Herbrand transforms of them are equal. They become

$$N\ Ae \rightarrow N\ Af$$

$$N\ Af \ — \ N\ Ae$$

where we let f be the Skolem function.

Neither of the sequents are provable.

V. OTHER MODAL LOGICS.

We introduce a transformation $\{\ \}_n$ on sequents by:

For a sequent $\Gamma \rightarrow \Delta$ the sequent $\{\Gamma \rightarrow \Delta\}_n$ is gotten from $\Gamma \rightarrow \Delta$ by inserting for every negative NF in $\Gamma \rightarrow \Delta$ $F \wedge NF \wedge --- \wedge \underbrace{NN---NF}_{n}$

THEOREM For any sequent $\Gamma \rightarrow \Delta$:

i) $\vdash_M \Gamma \rightarrow \Delta \iff \vdash_{LB} \{\Gamma \rightarrow \Delta\}_1$

ii) $\vdash_{S4} \Gamma \rightarrow \Delta \iff \exists n \vdash_{LB} \{\Gamma \rightarrow \Delta\}_n$

iii) $\vdash_{LB} \{\Gamma \rightarrow \Delta\}_n$ and $m \geq n \Longrightarrow \vdash_{LB} \{\Gamma \rightarrow \Delta\}_m$

Proof:

i) By induction over derivations in M we prove $\vdash_M \Gamma \rightarrow \Delta \Rightarrow \vdash_{LB} \{\Gamma \rightarrow \Delta\}_1$

Semantically we prove $\vdash_M \{\Gamma \rightarrow \Delta\}_1 \Longrightarrow \vdash_M \Gamma \rightarrow \Delta$

Then by $\vdash_{LB} \{\Gamma \rightarrow \Delta\}_1 \Longrightarrow \vdash_M \{\Gamma \rightarrow \Delta\}_1$

and we are done.

ii) By induction over derivations in S4 we prove:

There is an S4-derivation of length $n \longrightarrow \vdash_{LB} \{\Gamma \rightarrow \Delta\}_n$

Semantically $\vdash_{S4} \{\Gamma \rightarrow \Delta\}_m \Rightarrow \vdash_{S4} \Gamma \rightarrow \Delta$

Then $\vdash_{LB} \{\Gamma \rightarrow \Delta\}_n \Rightarrow \vdash_{S4} \{\Gamma \rightarrow \Delta\}_n$

iii) Follows by induction on derivations in LB. QED

From this we get the almost Herbrand theorems for M and S4 mentioned in the introduction. Using the usual imbedding of intuitionistic logic into S4 we get a similar theorem for LI.

THEOREM For any sequent $\Gamma \rightarrow \Delta$ =

i) $\vdash_M \Gamma \rightarrow \Delta \iff \exists n \vdash_{LB} HER_n \{\Gamma \rightarrow \Delta\}_1$

ii) $\vdash_{S4} \Gamma \rightarrow \Delta \iff \exists n \vdash_{LB} HER_n \{\Gamma \rightarrow \Delta\}_n$

VI. REFERENCES

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