INVARIANT WEIGHTS ON SEMI-FINITE VON NEUMANN ALGEBRAS

By

Nils H. Petersen

University of Copenhagen,
Copenhagen.

University of Oslo,
Oslo.

I. INTRODUCTION.

In [10] Størmer proves, that if φ is a faithful, normal state on a semifinite von Neumann algebra invariant w.r.t. a group of $*$-automorphisms of the algebra acting ergodically on the center, then there exists an invariant, faithful, normal, semifinite trace, and φ is a Radon-Nikodym derived of this trace. Hence if the group acts ergodically on the algebra, φ itself becomes a trace (and the algebra finite). The purpose of this paper is to examine the situation, where φ no longer is assumed to be a state but a semifinite weight. I refer to [1] and [7] for the general theory of weights (also contained in [12]) and to [2] and [12] for the theory of weights on von Neumann algebras and the connection between weights and Hilbert-algebras. For the general theory of Hilbertalgebras I refer to [11] and [12], as well as to [5] for general von Neumann algebra theory.

Basically the result is negative. The paper closes with an example of a II_∞ factor on a separable Hilbertspace and an ergodically acting group of $*$-automorphisms leaving a faithful, normal, semifinite weight invariant, but not the trace.

Before this it is proved that if a normal weight, invariant w.r.t. an ergodic group on a semifinite factor satisfies a condition, called L^1-continuity, then it is the trace and is the unique invariant, normal semifinite weight. The question whether the uniqueness always holds (without the assumption of L^1-continuity) is left open.

I use the notation from [5] and [12]. For a Hilbertalgebra Δ is always the modular operator, J the isometric (unitary) involution, $\#$ the involution of the Hilbertalgebra etc. For a weight φ, $\mathfrak{M}_\varphi$ denotes the linear span of the definition order-ideal $\mathfrak{M}_\varphi^+$. $\mathfrak{N}_\varphi = \{x \mid \varphi(x^*x) < +\infty\}$, etc. I take normal weights in the sense of ([12]) (φ is normal if it is the pointwise supremum of the normal, linear, positive functionals it majorizes).

I want to thank Erling Størmer both for his hospitality at the University of Oslo, and for guiding my work. Apart from general, helpful suggestions he formulated and proved Theorem III.3. Also I thank Alfons von Daele for helpful corrections and both him and Alan Hopenwasser for stimulating discussions as well as Francois Combes for fruitful conversations during his visit to the University of Oslo in December 71.

II. AUTOMORPHISMS AND HILBERTALGEBRAS

LEMMA II.1. Let $\mathfrak{A}$ be a Hilbert-algebra, the Hilbert space $\mathcal{H}$ its completion, and $M = \mathcal{L}(\mathfrak{A})$ the left von-Neumann algebra. Let u be a unitary operator on $\mathcal{H}$, so that for all $\xi \in \mathfrak{A}$: $u\pi(\xi)u^{-1} = \pi(u\xi)$ (esp. u maps $\mathfrak{A}$ onto $\mathfrak{A}$), then

i) u is a $\#$-automorphism of $\mathfrak{A}$

ii) u is an isometry of the Hilbert space $\mathcal{D}^\#$,

iii) u maps $\mathfrak{A}'$ onto $\mathfrak{A}'$

iv) u is a $\flat$-anti-automorphism of $\mathfrak{A}'$ and $\pi'(u\eta) = u\pi'(\eta)u^{-1}$ for all $\eta \in \mathfrak{A}'$

v) u is an isometry of the Hilbert space $\mathcal{D}^\flat$,

vi) $u\Delta u^{-1} = \Delta$, $uJu^{-1} = J$.

vii) If $\xi \in \mathcal{H}$ is left-(right-) bounded, then so is $u\xi$, and
$u\pi(\xi)u^{-1} = \pi(u\xi)$ (resp.
$u\pi'(\xi)u^{-1} = \pi'(u\xi)$).

