Spectra of states, and asymptotically abelian C*-algebras

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1. Introduction. If $\mathfrak{A}$ is an asymptotically abelian C*-algebra and ρ is an extremal invariant state with cyclic representation π_ρ, the structure of ρ and $\pi_\rho(\mathfrak{A})''$ is quite well understood if $\pi_\rho(\mathfrak{A})''$ is a semi-finite von Neumann algebra [8,13,15,16]. It is the purpose of the present paper to study the general case when $\pi_\rho(\mathfrak{A})''$ may also be of type III. This is best done if we define the spectrum $\mathrm{Spec}(\rho)$ of a state ρ of a C*-algebra to be - roughly - the set of real numbers u such that there is $A \in \mathfrak{A}$ with $\rho(A^*A) = 1$ such that $u\rho(BA)$ is approximately equal to $\rho(AB)$ for all $B \in \mathfrak{A}$ (Definition 2.1). For exampel; ρ is a trace if and only if $\mathrm{Spec}(\rho) = \{1\}$, and if ρ is a pure state and not a homomorphism then $\mathrm{Spec}(\rho) = \{0,1\}$. If x_ρ is the cyclic vector such that $\rho(A) = (\pi_\rho(A)x_\rho, x_\rho)$ for $A \in \mathfrak{A}$, we may cut down $\pi_\rho(\mathfrak{A})''$ by the support E_ρ of the state ω_{x_ρ}, and define the modular operator of Tomita of x_ρ relative to this smaller von Neumann algebra. If we extend the modular operator to be 0 on the complement of E_ρ it turns out that its spectrum equals $\mathrm{Spec}(\rho)$ (Theorem 2.3). Together with the resent results of Connes [2, 3] this result gives us a useful tool for

studying the spectrum of ρ . Now assume $\mathfrak{A}$ is asymptotically abelian and that ρ is a strongly clustering invariant state, e.g. if ρ is an invariant factor state. Then our main result (Theorem 3.1) states that the nonzero elements in $\mathrm{Spec}(\rho)$ form a closed subgroup of the multiplicative group $\mathbb{R}^+$ of positive real numbers. Furthermore, if ω is a state of $\mathfrak{A}$ quasi-equivalent to ρ then $\mathrm{Spec}(\rho) \subset \mathrm{Spec}(\omega)$. This last statement shows in particular that $\mathrm{Spec}(\rho)$ is a $*$-isomorphic invariant for $\pi_\rho(\mathfrak{A})''$. Since every proper closed subgroup of $\mathbb{R}^+$ is cyclic we have obtained an isomorphism class for each $u \in [0,1]$, where 1 correspond to the group $\{1\}$ and 0 to $\mathbb{R}^+$. It seems that $\mathrm{Spec}(\rho)$ most often equals $\mathbb{R}^+$. This is in particular the case when $\mathfrak{A}$ is asymptotically abelian with respect to a one parameter group and ρ is an extremal KMS-state (Corollary 4.5).

We shall follow the theory of asymptotically abelian C*-algebras as developed in [15]. Thus we shall say a C*-algebra $\mathfrak{A}$ is asymptotically abelian with respect to a group G of *-automorphisms if there is a sequence $\{g_n\}_{n=1,2,\ldots,}$ in G such that $\lim_n \|[g_n(A),B]\| = 0$ for all $A,B \in \mathfrak{A}$. This definition is sufficiently general to take care of most cases of physical interest and extends in particular the original one of Doplicher, Kastler, and Robinson [5] and Ruelle [12], in which case G is the translation group $\mathbb{R}^n$. We refer the reader to [6] for a general survey of the theory of asymptotically abelian C*-algebras. It is unclear at the present whether our results can be generalized to other definitions of asymptotically abelian systems.

As indicated above the main part of our analysis will be concerned with the modular operator of Tomita. We refer the reader to the notes of Takesaki [17] for the theory of Tomita and Takesaki.

For the general theory of von Neumann algebras the reader is referred to the book of Dixmier [4]. We only remark that the strong -* topology on a von Neumann algebra is generated by the semi-norms $A \to \|Ax\| + \|A^*x\|$, and that the usual density theorems hold for this topology.

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2. The spectrum of a state. In this section we shall give two equivalent definitions of the spectrum of a state and then obtain some simple properties of the spectrum.

Definition 2.1. Let $\mathfrak{A}$ be a C*-algebra and ρ a state of $\mathfrak{A}$. Then the spectrum of ρ, denoted by $\mathrm{Spec}(\rho)$, is the set of real numbers u such that given $\epsilon > 0$ there is $A \in \mathfrak{A}$ for which $\rho(A^*A) = 1$ such that

$$| u\,\rho(BA) - \rho(AB)| < \epsilon\, \rho(B^*B)^{\frac{1}{2}}$$

for all $B \in \mathfrak{A}$.

We shall soon show that u must be non negative. A modification of the same argument shows that in the definition we might as well have assumed u to be a complex number. It is clear that the definition can be generalized to other linear functionals.

Let ρ and $\mathfrak{A}$ be as above. Let π_ρ be a representation of ρ on a Hilbert space $\mathcal{H}_\rho$ and x_ρ a unit vector in $\mathcal{H}_\rho$ cyclic for $\pi_\rho(\mathfrak{A})$ such that $\rho(A) = (\pi_\rho(A)x_\rho, x_\rho)$ for $A \in \mathfrak{A}$. Let $\mathcal{R}_\rho$ denote the von Neumann algebra $\pi_\rho(\mathfrak{A})''$. Let $E_\rho = [\mathcal{R}_\rho' x_\rho]$. Then x_ρ is a separating and cyclic vector for the von Neumann algebra $E_\rho \mathcal{R}_\rho E_\rho$ acting on $E_\rho \mathcal{H}_\rho$. Let Δ_ρ be the modular operator of x_ρ relative to $E_\rho \mathcal{R}_\rho E_\rho$, and consider it as an operator on $\mathcal{H}_\rho$ by defining it to be 0 on $(I - E_\rho)\mathcal{H}_\rho$.

Definition 2.2. With the above notation we call Δ_ρ the modular operator of the state ρ.

Theorem 2.3. Let $\mathfrak{A}$ be a C*-algebra and ρ a state of $\mathfrak{A}$ with modular operator Δ_ρ. Then $\mathrm{Spec}(\rho) = \mathrm{Spec}(\Delta_\rho)$.

Proof: Suppose $u \neq 0$ and $u \in \mathrm{Spec}(\rho)$. In the notation introduced above drop the subscripts ρ, so $\mathcal{R} = \mathcal{R}_\rho, E = E_\rho$, $x = x_\rho$, $\Delta = \Delta_\rho$, $\pi = \pi_\rho$. We first show u belongs to the spectrum of ω_x considered as a state on $E\mathcal{R}E$. Since $\pi(\mathfrak{A})$ is dense in $\mathcal{R}$ in the strong-* topology it is clear that u belongs to the spectrum $\mathrm{Spec}(\omega_x)$ of ω_x as a state of $\mathcal{R}$.

Let $\delta > 0$ be given. Choose ϵ, $0 < \epsilon < 1$, so small that $2|u|^{-1} \max\{\epsilon, \epsilon(u+\epsilon)\} \leq \delta$. We assert that if $A \in \mathcal{R}$ is such that $\|Ax\| = 1$ and

1) $|u(Ax,B^*x) - (Bx,A^*x)| < \epsilon\|Bx\|$

for all $B \in \mathcal{R}$, then $\|EAEx\|^2 > 1-\delta$.