PROOF: i) $\pi(u(\xi_1\cdot\xi_2)) = u\pi(\xi_1\cdot\xi_2)u^{-1} = u\pi(\xi_1)u^{-1}u\pi(\xi_2)u^{-1} = \pi(u\xi_1)\pi(u\xi_2) = \pi((u\xi_1)\cdot(u\xi_2))$ for all $\xi_1,\xi_2 \in \mathfrak{A}$.
so $u(\xi_1\cdot\xi_2) = (u\xi_1)\cdot(u\xi_2)$.
Similarly $u\xi^\# = (u\xi)^\#$ for all $\xi \in \mathfrak{A}$.

ii) For $\xi \in \mathfrak{A}$ $\|u\xi\|_\#^2 = \|u\xi\|^2 + \|(u\xi)^\#\|^2 = \|\xi\|^2 + \|u\xi^\#\|^2 = \|\xi\|^2 + \|\xi^\#\|^2 = \|\xi\|_\#^2$.

Since $\mathfrak{A}$ is dense in the Hilbert space $\mathcal{D}^\#$, $u|_{\mathfrak{A}}$ has a unique isometric extension to $\mathcal{D}^\#$, but as this will be isometric in the norm from $\mathcal{H}$, this extension must coincide with u itself.

vii) Let $\eta \in \mathcal{H}$ be right bounded.

For all $\xi \in \mathfrak{A}$: $\pi(\xi)u\eta = uu^{-1}\pi(\xi)u\eta = u\pi(u^{-1}\xi)\eta = u\pi'(\eta)u^{-1}\xi$, so that $u\eta$ is right bounded and $\pi'(u\eta) = u\pi'(\eta)u^{-1}$.

Let $\xi \in \mathcal{H}$ be left bounded.

For all $\eta \in \mathfrak{A}'$: $\pi'(\eta)u\xi = uu^{-1}\pi'(\eta)u\xi = u\pi'(u^{-1})\xi = u\pi(\xi)u^{-1}\eta$, as η is right bounded, so that vii) is proved.

vi) Let $\xi \in \mathfrak{A}$, then we have from i):

$$J\Delta^{\frac{1}{2}}\xi = \xi^{\#} = u^{-1}(u\xi)^{\#} = u^{-1}J\Delta^{\frac{1}{2}}u\xi = (u^{-1}Ju)(u^{-1}\Delta^{\frac{1}{2}}u)\xi .$$

As $\mathfrak{A}$ is dense in the Hilbert space $\mathcal{D}^{\#}$, $J\Delta^{\frac{1}{2}}$ = the closure of $(u^{-1}Ju)(u^{-1}\Delta^{\frac{1}{2}}u)|_{\mathfrak{A}}$. As u is isometric in $\mathcal{D}^{\#}$, the norm defined by $(u^{-1}Ju)(u^{-1}\Delta^{\frac{1}{2}}u)$ is the same as $\| \ \|_{\#}$, so that $J\Delta^{\frac{1}{2}} = (u^{-1}Ju)(u^{-1}\Delta^{\frac{1}{2}}u)$.

From the uniqueness of the Polar decomposition this gives $J = u^{-1}Ju$ and $\Delta^{\frac{1}{2}} = u^{-1}\Delta^{\frac{1}{2}}u$, so $\Delta = u^{-1}\Delta u$.

Especially it follows, that for all functions measurable w.r.t. the spectral measures of Δ, $uf(\Delta)u^{-1} = f(\Delta)$, so $u\Delta^{-\frac{1}{2}} = \Delta^{-\frac{1}{2}}u$ in particular. So u maps $\mathcal{D}^{b}$ into $\mathcal{D}^{b}$ (so onto), from vii) and the fact, that

$\eta \in \mathfrak{A}' \iff \eta$ right bounded and $\eta \in \mathcal{D}^{b}$ then

iii) follows; from vii) iv) follows as in the proof of i), and similarly v) as ii).

q.e.d.

NB! The lemma and the proof are basically the same as LEMMA 2 in [10].

Let now M be a von Neumann algebra, φ a faithful, normal, semi-finite weight on M^+. π_φ denotes the cyclic representation associated with φ, since φ is faithful it is an isometry of M on $\pi_\varphi(M)$. From [2] and [12] I have the following:

$\mathfrak{A}_\varphi = \mathfrak{M}_\varphi$, with the prehilbert structure of φ is a Hilbert algebra, so that

$\mathfrak{A}_\varphi'' = \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$ and $\mathcal{L}(\mathfrak{A}_\varphi) = \pi_\varphi(M)$, where $\mathcal{L}(\mathfrak{A}_\varphi)$ is the left von Neumann algebra of $\mathfrak{A}_\varphi$. $\mathcal{H}_\varphi$ is the completion of $\mathfrak{A}_\varphi$. Let ψ be the canonical weight on $\mathcal{L}(\mathfrak{A}_\varphi)$ ([2], [12]), from [2] or [12] it is then easy to see, that $\psi \circ \pi_\varphi = \varphi$.