For this let $\eta = \max\{\epsilon, \epsilon(u+\epsilon)\}$. Let $B = A^*$. Then 1) gives

2) $|u - \|A^*x\|^2| < \epsilon\|A^*x\|$,

hence $\|A^*x\|^2 < u + \epsilon\|A^*x\|$. If $\|A^*x\| \geq 1$ we have since $\epsilon < 1$

$$\|A^*x\| < \frac{u}{\|A^*x\|} + \epsilon \leq u + \epsilon .$$

Thus $\|A^*x\| \leq \max\{1, u+\epsilon\}$. Now apply 1) to $B = EA^*$. Then we have

3) $|u - \|EA^*x\|^2| < \epsilon\|EA^*x\| \leq \epsilon\|A^*x\| \leq \eta$

In particular, since η is arbitrarily small we have that $u \geq 0$. Now apply 1) to $B = EA^*E$. Then we have

4) $|u\|EAEx\|^2 - \|EA^*Ex\|^2| < \epsilon\|EA^*Ex\| \leq \eta$.

Since $u \neq 0$ we then have by 3) and 4)

$$\begin{aligned} 0 &\leq 1 - \|EAEx\|^2 = u^{-1}|u\|EAEx\|^2 - u| \\ &\leq u^{-1}|u\|EAEx\|^2 - \|EA^*Ex\|^2| + u^{-1}|\,\|EA^*Ex\|^2 - u| \\ &< u^{-1}\eta + u^{-1}\eta \leq \delta . \end{aligned}$$

The assertion follows. Note that if $B \in E\mathcal{R}E$ then

$$\begin{aligned} &|u(EAEx, B^*x) - (Bx, EA^*Ex)| = \\ &|u(Ax, B^*x) - (Bx, A^*x)| < \epsilon\|Bx\| . \end{aligned}$$

Since $1 \geq \|EAEx\|^2 > 1 - \delta$ it follows that $u \in \mathrm{Spec}(\omega_x|E\mathcal{R}E)$, as we wanted to show.

Restricting attention to $E\mathcal{R}E$ we may thus assume x is separating and cyclic for $\mathcal{R}$ (so $E = I$). Let J be the conjugation so that $J\Delta^{\frac{1}{2}}Bx = \Delta^{-\frac{1}{2}}JBx = B^*x$ for $B \in \mathcal{R}$ [17,Thm.7.1]. Since the Tomita algebra (called modular algebra in [17]) is strong-* dense in $\mathcal{R}$ we may assume A belongs to the Tomita algebra, and thus Ax belongs to the domain of $\Delta^{-\frac{1}{2}}$ (see e.g. proof of [17,Thm.10.1]). Then 1) becomes

$$|u(Ax, \Delta^{-\frac{1}{2}}JBx) - (Bx, J\Delta^{\frac{1}{2}}Ax)| < \epsilon\|Bx\| ,$$

or

$$|(u\Delta^{-\frac{1}{2}}Ax, JBJx) - (\Delta^{\frac{1}{2}}Ax, JBJx)| < \epsilon\|JBJx\| .$$

Since $J\mathcal{R}J = \mathcal{R}'$ by [17,Thm.12.1], and x is cyclic for $\mathcal{R}'$ we have

$$|(u\Delta^{-\frac{1}{2}}Ax - \Delta^{\frac{1}{2}}Ax, y)| < \epsilon\|y\|$$

for all $y \in \mathcal{H}$. Thus we have

$$\|\Delta^{-\frac{1}{2}}(uI - \Delta)Ax\| = \|u\Delta^{-\frac{1}{2}}Ax - \Delta^{\frac{1}{2}}Ax\| < \epsilon \ .$$

Now $\Delta^{-\frac{1}{2}}(u^{\frac{1}{2}}I + \Delta^{\frac{1}{2}}) \geq I$. Hence we have

$$\begin{aligned}\|(u^{\frac{1}{2}}I - \Delta^{\frac{1}{2}})Ax\| &\leq \|\Delta^{-\frac{1}{2}}(u^{\frac{1}{2}}I + \Delta^{\frac{1}{2}})(u^{\frac{1}{2}}I - \Delta^{\frac{1}{2}})Ax\| \\ &= \|\Delta^{-\frac{1}{2}}(uI - \Delta)Ax\| < \epsilon \ .\end{aligned}$$

Since Ax is a unit vector and ϵ is arbitrary $u^{\frac{1}{2}} \in \mathrm{Spec}(\Delta^{\frac{1}{2}})$, hence $u \in \mathrm{Spec}(\Delta)$.

Now suppose $u = 0 \in \mathrm{Spec}(\rho)$. If $0 \notin \mathrm{Spec}(\Delta)$ $E = I$, so x is separating and cyclic for $\mathcal{R}$. Furthermore since $0 \notin \mathrm{Spec}(\Delta)$ there exists $k > 0$ such that $\Delta^{\frac{1}{2}} \geq kI$. By 1) we can for each integer n find $A_n \in \mathcal{R}$ such that $\|A_n x\| = 1$ and

$$|(Bx, A_n^* x)| < {}^1/n \|Bx\|$$

for all $B \in \mathcal{R}$. Since x is cyclic we have $\|A_n^* x\| < {}^1/n$ for each n . Thus

$${}^1/n > \|A_n^* x\| = \|J\,\Delta^{\frac{1}{2}}A_n x\| = \|\Delta^{\frac{1}{2}}A_n x\| \geq k\|A_n x\| = k \ .$$

This is a contradiction for n sufficiently large. Therefore $0 \in \mathrm{Spec}(\Delta)$, and we have shown $\mathrm{Spec}(\rho) \subset \mathrm{Spec}(\Delta)$.

Conversely assume $u \in \mathrm{Spec}(\Delta)$. We assert that $0 \in \mathrm{Spec}(\Delta^{-\frac{1}{2}}(uI - \Delta))$. Indeed, if $u = 0$ then $0 \in \mathrm{Spec}(\Delta^{\frac{1}{2}}) = - \mathrm{Spec}(\Delta^{-\frac{1}{2}}(0I - \Delta))$, so the assertion holds for $u = 0$. If $u \neq 0$ choose a spectral projection F for Δ such that $F\Delta$ and $F\Delta^{-\frac{1}{2}}$ are bounded and $u \in \mathrm{Spec}(F\Delta)$. Let $\epsilon > 0$ and choose a unit vector $y \in F\mathcal{H}$ such that $\|(uI - \Delta)y\| < \epsilon/\|F\Delta^{-\frac{1}{2}}\|$. Then we have

$$\begin{aligned}\|\Delta^{-\frac{1}{2}}(uI - \Delta)y\| &= \|\Delta^{-\frac{1}{2}}F(uI - \Delta)y\| \\ &\leq \|\Delta^{-\frac{1}{2}}F\|\|(uI - \Delta)y\| < \epsilon \ .\end{aligned}$$

Thus $0 \in \mathrm{Spec}(\Delta^{-\frac{1}{2}}(uI - \Delta))$ as asserted. Now the Tomita algebra is dense in the domain of $\Delta^{-\frac{1}{2}}(uI - \Delta)$, (see proof of [17, Thm.10.1]). Therefore if $\epsilon > 0$ is given there exists A in the Tomita algebra such that $\|Ax\| = 1$ and

$$\|u\Delta^{-\frac{1}{2}}Ax - \Delta^{\frac{1}{2}}Ax\| < \varepsilon .$$

Therefore if $B \in \mathcal{R}$ we have

$$|u(Ax,B^*x) - (Bx,A^*x)| =$$
$$|(u\Delta^{-\frac{1}{2}}Ax,JBx) - (\Delta^{\frac{1}{2}}Ax,JBx)| < \varepsilon\|JBx\| = \varepsilon\|B\| .$$

Thus $u \in \mathrm{Spec}(\omega_x)$. Since $\pi(\mathfrak{A})$ is strong-* dense in $\mathcal{R}$, $u \in \mathrm{Spec}(\rho)$. The proof is complete.

<u>Corollary 2.4</u>. Let $\mathfrak{A}$ be a C*-algebra and ρ a state of $\mathfrak{A}$, $\rho(A) = (\pi_\rho(A)x_\rho,x_\rho)$ for $A \in \mathfrak{A}$. Then

i) $\mathrm{Spec}(\rho)$ is a closed subset of the non negative real numbers such that $1 \in \mathrm{Spec}(\rho)$.

ii) If $u \neq 0$, $u \in \mathrm{Spec}(\rho)$ then $u^{-1} \in \mathrm{Spec}(\rho)$.

iii) $\mathrm{Spec}(\rho) = \{1\}$ if and only if ρ is a trace.

iv) $\mathrm{Spec}(\rho) = \{0,1\}$ if and only if ω_{x_ρ} is a trace on $\pi_\rho(\mathfrak{A})'$ but ρ is not a trace on $\mathfrak{A}$.

<u>Proof</u>: i) Since $1 \in \mathrm{Spec}(\Delta_\rho)$ and $\mathrm{Spec}(\Delta_\rho)$ is a closed subset of the non negative reals, the same is true for ρ by Theorem 2.3.

ii) Since $u \neq 0$, $u \in \mathrm{Spec}(\Delta_\rho)$ implies $u^{-1} \in \mathrm{Spec}(\Delta_\rho)$ by [17,Thm.7.1], ii) follows from Theorem 2.3.

iii) If ρ is a trace then $\rho(AB) = \rho(BA)$ for all $A,B \in \mathfrak{A}$. Let $u \in \mathrm{Spec}(\rho)$. Then

$$|u\rho(BA) - \rho(AB)| = |u-1||\rho(AB)|$$

for all $A,B \in \mathfrak{A}$. If $u \neq 1$ let $\varepsilon = \frac{1}{2}|u-1|$. Choose $A \in \mathfrak{A}$ such that $\rho(A^*A) = 1$ and such that

$$|u-1||\rho(AB)| < \tfrac{1}{2}|u-1|\rho(B^*B)^{\frac{1}{2}}$$

for all $B \in \mathfrak{A}$. Thus $|\rho(AB)| < \frac{1}{2}\rho(B^*B)^{\frac{1}{2}}$ for all B. In particular if $B = A^*$ we get $1 = \rho(A^*A) = \rho(AA^*) < \frac{1}{2}\rho(AA^*)^{\frac{1}{2}} = \frac{1}{2}$, a contradiction. Thus $u = 1$.

Conversely, if $\mathrm{Spec}(\rho) = 1$ then by Theorem 2.3 $\mathrm{Spec}(\Delta_\rho) = \{1\}$, so ω_x is a trace on $\pi_\rho(\mathfrak{A})''$, see e.g. proof of [17, Thm. 13.1], hence ρ is a trace on $\mathfrak{A}$.

iv) Assume $\mathrm{Spec}(\rho) = \{0,1\}$. Then the spectrum of $\Delta_\rho E_\rho$ acting on $E_\rho \mathcal{H}_\rho$ is $\{1\}$, where $E_\rho = [\pi_\rho(\mathfrak{A})' x_\rho]$. Thus, as above, ω_{x_ρ} is a trace on $E_\rho \pi_\rho(\mathfrak{A})'' E_\rho$, hence a trace on $\pi_\rho(\mathfrak{A})'$. By iii) ρ is not a trace. Conversely, if ω_{x_ρ} is a trace on $\pi_\rho(\mathfrak{A})'$, but ρ is not a trace, then as above the spectrum of $\Delta_\rho E_\rho$ is $\{1\}$, hence $\mathrm{Spec}(\Delta_\rho) = \{0,1\}$, so by Theorem 2.3 $\mathrm{Spec}(\rho) = \{0,1\}$. The proof is complete.

<u>3. Asymptotically abelian C*-algebras</u>. This section is devoted to the main result on asymptotically abelian C*-algebras and its proof. Following [15] if $\mathfrak{A}$ is a C*-algebra and G a group of *-automorphisms of $\mathfrak{A}$, we say $\mathfrak{A}$ is <u>asymptotically abelian</u> with respect to G if there is a sequence $\{g_n\}_{n\geq 1}$ in G such that whenever $A,B \in \mathfrak{A}$ then

$$\lim_{n\to\infty} \|[g_n(A),B]\| = 0 ,$$

where $[\ ,\ ]$ is the Lie commutator. A G-invariant state ρ of $\mathfrak{A}$ is said to be <u>strongly clustering</u> (or strongly mixing) if for $A,B \in \mathfrak{A}$ we have

$$\lim_{n\to\infty} \rho(g_n(A)B) = \rho(A)\rho(B) .$$

We shall need a concept which is slightly more general than that of quasi-equivalence. If ρ and ω are states of $\mathfrak{A}$ we say ω is <u>quasi-contained</u> in ρ if the cyclic representation π_ω of ω is quasi-contained in that π_ρ of ρ ; in other words π_ω is quasi-equivalent to a subrepresentation of π_ρ . It is easy to see that ω is quasi-contained in ρ if and only if $\omega = \bar{\omega} \circ \pi_\rho$, where $\bar{\omega}$ is a normal state of $\pi_\rho(\mathfrak{A})''$.

<u>Theorem 3.1</u>. Let $\mathfrak{A}$ be a C*-algebra which is asymptotically abelian with respect to a group G of *-automorphisms. Suppose ρ is a strongly clustering G-invariant state. Then the nonzero elements in $\mathrm{Spec}(\rho)$ form a closed subgroup of the multiplicative group of positive real numbers. Furthermore, if ω is a state of $\mathfrak{A}$ quasi-contained in ρ then $\mathrm{Spec}(\rho) \subset \mathrm{Spec}(\omega)$.

We shall first prove a few lemmas. Let as in the proof of Theorem 2.3 π be a *-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$, x a unit vector in $\mathcal{H}$ cyclic for $\pi(\mathfrak{A})$ such that $\rho(A) = (\pi(A)x,x)$ for $A \in \mathfrak{A}$. Let $\mathcal{R} = \pi(\mathfrak{A})''$, let $g \to U_g$ be a unitary representation of G on $\mathcal{H}$ such that $U_g x = x$ and $\pi(g(A)) = U_g \pi(A) U_g^{-1}$ for $g \in G$, $A \in \mathfrak{A}$. Let E_0 be the orthogonal projection on $\{y \in \mathcal{H} : U_g y = y \text{ for all } g \in G\}$. Then $E_0 = [x]$ is the one dimensional projection on the subspace spanned by x , since ρ is extremal G-invariant by [15,Thm.4.4] and therefore $E_0 = [x]$ by [15,Thm.3.3]. Let $\{g_n\}$ be a sequence in G such that $\lim_n \|[g_n(A),B]\| = 0$ and $\lim_n \rho(g_n(A)B) = \rho(A)\rho(B)$. Then by [15,Thm.4.4] $U_{g_n} \to [x]$ weakly, and if $A \in \mathfrak{A}$ then $U_{g_n} \pi(A) U_{g_n}^{-1} \to \rho(A)I$ weakly. Let $E = [\mathcal{R}'x]$ be the support of ω_x on $\mathcal{R}$. Let Δ be the modular operator of the state ρ (Definition 2.2) and J the conjugation of the Hilbert space $E\mathcal{H}$ defined by x, so $JE\mathcal{R}EJ = E\mathcal{R}'$ by [17,Thm.12.1]. Extend J to all of $\mathcal{H}$ by defining it to be 0 on $(I-E)\mathcal{H}$. Thus $J = JE = EJ$. Since ω_x is invariant under the automorphisms $T \to U_g T U_g^{-1}$ its support E is invariant. Therefore $EU_g = U_g E$ for all $g \in G$.