Assume G is a group of $*$-automorphisms of M, and that φ is invariant w.r.t. G. As in ([2] and [4]) we use the obvious generalization of the Gelfand-Naimark-Segal construction, namely representing G on $\mathcal{H}_\varphi$ in the following way; $\mathcal{G}$ is the group of $*$-automorphisms of $\mathcal{L}(\mathfrak{A}_\varphi)$, $\{\pi_\varphi \circ g \circ \pi_\varphi^{-1} \mid g \in G\}$. Each $\alpha_g \in \mathcal{G}$ is implemented by the unitary operator

$$u_g \xi = g(\xi), \quad \text{where} \quad \xi \in \mathfrak{A}_\varphi = \mathfrak{M}_\varphi$$
$$\text{and} \quad \alpha_g = \pi_\varphi \circ g \circ \pi_\varphi^{-1}$$

Since for $x \in \pi_\varphi(\mathfrak{M}_\varphi) = \pi(\mathfrak{M}_\varphi) = \pi(\mathfrak{A}_\varphi)$, $x = \pi(\zeta)$

$u_g x u_g^{-1} \xi = u_g \pi(\zeta) g^{-1}(\xi) = g(\zeta)\cdot\xi = \pi(g(\zeta))\xi = \alpha_g(\pi_\varphi(\zeta))\xi = \alpha_g(x)\xi$, as $\pi_\varphi = \pi$ on $\mathfrak{M}_\varphi = \mathfrak{A}_\varphi$, so that

$$u_g x u_g^{-1} = \alpha_g(x), \quad x \in \pi(\mathfrak{A}_\varphi),$$

since $\pi(\mathfrak{A}_\varphi)$ is strongly dense in $\mathcal{L}(\mathfrak{A}_\varphi)$ it follows, that u_g implements α_g.

From the above calculation it also follows that $u_g \pi(\zeta) u_g^{-1} = \pi(g(\zeta)) = \pi(u_g \xi)$. So the following proposition is merely a summation of known facts:

PROPOSITION II.2. Let M be a von Neumann algebra, φ a faithful, normal, semifinite weight on M^+, invariant w.r.t. a group G of $*$-automorphisms of M. Then G has a faithful unitary representation on $\mathcal{H}_\varphi$ the completion of the Hilbert-algebra $\mathfrak{A}_\varphi$, $g \in G \to u_g$, so that

$$u_g \pi(\xi) u_g^{-1} = \pi(u_g \xi) \quad \text{for all} \quad g \in G \quad \text{and} \quad \xi \in \mathfrak{A}_\varphi .$$

Furthermore $\pi_\varphi(M) = \mathcal{L}(\mathfrak{A}_\varphi)$ and $\varphi = \psi \circ \pi_\varphi$, where π_φ is the representation of M on $\mathcal{H}_\varphi$ induced by φ, and ψ is the canonical weight on $\mathcal{L}(\mathfrak{A}_\varphi)$.

III. INVARIANT WEIGHTS AND TRACES

DEFINITION III.1. Let M be a semifinite von Neumann algebra, τ a faithful, normal, semifinite trace. Let φ be a normal weight on M^+. We say φ is <u>L^1-continuous</u> if for any sequence of elements A_n belonging to the unitball M_1^+, $\|A_n\|_1 \to 0$ implies $\varphi(A_n) \to 0$. (Størmer)

LEMMA III.2. In the above situation φ is semifinite; in fact $\mathcal{M}_\varphi^+ \supset \mathcal{M}_\tau^+$.

PROOF: Let $A \in M_1^+$ be in $\mathcal{M}_\tau^+$, then $A_n = \frac{1}{n}A$ is a sequence with $\|\frac{1}{n}A\|_1 \to 0$, so $\varphi(A_n) \to 0$, so $\varphi(A) < +\infty$, i.e. $A \in \mathcal{M}_\varphi^+$.

q.e.d.

REMARK: 1) In ([3], REMARQUES 4.11 (c)) Combes gives an example showing that there exist normal, semifinite weights, not strictly semifinite. The weights mentioned are all L^1-continuous, as they are derivatives of the trace on $\mathcal{B}(\mathcal{H})$. So L^1-continuity does not imply strictly semifiniteness. The other implication is not true either, which the example in the next section will show.