Lemma 3.2. Let $A \in \pi(\mathfrak{A})$. Let $y \in \mathcal{H}$. Then

$$\lim_{n\to\infty} \|U_{g_n}^{-1} A U_{g_n} y\| = \|Ax\| \|y\|$$

Proof. For $B, C \in \mathfrak{A}$ we have

$$\lim_n \|[g_n^{-1}(C), B]\| = \lim_n \|[B, g_n^{-1}(C)]\| = \lim_n \|[g_n(B), C]\| = 0$$

and $\lim \rho(g_n^{-1}(C)B) = \lim \rho(B g_n^{-1}(C)) = \lim \rho(g_n(B)C) = \rho(B)\rho(C)$, so that the sequence $\{g_n^{-1}\}$ have the same properties as the sequence $\{g_n\}$. Thus for $B \in \mathfrak{A}$ we have weak $\lim_n U_{g_n}^{-1}\pi(B)U_{g_n} = \rho(B)I$. Thus we have for $A \in \pi(\mathfrak{A})$

$$\begin{aligned}\lim_n \|U_{g_n}^{-1} A U_{g_n} y\|^2 &= \lim(U_{g_n}^{-1} A U_{g_n} y, U_{g_n}^{-1} A U_{g_n} y) \\ &= \lim(U_{g_n}^{-1} A^* A U_{g_n} y, y) \\ &= \omega_x(A^*A)(y,y) \\ &= \|Ax\|^2 \|y\|^2 .\end{aligned}$$

The proof is complete.

Lemma 3.3. Let $\epsilon > 0$ be given. Let $A \in \pi(\mathfrak{A})$ be chosen so that $1 = \|Ax\| < \|EAx\| + \epsilon$. Let $y \in E\mathcal{H}$.
Then we have

$$\lim_n \big| \|E U_{g_n} A U_{g_n}^{-1} y\| - \|y\| \big| < \epsilon \|y\| .$$

Proof. We first consider the case when $y = B'x$ with $B' \in \mathcal{R}'$. Since $U_g^{-1} E U_g = E$ for $g \in G$ and weak $\lim_n U_{g_n} = [x]$ we have

$$\begin{aligned}
\lim_n \|E U_{g_n} A U_{g_n}^{-1} B'x\|^2 &= \lim (U_{g_n} A^* U_{g_n}^{-1} E U_{g_n} A U_{g_n}^{-1} B'x, B'x) \\
&= \lim (U_{g_n} A^* E A U_{g_n}^{-1} B'x, B'x) \\
&= \lim (U_{g_n} A^* E A U_{g_n}^{-1} x, B'^* B'x) \\
&= \lim (U_{g_n} A^* E A x, B'^* B'x) \\
&= ([x] A^* E A x, B'^* B'x) \\
&= \|E A x\|^2 \|B'x\|^2 .
\end{aligned}$$

Now if $w, y, z \in \mathcal{H}$ then

1) $$\big|\, \|w\| - \|y\| \,\big| \leq \|w - z\| + \big|\, \|z\| - \|y\| \,\big| .$$

Indeed, if $\|w\| \geq \|y\|$ then $0 \leq \|w\| - \|y\| \leq \|w - z\| + \|z\| - \|y\| \leq \|w - z\| + \big|\, \|z\| - \|y\| \,\big|$, and if $\|w\| \leq \|y\|$ then $\|y\| - \|w\| \leq \|y\| - \|z\| + \|w - z\| \leq \|w - z\| + \big|\, \|z\| - \|y\| \,\big|$.

If $y \in E\mathcal{H}$ let $\delta > 0$ be given. Since $E = [\mathcal{R}'x]$ we can choose $B' \in \mathcal{R}'$ such that $\|B'x\| = \|y\|$ and $\|B'x - y\| < \delta/2\|A\|$. From the case $y = B'x$ we can choose n_1 so large that if $n \geq n_1$ then

$$\big|\, \|E U_{g_n} A U_{g_n}^{-1} B'x\| - \|E A x\| \, \|B'x\| \,\big| < \delta/2 .$$

Thus by 1), since $\|B'x\| = \|y\|$, we have for $n \geq n_1$

$$\begin{aligned}
&\big|\, \|E U_{g_n} A U_{g_n}^{-1} y\| - \|y\| \,\big| \leq \\
&\leq \|E U_{g_n} A U_{g_n}^{-1} (y - B'x)\| + \big|\, \|E U_{g_n} A U_{g_n}^{-1} B'x\| - \|y\| \,\big| \\
&\leq \|A\| \|y - B'x\| + \big|\, \|E A x\| - 1 \,\big| \, \|y\| + \delta/2 \\
&< \delta/2 + \epsilon \|y\| + \delta/2 = \delta + \epsilon \|y\| .
\end{aligned}$$

Since δ is arbitrary the lemma follows.

Lemma 3.4. Let $u \in \mathrm{Spec}(\Delta E)$, where ΔE is considered as an operator on $E\mathcal{H}$. Let $\varepsilon > 0$. Then there is A in $\pi(\mathfrak{A})$ with the following properties:

i) $\|Ax\| = 1$.

ii) $\|EAx\| > 1 - \varepsilon$.

iii) $\|u^{\frac{1}{2}}Ax - JA^*Jx\| < \varepsilon$.

iv) If y is a unit vector in $\mathcal{H}$ then there is n_1 such that if $n \geq n_1$ then

$$\|E(u^{\frac{1}{2}} U_{g_n} A U_{g_n}^{-1} y - J U_{g_n} A^* U_{g_n}^{-1} J y)\| < (2u^{\frac{1}{2}} + 3)\varepsilon .$$

Proof. Since $u \in \mathrm{Spec}(\Delta E)$ there is by [2] B in $E\mathcal{R}E$ such that $\|Bx\| = 1$ and $\|u^{\frac{1}{2}}Bx - JB^*Jx\| < \varepsilon/2$. Since $\pi(\mathfrak{A})$ is strong-* dense in $\mathcal{R}$ and $E \in \mathcal{R}$ we can find $A \in \pi(\mathfrak{A})$ such that $\|(A - B)x\| < \min\{\varepsilon, \varepsilon/4u^{\frac{1}{2}}\}$, $\|(A^* - B^*)x\| < \varepsilon/4$, and $\|Ax\| = 1$. Then $1 = \|Bx\| \leq \|EAx\| + \|EAx - Bx\| \leq \|EAx\| + \|Ax - Bx\| < \|EAx\| + \varepsilon$, so i) and ii) hold.

iii) follows since we have

$$\begin{aligned} &\|u^{\frac{1}{2}}Ax - JA^*Jx\| \leq \\ &\leq \|u^{\frac{1}{2}}Ax - u^{\frac{1}{2}}Bx\| + \|u^{\frac{1}{2}}Bx - JB^*Jx\| + \|JB^*Jx - JA^*Jx\| \\ &< u^{\frac{1}{2}}\|(A-B)x\| + \varepsilon/2 + \|(B^*-A^*)x\| \\ &< u^{\frac{1}{2}}\, \varepsilon/4u^{\frac{1}{2}} + \varepsilon/2 + \varepsilon/4 = \varepsilon , \end{aligned}$$

if $u \neq 0$, and trivially if $u = 0$.

In order to show iv) we first assume $y = Cx$ with $C = \Pi(\mathfrak{A})$. Let $z = u^{\frac{1}{2}}Ax - JA^*Jx$. Then by iii) $\|z\| < \varepsilon$. By Lemma 3.2 and definition of $\mathfrak{A}$ being asymptotically abelian we can choose an integer n_1 so that if $n \geq n_1$ then

$$\|[U_{g_n} A U_{g_n}^{-1}, C]\| < \varepsilon$$

$$\|U_{g_n}^{-1} C U_{g_n} z\| < \|Cx\| \|z\| + \varepsilon = \|z\| + \varepsilon < 2\varepsilon \ .$$

Let $A_n = U_{g_n} A U_{g_n}^{-1}$. Since $J = JE = EJ$ we have $JA_nJ = JEU_{g_n} A U_{g_n}^{-1} EJ \in JE\mathcal{R}EJ = E\mathcal{R}'$. In particular, $JA_nJECE = ECEJA_nJ$. As remarked before Lemma 3.2 $EU_g = U_gE$ for all $g \in G$. Thus, since $U_gx = x$ for g , it follows from [16,Lem.2] than $JU_g = U_gJ$ for all g . We therefore have

$$\begin{aligned}
&\|E(u^{\frac{1}{2}}A_nCx - JA_n^*JCx\| \leq \\
&\leq u^{\frac{1}{2}}\|E[A_n,C]x\| + \|E(Cu^{\frac{1}{2}}A_nx - JA_n^*JECEx)\| \\
&< u^{\frac{1}{2}}\varepsilon + \|EC(u^{\frac{1}{2}}A_nx - JA_n^*Jx)\| \\
&\leq u^{\frac{1}{2}}\varepsilon + \|U_{g_n}^{-1} C U_{g_n}(u^{\frac{1}{2}}Ax - JA^*Jx)\| \\
&< u^{\frac{1}{2}}\varepsilon + 2\varepsilon = (u^{\frac{1}{2}} + 2)\varepsilon \ ,
\end{aligned}$$

if $n \geq n_1$. Now let y be an arbitrary unit vector in $\mathcal{H}$. Since x is cyclic for $\pi(\mathfrak{A})$ we can choose C in $\pi(\mathfrak{A})$ such that $\|Cx\| = 1$ and $\|Cx - y\| < \varepsilon/\|A\|$. Let n_1 be as above. Then for $n \geq n_1$ we have

$$\begin{aligned}
&\|E(u^{\frac{1}{2}}A_ny - JA_n^*Jy)\| \leq \\
&\leq \|Eu^{\frac{1}{2}}A_n(y-Cx)\| + \|E(u^{\frac{1}{2}}A_nCx - JA_n^*JCx)\| + \\
&+ \|EJA_n^*J(Cx-y)\| \\
&< u^{\frac{1}{2}}\|A_n\|\|y - Cx\| + (u^{\frac{1}{2}}+2)\varepsilon + \|JA_n^*J\| \|Cx-y\| \\
&< u^{\frac{1}{2}}\varepsilon + (u^{\frac{1}{2}}+2)\varepsilon + \varepsilon = (2u^{\frac{1}{2}} + 3)\varepsilon \ .
\end{aligned}$$

The proof is complete.

<u>Lemma 3.3</u>. Let $u,v \in \mathrm{Spec}(\Delta E)$. Let $\varepsilon > 0$. Then there exist $A,B \in \pi(\mathfrak{A})$ and an integer n_2 such that if $n \geq n_2$ then

i) $\|Ax\| = \|Bx\| = 1$.

ii) $\left| \| E\, B\, U_{g_n} A\, U_{g_n}^{-1} x \| - 1 \right| < 2\varepsilon$

iii) $\| E((uv)^{\frac{1}{2}} B\, U_{g_n} A\, U_{g_n}^{-1} x - J(B\, U_{g_n} A\, U_{g_n}^{-1})^* Jx) \| < (2(uv)^{\frac{1}{2}} + 2v^{\frac{1}{2}} + 1)\varepsilon .$

Proof: Let A be chosen so that i), ii), iii) in Lemma 3.4 hold. Apply Lemma 3.4 once more to find $B \in \pi(\mathfrak{A})$ such that $\|Bx\| = 1$ and if $w = v^{\frac{1}{2}} Bx - J B^* Jx$ then $\|w\| < \varepsilon / \|A\|$. Now from Lemma 3.4 and its proof there is an integer n_1 such that if $n \geq n_1$ and $A_n = U_{g_n} A\, U_{g_n}^{-1}$ then

$$\| E(u^{\frac{1}{2}} A_n Bx - J A_n^* J Bx) \| < (u^{\frac{1}{2}} + 2)\varepsilon .$$

Also from the proof we have $\|[A_n, B]\| < \varepsilon$ for $n \geq n_1$. Thus for $n \geq n_1$ we have

$$\begin{aligned} &\| E((uv)^{\frac{1}{2}} B\, A_n x - J(B\, A_n)^* Jx) \| \leq \\ &\leq \| E(uv)^{\frac{1}{2}} [B, A_n] x \| + \| E(uv)^{\frac{1}{2}} A_n Bx - J A_n^* B^* Jx \| \\ &< (uv)^{\frac{1}{2}} \varepsilon + v^{\frac{1}{2}} \| E(u^{\frac{1}{2}} A_n Bx - J A_n^* J Bx) \| + \\ &+ \| E\, J\, A_n^* J(v^{\frac{1}{2}} Bx - J B^* Jx) \| \\ &< (uv)^{\frac{1}{2}} \varepsilon + v^{\frac{1}{2}} (u^{\frac{1}{2}} + 2)\varepsilon + \|A_n\| \|w\| \\ &< (2(uv)^{\frac{1}{2}} + 2v^{\frac{1}{2}} + 1)\varepsilon , \end{aligned}$$

and iii) is proved.

To show ii) we choose by Lemma 3.3 $n_2 \geq n_1$ such that if $n \geq n_2$ then

$$\left| \| E\, A_n Bx \| - 1 \right| = \left| \| E\, A_n Bx \| - \|Bx\| \right| < \varepsilon .$$

Thus we have

$$\left| \| E\, B\, A_n x \| - 1 \right| \leq \| E[B, A_n] x \| + \left| \| E\, A_n Bx \| - 1 \right| < \varepsilon + \varepsilon = 2\varepsilon .$$

Thus ii) follows, and the proof is complete.

<u>Proof of Theorem 3.1</u>. We first show that $\mathrm{Spec}(\rho)\setminus\{0\}$ is a multiplicative group of positive real numbers. By Corollary 2.4 $1 \in \mathrm{Spec}(\rho)\setminus\{0\}$, and if $u \in \mathrm{Spec}(\rho)\setminus\{0\}$ then so is u^{-1}. Therefore it remains to show $\mathrm{Spec}(\rho)$ is closed under multiplication. Let $u,v \in \mathrm{Spec}(\rho)$, $u \neq 0 \neq v$. By Theorem 2.3 $u,v \in \mathrm{Spec}(\Delta E)$. By Lemma 3.5 if $\epsilon > 0$ there is $S \in E\mathcal{R}E$ (e.g. $S = EBA_nE$) such that $|\,\|Sx\|-1| < 2\epsilon$ and

$$\|(uv)^{\frac{1}{2}}Sx - JS^*Jx\| < (2(uv)^{\frac{1}{2}} + 2v^{\frac{1}{2}} + 1)\epsilon \ .$$

Since ϵ is arbitrary it follows from [2] that $uv \in \mathrm{Spec}(\Delta)$, hence $uv \in \mathrm{Spec}(\rho)$ by Theorem 2.3, and $\mathrm{Spec}(\rho)\setminus\{0\}$ is a multiplicative group. By Corollary 2.4 $\mathrm{Spec}(\rho)$ is a closed subset of the non negative real numbers. Thus $\mathrm{Spec}(\rho)\setminus\{0\}$ is a closed subgroup of the positive real numbers.

We next show that if ω is a state of $\mathfrak{A}$ quasi-contained in ρ then $\mathrm{Spec}(\rho) \subset \mathrm{Spec}(\omega)$. Then $\omega = \bar{\omega}\circ\pi$ with $\bar{\omega}$ a normal state of $\mathcal{R}$. We first assume $\bar{\omega}$ has support E. Since x is separating and cyclic for $E\mathcal{R}E$, $\bar{\omega} = \omega_y$ with y a unit vector which is separating for $E\mathcal{R}E$ [4,Thm.4,p.233]. Let $u \neq 0$, $u \in \mathrm{Spec}(\rho)$. Then as above $u \in \mathrm{Spec}(\Delta E)$. By Lemma 3.4 there is $A \in \pi(\mathfrak{A})$ such that $\|Ax\| = 1$, $\|EAx\| > 1-\epsilon$, and if $A_n = U_{g_n} A U_{g_n}^{-1}$ then there is n_1 such that if $n \geq n_1$ then