2) As the trace on $\mathcal{B}(\mathcal{H})$ majorizes the norm, every state on $\mathcal{B}(\mathcal{H})$ is L^1-continuous. So L^1-continuity does not imply normality.

THEOREM III.3. Let M be a semifinite von Neumann algebra with center $\mathcal{C}$, G a group of $*$-automorphisms of M, leaving $\mathcal{C}$ elementwise fixed. Let τ be a faithful, normal, semifinite trace on M^+, and φ a faithful, L^1-continuous and G-invariant weight on M^+ . Let Ψ be a centervalued trace on M^+ , faithful, normal and semifinite. Then Ψ is G-invariant.

PROOF: As in ([5], Chap.III.§4) we identify $\mathscr{C}$ with $L^\infty_{\mathbb{C}}(Z,\nu)$ where Z is locally compact Hausdorff and ν a positive measure on Z. Let $\tilde{\mathscr{C}}^+$ be the positive measurable functions on Z (finite or not).

For all $g \in G$ $\Psi \circ g$ is again a faithful, normal and semifinite centervalued trace on M^+ , so that by ([5], Chap.III.§4, Théorème 2) there exists a unique $Q_g \in \tilde{\mathscr{C}}^+$, $0 < Q_g(\zeta) < +\infty$ l.a.e. on Z , so that

$$\Psi(g(A)) = Q_g \cdot \Psi(A) \text{ , for all } A \in M^+ .$$

By the uniqueness we get $Q_g \cdot Q_h = Q_{g \cdot h}$ l.a.e. for $g,h \in G$.

Assume that Ψ is not invariant, so that for some $g \in G$, $Q_g \neq 1$. Then there exists a $\delta > 0$, a measurable set Y (not of measure 0) and possibly a new g so that $Q_g(\zeta) < 1 - \delta$ for $\zeta \in Y$. Let F be the projection corresponding to 1_Y , $F \in \mathscr{C}$.

We can choose a non-zero projection $E \in M$, $E \leq F$ so that $0 \neq \tau(E) < +\infty$.

For all $\epsilon > 0$ we can find $n \in \mathbb{N}$, so that

$$0 < Q_g^n(\zeta)F(\zeta) < \epsilon . \qquad \zeta \in Z ,$$

that is $Q_g^n \cdot F < \epsilon \cdot F$.

By ([5], Chap.III.§4, Proposition 4) there exists a normal trace ψ on $\tilde{\mathscr{C}}^+$, so that $\tau = \psi \circ \Psi$.

$$\tau(g^n(E)) = \psi(\Psi(g^n(E))) = \psi(Q_g^n \Psi(E)) = \psi(Q_g^n F \Psi(E)) \leq$$

$$\leq \psi(\epsilon F \Psi(E)) = \psi(\epsilon \Psi(E)) = \epsilon \; \tau(E) .$$

so $\tau(g^n(E)) \to 0$; that implies that $\varphi(g^n(E)) = \varphi(E) \to 0$. As φ is faithful this implies $E = 0$, a contradiction.
(Størmer) q.e.d.

Note that the proof is very similar to the proof of LEMMA 2.1. in ([9]).

COROLLARY III.4. In the situation in THEOREM III.3. every normal, faithful, semifinite trace on M^+ is G-invariant.

REMARKS: 1) If φ is majorized by a trace it is L^1-continuous.
2) If φ is a normal state then φ is L^1-continuous. See ([8], LEMMA 2.1).

Note in the following theorem that when G acts ergodically, then φ invariant implies that φ is faithful.

THEOREM III.5. Let M be a semifinite von Neumann algebra, G an ergodically acting group of *-automorphisms of M . Let τ be a normal, semifinite, G-invariant trace on M^+ . Let φ be a normal, semifinite G-invariant weight on M^+ . Then φ is a trace.