2) $$\|u^{\frac{1}{2}}EA_ny - JA_n^*Jy\| < (2u^{\frac{1}{2}}+3)\epsilon \ .$$

By Lemma 3.3 there is $n_2 \geq n_1$ such that if $n \geq n_2$ then

3) $$|\,\|EA_n y\| - 1| < \epsilon .$$

Choose $B \in \pi(\mathfrak{A})$ such that $\|Bx - y\| < \min\{\epsilon, \epsilon/\|A\|\}$. Since $\mathfrak{A}$ is asymptotically abelian there is $n_3 \geq n_2$ such that if $n \geq n_3$ then $\|[A_n^*,B]\| < \epsilon$. Thus we have

$$\|EA_n^* Ey - u^{\frac{1}{2}} J A_n Jy\| =$$

$$= \|EA_n^* y - u^{\frac{1}{2}} J A_n Jy\|$$

$$\leq \|EA_n^*(y-Bx)\| + \|EA_n^*Bx - u^{\frac{1}{2}} J A_n JBx\| + u^{\frac{1}{2}}\|J A_n J(y-Bx)\|$$

$$< \epsilon + \|EA_n^*Bx - u^{\frac{1}{2}} EBEJA_n Jx\| + u^{\frac{1}{2}}\epsilon$$

$$\leq (u^{\frac{1}{2}}+1)\epsilon + \|[A_n^*,B]x\| + \|EBA_n^* x - u^{\frac{1}{2}} EBEJA_n Jx\|$$

$$< (u^{\frac{1}{2}}+2)\epsilon + \|EBU_{g_n}(A^*x - u^{\frac{1}{2}} JAJx)\|$$

$$\leq (u^{\frac{1}{2}}+2)\epsilon + \|BU_{g_n}(A^*x - u^{\frac{1}{2}} JAJx)\| .$$

By Lemma 3.2 this converges to

$$(u^{\frac{1}{2}}+2)\epsilon + \|Bx\|\,\|A^*x - u^{\frac{1}{2}} JAJx\|$$

$$= (u^{\frac{1}{2}}+2)\epsilon + \|Bx\|\,\|JA^*Jx - u^{\frac{1}{2}} Ax\|$$

$$< (u^{\frac{1}{2}}+2)\epsilon + \epsilon\|Bx\| .$$

Since $\|Bx\| < \|y\| + \epsilon = 1+\epsilon$, we have that there exists $n_4 \geq n_3$ such that if $n \geq n_4$ then

4) $\|EA_n^*Ey - u^{\frac{1}{2}} JA_nJy\| < (u^{\frac{1}{2}}+2)\epsilon + \epsilon(1+\epsilon) + \epsilon = (u^{\frac{1}{2}}+4+\epsilon)\epsilon$.

By 2) we have

5) $\|u^{\frac{1}{2}} EA_nEy - JA_n^*Jy\| < (2u^{\frac{1}{2}}+3)\epsilon$.

Let $P = [E\mathcal{R}Ey]$. Then $P \in E\mathcal{R}'$, and y is separating and cyclic for $E\mathcal{R}EP$. By 5) we have

$$\|u^{\frac{1}{2}} PEA_nEy - PJA_n^*JP\| < (2u^{\frac{1}{2}}+\epsilon)\epsilon .$$

By 4) we have

$$\|(PEA_nE)^*y - u^{\frac{1}{2}}(PJA_n^*JP)^*y\| \leq \|EA_n^*EPy - u^{\frac{1}{2}} JA_nJPy\|$$

$$= \|EA_n^*Ey - u^{\frac{1}{2}} JA_nJy\| < (u^{\frac{1}{2}}+4+\epsilon)\epsilon .$$

Finally $\|PEA_nEy\| = \|EA_nEPy\| = \|EA_nEy\| = \|EA_ny\|$, so by 3)

$|\|PEA_n E y\| - \|y\|| < \epsilon$. Therefore by [2] u belongs to the spectrum of Δ_ω , hence by Theorem 2.3 $u \in \mathrm{Spec}(\omega_y) = \mathrm{Spec}(\bar{\omega})$. Since $\mathrm{Spec}(\omega) = \mathrm{Spec}(\bar{\omega})$, $u \in \mathrm{Spec}(\omega)$.

In particular we have shown that $\mathrm{Spec}(\Delta E) = \bigcap_\varphi \mathrm{Spec}(\Delta_\varphi)$, where the intersection is taken over all faithful normal states φ of $E\mathcal{R}E$. By definition $\cap\, \mathrm{Spec}(\Delta_\varphi)$ equals the invariant $S(E\mathcal{R}E)$ defined by Connes [2]. If $E\mathcal{R}E$ is semi-finite then $S(E\mathcal{R}E)$ is either $\{1\}$, or $\{0,1\}$. Thus either ρ is a trace or $\omega_x|\mathcal{R}'$ is a trace by Corollary 2.4. If $\mathcal{R}$ is finite let $\mathcal{C}$ denote its center. Let Φ be the centervalued trace on $\mathcal{R}$ [4, Thm.3,p.267]. By uniqueness of Φ [4,Thm.3,p.267], $U_g\Phi(U_g^{-1}\,T\,U_g)U_g^{-1} = \Phi(T)$ for all $T \in \mathcal{R}$. Thus $\omega_x(\Phi(U_g\,T\,U_g^{-1})) = \omega_x(U_g\Phi(T)U_g^{-1}) = \omega_x(\Phi(T))$, so that $(\omega_x|\mathcal{C}) \circ \Phi$ is a G-invariant normal state. By uniqueness of ω_x [15,Thm.3.3] $\omega_x = (\omega_x|\mathcal{C}) \circ \Phi$, so ω_x is a trace, hence so is ρ , and $\mathrm{Spec}(\rho) = 1$ by Corollary 2.4. Thus if $\mathcal{R}$ is finite $\mathrm{Spec}(\rho) = \{1\} = S(\mathcal{R})$, and if $\mathcal{R}$ is not finite then $S\mathcal{R} = \{0,1\} = \mathrm{Spec}(\rho)$. Therefore in either case $\mathrm{Spec}(\rho) = S(\mathcal{R})$ in case $S(\mathcal{R})$ is defined, and $\mathrm{Spec}(\rho) \subset \mathrm{Spec}(\omega)$ for any state of $\mathfrak{A}$ quasi-contained in ρ .

We now consider the case when $E\mathcal{R}E$ is not semi-finite. Then $\mathcal{R}$ is not semi-finite, hence is of type III since the automorphisms $T \to U_g T U_g^{-1}$ act ergodically on the center $\mathcal{C}$ of $\mathcal{R}$ [15,Thm.3.3]. Then as remarked in [3], $0 \in \mathrm{Spec}(\omega)$ for all ω, hence we may assume $u \in \mathrm{Spec}(\rho)$, $u \neq 0$. Furthermore, since $\mathcal{R}$ is of type III, every normal state of $\mathcal{R}$ is a vector state [4,Cor.9,p.322]. Let ω_y be a vector state of $\mathcal{R}$. Let F be its support, $F = [\mathcal{R}'y]$. Since $[\mathcal{R}y] \leq I = [\mathcal{R}x]$ we have $[\mathcal{R}y] \precsim [\mathcal{R}x]$, hence by [4,Thm.2,p.231] $[\mathcal{R}'y] \precsim [\mathcal{R}'x]$, or $F \precsim E$. Therefore there is a partial isometry V in $\mathcal{R}$ such that $V^*V = E_1 \leq E$, $VV^* = F$. Since $E\mathcal{R}E$ has a separating vector E is countably decomposable [4,Prop.6,p.6]. Now the central carrier C_F of F equals that of E_1 . Thus $EC_F \sim E_1 \sim F$ by [4,Cor.5,p.320]. Therefore $F\mathcal{R}F \simeq E_1\mathcal{R}E_1 \simeq E\mathcal{R}EC_F$. Suppose we have shown $\mathrm{Spec}(\rho) \subset S(E\mathcal{R}EC_F)$. Then $\mathrm{Spec}(\rho) \subset S(F\mathcal{R}F)$,hence $\mathrm{Spec}(\rho) \subset \mathrm{Spec}(\omega_y)$, and $\mathrm{Spec}(\rho) \subset \mathrm{Spec}(\omega)$ for any state ω of $\mathfrak{A}$ quasi-contained in ρ . It therefore remains to consider the case when $y \in EQ$ where Q is a central projection in $\mathcal{R}$ and the support F of ω_y equals EQ .

Let z be a vector in $E(I-Q)\mathcal{H}$ which is separating for $E\mathcal{R}E(I-Q)$, e.g. let $z = (I-Q)x$. Then $y+z$ is separating for $E\mathcal{R}E$ and $y+z \in E\mathcal{H}$. By 4) and 5) there exist a constant k and an integer n_4 such that if $n \geq n_4$ then

$$\|EA_n^*E(y+z) - u^{\frac{1}{2}}JA_nJ(y+z)\| < k\varepsilon$$

$$\|u^{\frac{1}{2}}EA_nE(y+z) - JA_n^*J(y+z)\| < k\varepsilon .$$

By 3) we further have

$$|\,\|EA_nE(y+z)\| - \|y+z\|\,| < \varepsilon\|y+z\| .$$

Thus we have

$$\|QEA_n^*Ey - u^{\frac{1}{2}}QJA_nJy\|$$

$$= \|QEA_n^*(y+z) - u^{\frac{1}{2}}QJA_nJ(y+z)\| < k\varepsilon$$

and similarly

$$\|u^{\frac{1}{2}}EA_nEQy - JA_n^*JQy\| < k\varepsilon .$$

Finally, by Lemma 3.3 $\|EA_nEQy\| = \|EA_ny\|$ converges to $\|y\|$. As in the case when support ω_y was E we let $P = [E\mathcal{R}EQy]$. Then $P \in EQ\mathcal{R}'$. If we let $S = PQEA_nE$ and $T = PQJA_n^*JP$ then $S \in PQE\mathcal{R}E$ and $T \in (PQE\mathcal{R}E)'$ and for sufficiently large $n > n_4$ we have

$$\|S^*y - u^{\frac{1}{2}}T^*y\| < k\varepsilon ,$$

$$\|u^{\frac{1}{2}}Sy - Ty\| < k\varepsilon ,$$

and $|\|Sy\| - \|y\|| < \varepsilon$.

Thus by [2] $u \in \mathrm{Spec}(\Delta_{\omega_y})$, so by Theorem 2.3 $u \in \mathrm{Spec}(\omega_y)$. This completes the proof of the theorem.

4. Applications. We note some consequences of Theorem 3.1. Throughout this section we use our previous notation, so if $\mathfrak{A}$ is a C*-algebra and ρ a state of $\mathfrak{A}$, then π_ρ is a representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_\rho$, and x_ρ a unit vector in $\mathcal{H}_\rho$ cyclic for $\pi_\rho(\mathfrak{A})$ such that $\rho(A) = \omega_{x_\rho}(\pi_\rho(A))$ for all $A \in \mathfrak{A}$. Suppose $\mathfrak{A}$ is asymptotically abelian with respect to a group G . Then if ρ is a G-invariant factor state, i.e. $\pi_\rho(\mathfrak{A})''$ is a factor, then ρ is strongly clustering by [15,Cor. 4.5]. Hence we have the following corollary of Theorem 3.1.

Corollary 4.1. Let $\mathfrak{A}$ be a C*-algebra which is asymptotically abelian with respect to a group G . Suppose ρ is a G-invariant factor state. Then $\mathrm{Spec}(\rho)\setminus\{0\}$ is a closed subgroup of the multiplicative group of positive real numbers, and if ω is a state of $\mathfrak{A}$ which is quasi-equivalent to ρ then $\mathrm{Spec}(\rho) \subset \mathrm{Spec}(\omega)$.

If $\mathcal{R}$ is a von Neumann algebra we extend the notion $S(\mathcal{R})$ defined by Connes [2] slightly and let $S'(\mathcal{R})$ denote $\cap\,\mathrm{Spec}(\Delta_\varphi)$, where φ runs through the set of all normal states of $\mathcal{R}$ (In the definition of $S(\mathcal{R})$ only faithful normal states are considered.) $S'(\mathcal{R})$ is, just as $S(\mathcal{R})$, a *-isomorphic invariant for $\mathcal{R}$.

If $\mathfrak{A}$ is a C*-algebra and ρ and φ two states of $\mathfrak{A}$

they are called <u>algebraically</u> <u>equivalent</u> if $\pi_\rho(\mathfrak{A})''$ is *-isomorphic to $\pi_\varphi(\mathfrak{A})''$, see [10].

<u>Corollary 4.2</u>. Let $\mathfrak{A}$ be a C*-algebra which is asymptotically abelian with respect to a group G . Suppose ρ and φ are strongly clustering G-invariant states. Then $S'(\pi_\rho(\mathfrak{A})'') = \mathrm{Spec}(\rho)$, and if $\mathrm{Spec}(\varphi) \neq \mathrm{Spec}(\rho)$ then ρ and φ are not algebraically equivalent.

<u>Proof</u>. The first statment is immediate from Theorem 3.1. If $\mathrm{Spec}(\varphi) \neq \mathrm{Spec}(\rho)$ we therefore have that $S'(\pi_\rho(\mathfrak{A})'') \neq S'(\pi_\varphi(\mathfrak{A})'')$, hence $\pi_\rho(\mathfrak{A})''$ and $\pi_\varphi(\mathfrak{A})''$ are not *-isomorphic.

If $\mathfrak{A}$ is G-abelian with respect to a group G of *-automorphisms, see [9], and if ρ is an extremal G-invariant state then by [16,Cor.4] $\pi_\rho(\mathfrak{A})''$ is semi-finite if and only if ω_{x_ρ} is a trace on $\pi_\rho(\mathfrak{A})'$. For G-invariant factor states sharper results of this kind can be found in [15]. The next corollary should be viewed as an extension of these results to the case when $\pi_\rho(\mathfrak{A})''$ is of type III. Recall from [2] that if a countably decomposable von Neumann algebra $\mathcal{R}$ is semi-finite then $S(\mathcal{R}) \subset \{0,1\}$. Thus in general the same is true for $S'(\mathcal{R})$.

<u>Corollary 4.3</u>. Let $\mathfrak{A}$ be a C*-algebra which is asymptotically abelian with respect to a group G . Suppose ρ is a strongly clustering state. Then $\mathrm{Spec}(\rho)$, which equals $S'(\pi_\rho(\mathfrak{A})'')$, is one of the following sets:

i) $\mathrm{Spec}(\rho) = \{1\}$, in which case ρ is a trace.

ii) $\mathrm{Spec}(\rho) = \{0,1\}$, in which case ω_{x_ρ} is a trace on $\pi_\rho(\mathfrak{A})'$, but ρ is not a trace.

iii) $\mathrm{Spec}(\rho)$ is the closure of the cyclic group $\{u^n\}$ generated by a number $u \in (0,1)$.

iv) $\mathrm{Spec}(\rho)$ is the non negative real numbers.

Proof. i) and ii) follow from Corollary 2.4. By Theorem 3.1 $\mathrm{Spec}(\rho)\setminus\{0\}$ is a closed subgroup of the positive real numbers. Hence the only possibilities left are iii) and iv).

At this state it should be pointed out that not all factors can be obtained as $\pi_\rho(\mathfrak{A})''$ for ρ a G-invariant factor state of an asymptotically abelian C*-algebra. This can even be done for ITPFI-factors, i.e. infinite tensor products of finite type I factors.

Corollary 4.4. There exist ITPFI-factors which are not of the form $\pi_\rho(\mathfrak{A})''$, where ρ is a G-invariant factor state of an asymptotically abelian C*-algebra $\mathfrak{A}$.

Proof. By [1,Thm.10.10] there exist non denumerably many mutually non-isomorphic ITPFI-factors $\mathcal{R}$ with asymptotic ratio set equal to $\{0,1\}$. By [3] the asymptotic ratio set of $\mathcal{R}$ equals $S(\mathcal{R})$. Thus $S(\mathcal{R}) = \{0,1\}$. Since $\mathcal{R}$ is of type III it cannot be of the form $\pi_\rho(\mathfrak{A})''$, where ρ is a G-invariant factor state of an asymptotically abelian C*-algebra $\mathfrak{A}$, by an application of [15, Cor.4.5] and Corollary 4.3.