PROOF: Consider the standardrepresentation on $\mathcal{H}_\varphi$. Let as in ([12],§13) $\mathfrak{n}$ be the set of all leftbounded elements, ξ , in $\mathcal{H}_\varphi$ such that $\pi(\xi) \in n_\tau$, the definition ideal of τ . From ([12], §13, 13.33) we have the Polar decomposition of the closure π of $\pi|\mathfrak{n}$: $\pi = \Lambda \circ K'$, K' positive selfadjoint on $\mathcal{H}_\varphi$, and Λ a unitary operator from $\mathcal{H}_\varphi$ onto $\mathcal{H}_\tau$, the Hilbert space corresponding to τ . As τ is invariant, the operator V_g defined on n_τ by $V_g(x) = u_g x u_g^{-1}$ (by PROPOSITION II.2 we identify M and $\mathcal{L}(\mathfrak{A}_\varphi)$) extends to a unitary operator on $\mathcal{H}_\tau$, for all $g \in G$. For all $g \in G$ $V_g \circ \Lambda \circ u_g^{-1}$ is then unitary from $\mathcal{H}_\varphi$ onto $\mathcal{H}_\tau$. Further $u_g K' u_g^{-1}$ is positive selfadjoint on $\mathcal{H}_\varphi$ and for $\xi \in \mathfrak{N}$, $g \in G$:

$$\pi(\xi) = \pi(u_g u_g^{-1}\xi) = u_g \pi(u_g^{-1}\xi)u_g^{-1} = V_g(\pi(u_g^{-1}\xi)) =$$

$$V_g \circ \Lambda(K' u_g^{-1}\xi) = (V_g \circ \Lambda \circ u_g^{-1}) \circ (u_g K' u_g^{-1})\xi ,$$ where we have used that u_g maps $\mathfrak{N}$ onto $\mathfrak{N}$ (LEMMA II.1 and since τ is invariant).

Since u_g maps $\mathfrak{N}$ onto $\mathfrak{N}$ and is unitary, it is easy to see, that $u_g K' u_g^{-1}$ and K' have the same domain (as K' is the closure of $K'|\mathfrak{N}$) and that $u_g K' u_g^{-1}$ is the closure of $u_g K' u_g^{-1}|\mathfrak{N}$. So we get:

$$\pi = (V_g \circ \Lambda \circ u_g^{-1}) \circ (u_g K' u_g^{-1}) .$$

But from the uniqueness of the Polar decomposition it then follows that

$$u_g K' u_g^{-1} = K' .$$

As the u_g's act ergodically on $\mathcal{L}(\mathfrak{A}_\varphi)'$ as well, $K' = 1$. From ([12], §13, 13.35 and 13.40) it follows that $\Delta = 1$, so that φ is a trace.

q.e.d.

Combining THEOREM III.3. and III.5. we get:

THEOREM III.6. Let M be a semifinite factor. Let G be an ergodically acting group of *-automorphisms of M. Suppose φ is a normal L^1-continuous G-invariant weight on M^+. Then φ is the trace and furthermore φ is the unique normal, semifinite G-invariant weight on M^+.

IV. AN EXAMPLE

THEOREM IV.1. There exists a II_∞- factor, $\mathcal{B}$, on a separable Hilbert space, a faithful, normal, strictly semifinite weight, ψ on $^+$, an ergodic acting group of $*$-automorphisms of $\mathcal{B}$ leaving ψ invariant, but which does not leave the trace on $\mathcal{B}$ invariant.

PROOF: Throughout the proof we will use the notation from (5), CHAPITRE I, §9. The factor $\mathcal{B}$ is chosen to be the factor of type II_∞ constructed in Théorème 1 ((5), I, §9). As the group G used in the construction, we specify $G = \mathbb{Q}$, the rational numbers.

The trace φ on $\mathcal{B}^+$ is defined by : For $A \in \mathcal{B}^+$, A has a matrix of the form

$R_{s,t} = T_{s-t}\,\mathcal{U}_{s-t}$, with $T_{s-t} \in \mathcal{O}\!\!\mathcal{L}$ (here $= L^\infty_{\mathbb{C}}(\mathbb{R},\nu)$ with ν the Lebesgue-measure). $s,t \in \mathbb{Q}$.

T_0 corresponds to a L^∞-function on $\mathbb{R}$, f_0 , and

$$\varphi(A) = \int_{\mathbb{R}} f(\zeta)d\nu(\zeta) .$$

(This is well-defined, since $f \geq 0$.)