Let $\mathfrak{A}$ be a C*-algebra and $\{\sigma_t : t \in \mathbb{R}\}$ be a one parameter automorphism group of $\mathfrak{A}$. Let ρ be an invariant state. Then

ρ is said to be a KMS - state if there is a constant $\beta > 0$ such that for each pair $A, B \in \mathfrak{A}$ there is a function F holomorphic in the strip $0 < \operatorname{Im} z < \beta$ and with continuous boundary values

$$F(t) = \rho(\sigma_t(A)B) \quad \text{and} \quad F(t+i\beta) = \varphi(B\sigma_t(A)) .$$

(it is not necessary to assume ρ invariant, since this follows automatically). In quantum statistical mechanics it is sometimes of interest to study KMS -states of one parameter groups with respect to which the C*-algebra is asymptotically abelian. The next result is an extension of [18,IV.4,Lem.1' and 2], which are incorrectly stated, as the possibility that homomorphisms may occur is left out.

Corollary 4.5. Let $\mathfrak{A}$ be a C*-algebra which is asymptotically abelian with respect to a one parameter group of automorphisms $\{\sigma_t\}$. Suppose ρ is an extremal KMS-state of $\mathfrak{A}$. Then either ρ is a homomorphism onto the complex numbers or $\operatorname{Spec}(\rho)$ is the non negative real numbers.

Proof. By [17,Thm.13.3] x_ρ is separating and cyclic for $\mathcal{R} = \pi_\rho(\mathfrak{A})''$. Since ρ is an extremal KMS-state $\mathcal{R}$ is a factor by [17,Thm.15.4]. Since $\mathfrak{A}$ is asymptotically abelian with respect to $\{\sigma_t\}$, ρ is strongly clustering by [15,Cor.4.5]. Suppose ρ is not a homomorphism. Suppose $\operatorname{Spec}(\rho)$ is not the non negative real numbers. Since x_ρ is separating for $\mathcal{R}$ $\operatorname{Spec}(\rho) \neq \{0,1\}$ by Corollary 2.4. Thus by Corollary 4.3 $\operatorname{Spec}(\rho) \setminus \{0\}$ is the cyclic group generated by a number $u \in (0,1]$. Let F be the spectral projection of the modular operator Δ_ρ of ρ onto the subspace $\{y \in \mathcal{H}_\rho : \Delta_\rho y = uy\}$. $F \neq 0$ since u is an isolated point in $\operatorname{Spec}(\Delta_\rho)$, which by Theorem 2.3 equals $\operatorname{Spec}(\rho)$.

Let $\mathcal{A}$ denote the abelian von Neumann algebra generated by the spectral projections of Δ_ρ . Then F is a minimal projection in $\mathcal{A}$, so $F\mathcal{A}' = F\mathcal{B}(\mathcal{H}_\rho)F$, where $\mathcal{B}(\mathcal{H}_\rho)$ denotes the bounded operators on $\mathcal{H}_\rho$. Since $\dim \mathcal{H}_\rho \geq 2$ by assumption there is therefore a nonzero projection $P \in \mathcal{A}'$ orthogonal to $[x_\rho]$ and minimal in $\mathcal{A}'$. Since $\{\sigma_t\}$ is an abelian group and $\pi_\rho(\sigma_t(A)) = \Delta_\rho^{it}\pi_\rho(A)\Delta_\rho^{-it}$, $\Delta_\rho^{it}x_\rho = x_\rho$ for all t , we have obtained a contradiction, since by [15,Cor.4.6] $[x_\rho]$ is the unique nonzero minimal projection in $\mathcal{A}'$. Thus $\mathrm{Spec}(\rho)$ equals the non negative real numbers. The proof is complete.

<u>Remarks</u>. The factors studied by Powers [10,11] having what he called property L_λ $(0 \leq \lambda \leq \frac{1}{2})$ in [11], correspond to case iii) in Corollary 4.3 with $u = \lambda/1-\lambda$. His factors where constructed from product states of the CAR-algebra, for which all factors were equal. These states are strongly clustering with respect to the group of finite permutations of the factors [14]. It should be remarked that Connes' proof [2] that the factors of Powers are non isomorphic, is much easier and direct than an application of the theory developed in this paper.

The case iv) in Corollary 4.3 seems to be most common. For example consider the infinite tensor product $\mathfrak{A} = \bigotimes_{i=1}^{\infty} M_i$, where each M_i equals the 3×3 matrices over the complex numbers, and consider the group of finite permuations of the factors of $\mathfrak{A}$. The extremal invariant states are all of the form $\rho = \otimes\rho_i$ with ρ_i all the same state of M_i , and they are all strongly clustering [14,Thm.2.7]. Supose $\rho_i(A) = \mathrm{Tr}(HA)$ for all $A \in M_i$, where Tr is the usual trace on the 3×3 matrices, and H is a positive matrix with $\mathrm{Tr}(H) = 1$. If H has the eigenvalues

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \neq 0$ such that the quotients λ_i/λ_j are not all contained in the same cyclic subgroup of the positive real numbers, then $\mathrm{Spec}(\rho) \setminus \{0\}$ is not a cyclic group. Hence by Corollary 4.3 $\mathrm{Spec}(\rho)$ is the non negative real numbers, and we have case iv) in the corollary.

An example in which the situation in Corollary 4.5 holds, has been exhibited by Herman and Takesaki [7,§3,Theorem 1].

References

1. H. Araki and E.J. Woods, A classification of factors, Publ. RIMS, Kyoto Univ. Ser.A, 4 (1968), 51-130.

2. A. Connes, Un nouvel invariant pour les algèbres de von Neumann. Comptes rendus, Ser.A, 273 (1971), 900-903.

3. ———, Calcul des deux invariants d'Araki et Woods par la théorie de Tomita et Takesaki. Comptes rendus, Ser.A, 274 (1972), 175-177.

4. J. Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars, Paris 1957.

5. S. Doplicher, D. Kastler, and D.W. Robinson, Covariance algebras in field theory and statistical mechanics, Comm. Math.Phys. 3 (1966), 1-28.

6. S. Doplicher, D. Kastler, and E. Størmer, Invariant states and asymptotic abelianness. J.Funct.Anal. 3 (1969), 419-434.

7. R.H. Herman and M. Takesaki, States and automorphism groups of operator algebras, Comm.Math.Phys. 19 (1970), 142-160.

8. N. Hugenholtz, On the factor type of equilibrium states in quantum statistical mechanics, Comm.Math.Phys. 6 (1967), 189-193.

9. O. Lanford and D. Ruelle, Integral representations of invariant states on B*-algebras, J.Math.Phys. 8 (1967), 1460-1463.

10. R.T. Powers, Representations of uniformly hyperfinite algebras and their associated von Neumann rings, Ann. of Math. 86 (1967), 138-171.

11. ———, UHF algebras and their applications to representations of the anticommutation relations, Cargèse lectures in physics, Vol.4. Gordon and Breach (1970), 137-168.

12. D. Ruelle, States of physical systems, Comm.Math.Phys. 3 (1966), 133-150.

13. E. Størmer, Types of von Neumann algebras associated with extremal invariant states, Comm.Math.Phys. 6 (1967), 194-204.

14. ————, Symmetric states of infinite tensor products of C*-algebras, J.Funct.Anal. 3 (1969), 48-68.

15. ————, Asymptotically abelian systems, Cargèse lectures in physics, Vol.4. Gordon and Breach (1970), 195-213.

16. ————, Automorphisms and invariant states of operator algebras, Acta math. 127 (1971), 1-9.

17. M. Takesaki, Tomita's theory of modular hilbert algebras and its applications, Lecture Notes in Mathematics 128. Springer-Verlag 1970.

18. M. Winnink, An application of C*-algebras to quantum statistical mechanics of systems in equilibrium, Thesis, Groningen 1968.