Let now a be a positive, non zero rational number $\neq 1$. Define Ω_a on $L^2_{\mathbb{C}}(\mathbb{R},\nu)$ by

$$\Omega_a f(\zeta) = a^{\frac{1}{2}} f(a\zeta) , \qquad \text{for } \zeta \in \mathbb{R} .$$

Then the following is immediate:

Ω_a is unitary, and for $s \in \mathbb{Q}$

$$\Omega_a^{-1}\,\mathcal{U}_s\Omega_a = \mathcal{U}_{\frac{s}{a}} , \quad \text{and} \quad \Omega_a^{-1} = \Omega_{a^{-1}}$$

and, for $g \in L^\infty_{\mathbb{C}}$ T_g being the corresponding operator in $\mathfrak{A}$,

$$\Omega_a^{-1} T_g \Omega_a = T_{g_a} \quad , \text{ where } \quad g_a(\zeta) = g(a^{-1}\zeta) \ .$$

Now we define $\tilde{\Omega}_a$ on $\tilde{\mathcal{H}}$ by the matrix:

$$R_{s,t} = \begin{cases} \Omega_a & s = at \\ 0 & \text{else.} \end{cases}$$

It is clear, that $\tilde{\Omega}_a$ is unitary and maps $\mathcal{H}_t$ on $\mathcal{H}_{at}$ for $t \in \mathbb{Q}$. (Note, that $\tilde{\Omega}_a$ obviously is not in $\mathcal{B}$)

Claim 1: $\tilde{\Omega}_a$ implements a $*$-automorphism of $\mathcal{B}$.

Since $\mathcal{B}$ is the weak clusure of $\mathcal{B}_0$ it is enough to show, that for $S \in \mathcal{B}_0$, $\tilde{\Omega}_a^{-1} S \, \tilde{\Omega}_a$ is again in $\mathcal{B}_0$:

Let $S = \Phi(T_g)\tilde{\mathcal{U}}_y$, $g \in L^\infty_{\mathbb{C}}$, $y \in \mathbb{Q}$.

The matrix of S is then:

$$R_{s,t} = \begin{cases} T_g \, \mathcal{U}_y & s - t = y \\ 0 & \text{else.} \end{cases}$$

The matrix of $\tilde{\Omega}_a^{-1}\Phi(T_g)\tilde{\mathcal{U}}_y\tilde{\Omega}_a$ is defined by:

The matrix element with indices $s,t = J_s^*\tilde{\Omega}_a^{-1}\Phi(T_g)\,\tilde{\mathcal{U}}_y\tilde{\Omega}_a J_t =$

$$J_s^*\tilde{\Omega}_a^{-1} J_{as} J_{as}^* \Phi(T_g)\,\tilde{\mathcal{U}}_y J_{at} J_{at}^* \tilde{\Omega}_a J_t = \Omega_a^{-1} J_{as}^* \Phi(T_g)\,\tilde{\mathcal{U}}_y J_{at}\Omega_a =$$

$$\left\{ \begin{matrix} \Omega_a^{-1} T_g \, \mathcal{U}_y \Omega_a & \text{if} \quad a(s-t) = y \\ 0 & \text{else} \end{matrix} \right\} =$$

$$\left\{ \begin{matrix} \Omega_a^{-1} T_g \Omega_a \Omega_a^{-1} \, \mathcal{U}_y \Omega_a & \text{if} \quad a(s-t) = y \\ 0 & \text{else} \end{matrix} \right\} =$$

$$\left\{ \begin{matrix} T_{g_a} \, \mathcal{U}_{\frac{y}{a}} & a(s-t) = y \\ 0 & \text{else} \end{matrix} \right\} =$$

$$\left\{ \begin{matrix} T_{g_a} \, \mathcal{U}_z & s-t = z \\ 0 & \text{else} \end{matrix} \right\} \quad (\text{where} \quad z = \tfrac{y}{a}) \ .$$

But this is the matrix of

$$\Phi(T_{g_a})\,\tilde{\mathcal{U}}_z\ , \quad \text{so}$$

$$\tilde{\Omega}_a^{-1}\Phi(T_g)\,\tilde{\mathcal{U}}_y\tilde{\Omega}_a = \Phi(T_{g_a})\,\tilde{\mathcal{U}}_{\frac{y}{a}}$$

as $\mathcal{B}_o$ consists of sums of operators of this type, $\mathcal{B}_o$ is left invariant, so that the claim follows.

<u>Claim 2</u>: φ is not invariant under this automorphism of $\mathcal{B}$.

Let S be as in proof of claim 1, with $y = 0$. $R_{o,o} = T_g$, with some $g \in L^\infty_{\mathbb{C}}$. $\tilde{\Omega}_a^{-1} S\,\tilde{\Omega}_a = \Phi(T_{g_a})$, that is the $(0,0)$ matrix-element is T_{g_a}, so

$$\varphi(S) = \int_{\mathbb{R}} g(\zeta)d\nu(\zeta) \quad \text{and}$$

$$\varphi(\tilde{\Omega}_a^{-1} S\,\tilde{\Omega}_a) = \int_{\mathbb{R}} g(a^{-1}\zeta)d\nu(\zeta) = a\int_{\mathbb{R}} g(\zeta)d\nu(\zeta)\ .$$

As $a \neq 1$, claim 2 follows by chosing a $g \in L^\infty_{\mathbb{C}}$, integrable w.r.t. Lebesgue-measure and not a zero-function.

Now we define the weight ψ on $\mathcal{B}^+$.

As on page IV.1, for $A \in \mathcal{B}^+$, define

$$\psi(A) = \int_{\mathbb{R}} f(\zeta)\cdot\frac{1}{|\zeta|}\,d\nu(\zeta)\ , \text{ where } f \in L^\infty_{\mathbb{C}}$$

is derived as on page IV. .

Consider the intervals $[n,n+1[$, where $n \in Z$ and $n \geq 1$ or ≤ -2, and $[\frac{1}{n+1}, \frac{1}{n}[$ $[-\frac{1}{n}, -\frac{1}{n+1}[$, $n \in N$; they form a partition of $]-\infty,0[\,\cup\,]0,\infty[$. Calling them I_n (giving them some ordering) consider the pos. lin. normal functional on $\mathcal{B}$ defined by $\psi_n(A) = \int_{I_n} f(\zeta)\cdot\frac{1}{|\zeta|}\,d\nu(\zeta)$.

Consider the projection $\Phi(T_{\chi_{I_n}}) \in \mathcal{B}$, where χ_{I_n} is the characteristic function for I_n . Since $1-\Phi(T_{\chi_{I_n}}) = \Phi(1-T_{\chi_{I_n}}) = \Phi(T_{1-\chi_{I_n}})$, and $\psi_n(\Phi(T_{1-\chi_{I_n}})) = 0$, $\mathrm{Supp}\psi_n \subseteq \Phi(T_{\chi_{I_n}})$; so the ψ_n's have orthogonal support, and $\sum_n \psi_n = \psi$, so by ([3] Prop.4.2 and 4.5) ψ is strictly semifinite. Also it follows, that ψ is normal.

ψ is faithful, since if for some $S \in \mathcal{B}^+$ $\psi(S) = 0$, then $\psi_n(S) = 0$, so that, as $\psi_n(A) = \varphi(A\cdot\Phi(T_g))$ where $g(\zeta) = \frac{1}{|\zeta|}\cdot\chi_{I_n}(\zeta)$, $f\cdot g = 0$, so that f is zero a.e. on I_n , and so on $\mathbb{R}$. From the proof of ([5], Prop.1, Chap.I,§9) the proof that this for a positive S implies $S = 0$ carries over.

ψ is invariant with respect to the constructed *-automorphism of $\mathcal{B}$. To prove this, let $S \in \mathcal{B}^+$. S has matrix: $R_{s,t} = T_{s-t}\,\mathcal{U}_{s-t}$, $T_{s-t} \in \mathfrak{A}$. Let T_o correspond to the $L^\infty_{\mathbb{C}}$-function f .

Then $\psi(S) = \int_{\mathbb{R}} f(\zeta)\frac{1}{|\zeta|}\,d\nu(\zeta)$.

$\tilde{\Omega}_a^{-1} S\,\tilde{\Omega}_a$ has as its $(0,0)$-matrix element $J_o^*\tilde{\Omega}_a^{-1}J_oJ_o^*\,S\,J_oJ_o^*\tilde{\Omega}_aJ_o = \Omega_a^{-1}T_o\Omega_a$, which from the beginning of the proof corresponds to the $L^\infty_{\mathbb{C}}$-function f_a .

So $\psi(\tilde{\Omega}_a^{-1} S\,\tilde{\Omega}_a) = \int_{\mathbb{R}} f_a(\zeta)\frac{1}{|\zeta|}\,d\nu(\zeta) = \int_{\mathbb{R}} f(a^{-1}\zeta)\frac{1}{|\zeta|}\,d\nu(\zeta) =$

$$\int_{\mathbb{R}} f(\zeta)\frac{1}{a|\zeta|}\cdot a\,d\nu(\zeta) = \int_{\mathbb{R}} f(\zeta)\frac{1}{|\zeta|}\,d\nu(\zeta) = \psi(S) .$$

Consider now a unitary operator $\tilde{\mathcal{V}}$ from $\Phi(\mathfrak{A}) = \tilde{\mathfrak{A}}$. Its matrix has the form

$$Q_{s,t} = \begin{cases} T_u & s = t \\ 0 & \text{else} \end{cases} \qquad \text{where } T_u \text{ is a unitary from } \mathfrak{A}.$$

Set S be in $\mathcal{B}$, with matrix $R_{s,t}$. The $(0,0)$-element in the matrix of $\tilde{\mathcal{V}}^{-1}S\,\tilde{\mathcal{V}}$ is then: $J_e^*\tilde{\mathcal{V}}^{-1}S\,\tilde{\mathcal{V}}J_e = T_u^{-1}R_{o,o}T_u = R_{o,o}$, since $R_{o,o} \in \mathfrak{A}$, which is abelian. From this it follows, that the *-automorphism of M , that $\tilde{\vartheta}$ implements leaves ψ invariant.

Consider now the group of *-automorphisms of M spanned by the *-automorphisms implemented by $\tilde{\Omega}_a$, $a \in \mathbb{Q}_+$ and all the unitaries from $\tilde{\mathfrak{A}}$. It is clear that this group leaves ψ invariant, but not φ . So to prove the theorem it is enough to prove that it acts ergodicly on M .

Assume that $S \in \mathfrak{B}$ is invariant under the group. Then S commutes with $\tilde{\mathfrak{A}}$, so that by the proof of Théorème 1 and by Lemme 2 in ([5], Chap.I,§9) S itself belongs to $\tilde{\mathfrak{A}}$. So if S has the matrix: $R_{s,t}$,

$$R_{s,t} = \begin{cases} T_f & s = t \\ 0 & \text{else} \end{cases} \qquad \text{with } f \in L^{\infty}_{\mathbb{C}}(\mathbb{R}, \nu) \; .$$

But from P.IV.4. $\tilde{\Omega}_a^{-1} S \tilde{\Omega}_a$ has as its (0,0)-element in its matrix T_{f_a} . So for all $a \in \mathbb{Q}_+$, $f = f_a$ almost everywhere, so f is constant a.e. and so S is a constant, so that the group acts ergodically.

q.e.d.

References

[1] F. Combes: Poids sur une C^*-algèbre. J.Math.Pures et Appl., 9e Série, t.47, 1968, p. 57-100.

[2] F. Combes: Poids associé a une algèbre Hilbertienne a gauche. Compositio Mathematica, vol.23, Fasc.1, 1971, p. 49-77.

[3] F. Combes: Poids et espérances conditionelles dans les algèbres de von Neumann. Bull.Soc.Math.France, 99, 1971, p. 73-112.

[4] A. van Daele: The upper envelope of invariant functionals majorized by an invariant weight. Preprint Series No.29, Dec. 1971, University of Oslo.

[5] J. Dixmier: Les algèbres d'operateurs dans l'espace Hilbertien. Gauthier-Villars Paris, 2e edition, 1969.

[6] N. Dunford and J. Schwarz: Linear Operators II. Interscience Publishers, New York (1963).

[7] G. Kjærgård Pedersen: Measure Theory for C^*-algebras. Math.Scand. 19 (1966), p. 131-145.

[8] E. Størmer: Types of von Neumann algebras associated with extremal invariant states. Commun.Math.Phys.6, p. 194-204, (1967).

[9] E. Størmer: States and invariant maps of operator algebras. J. Funtional analysis 5, p. 44-65, (1970)

[10] E. Størmer: Automorphisms and invariant states of operator-algebras. Acta Mathematica, vol.127, (1971).

[11] M. Takesaki: Tomita's theory of modular Hilbert algebras and its applications. Springer-Verlag, Lecture notes in mathematics, 128, (1970).

[12] M. Takesaki: Lecture notes U.C.L.A. on operator algebras.

[13] M. Tomita: Standard forms of von Neumann algebras. The V^{th} functional analysis symposium of the Math.Soc. of Japan, Sendai, (1967).