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A THEOREM OF FINITENESS FOR MODULES WHICH ARE FLAT AND PURE OVER THE BASE SCHEME

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INTRODUCTION

Let S be an affine noetherian scheme, X an S-scheme of finite type, $\mathcal{M}$ a coherent $\mathcal{O}_X$-module. In this paper we are interested in the structure of $\mathcal{M}$ as an $\mathcal{O}_S$-module. More precisely, if $f : X \to S$ denotes the structural morphism, we want to study the $\mathcal{O}_S$-module $f_*\mathcal{M}$. We will give some conditions under which, in order to show that $f_*\mathcal{M}$ is coherent, it is enough to show a certain "generic coherence" of this module.

This problem was posed to me by Michel Raynaud. Its solution is based upon results obtained by him and L.Gruson (see [1]).

1. Some Notations and Definitions.

The language to be used is that of EGA, and especially that of [1]. However, we state the following:

(1.1) If S is a scheme, s a point of S, we denote by k(s) the residue field at s.

(1.2) If X is an S-scheme, s a point of S, the fibre of X over s (i.e. the Spec(k(s))-scheme $X \times_S \operatorname{Spec}(k(s))$) will be denoted by $X \otimes k(s)$.

(1.3) Let X be an S-scheme, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. We put

$$\operatorname{Ass}(\mathcal{M}/S) = \bigcup_{s \in S} \operatorname{Ass}_{X \otimes k(s)}(\mathcal{M} \otimes k(s)),$$

where $\operatorname{Ass}_{X \otimes k(s)}(\mathcal{M} \otimes k(s))$ is defined in the usual way (see [2], 1.1).

(1.4) A pointed scheme (S,s) is a couple consisting of a scheme S and a point s of S. A morphism of pointed schemes

$$g : (S',s') \longrightarrow (S,s)$$

is a morphism of schemes $g: S' \to S$ such that $g(s') = s$.

(1.5) An elementary étale neighbourhood of a pointed scheme (S,s) is an étale morphism of pointed schemes

$$(S',s') \longrightarrow (S,s)$$

which induces an isomorphism of $k(s)$ with $k(s')$.

(1.6) A henselisation of a pointed scheme (S,s) is a morphism of pointed schemes

$$(\tilde{S},\tilde{s}) \longrightarrow (S,s)$$

where $\tilde{S}$ is the spectrum of a henselisation of the local ring $\mathcal{O}_{S,s}$, $\tilde{s}$ its closed point. We know that $(\tilde{S},\tilde{s})$ can be identified with a projective limite of affine elementary étale neighbourhoods of (S,s). (For a treatment of this subject, see EGA. IV, 18... or [4].)

2. A "Structure Lemma".

We now state as a lemma the following result of Raynaud and Gruson:

LEMMA (2.1): Let A be a noetherian ring, $f : X \to S = \mathrm{Spec}(A)$ a morphism of finite type. $\mathcal{M}$ a coherent $\mathcal{O}_X$-module which is flat over A.
Let $x \in X$, $s = f(x)$. There exist an affine elementary étale neighbourhood (Y,y) of (X,x) and an affine elementary étale neighbourhood $(S' = \mathrm{Spec}(A'),s')$ of (S,s) such that

(i) the following diagram of pointed schemes commutes:

$$\begin{array}{ccc} (Y,y) & \longrightarrow & (X,x) \\ \downarrow & & \downarrow \\ (S',s') & \longrightarrow & (S,s) \end{array}$$

(ii) if $\mathcal{N}$ denotes the inverse image of $\mathcal{M}$ on Y, $\Gamma(Y,\mathcal{N})$ is a free A'-module.

Proof: The lemma is an immediate corollary of [1], 3.3.2, taking into account the result of Bass which asserts that "big" projective modules are free ([3], Cor. 4.5).

Thus we know that. if $\mathcal{M}$ is S-flat, $\mathcal{M}$ is, locally for the étale topology on X and on S, free over S.

Now one could ask if this implies that $\mathcal{M}$ can be realized as a sub-module of a free $\mathcal{O}_S$-module, locally for the étale topology on S. It turns out that a sufficient condition for this to be true, is that $\mathcal{M}$ is "relatively pure", in the sense of Raynaud (1 , 3.3.3):

DEFINITION (2.2): Let S be a scheme, X an S-scheme locally of finite type, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module of finite type.

(i) Let s be a point of S, $(\tilde{S},\tilde{s})$ a henselisation of (S,s), $\tilde{X} = X \times_S \tilde{S}$, $\tilde{\mathcal{M}} = \mathcal{M} \times_S \tilde{S}$.
We say that $\mathcal{M}$ is pure along $X \otimes k(s)$ if, for all $x \in \mathrm{Ass}(\tilde{\mathcal{M}}/\tilde{S})$, the intersection of the closure of x in $\tilde{X}$ with $\tilde{X} \otimes k(\tilde{s})$ is non-empty.

(ii) $\mathcal{M}$ is said to be S-pure if $\mathcal{M}$ is pure along $X \otimes k(s)$ for all $s \in S$.

Remark (2.3): The condition of (i) is equivalent to the condition that any open set of $\tilde{X}$ containing the closed fibre $\tilde{X} \otimes k(\tilde{s})$, contains also $\mathrm{Ass}(\tilde{\mathcal{M}}/\tilde{S})$.

3. A Theorem of Finiteness.

Under the conditions of (2.1), if $\mathcal{M}$ is also A-pure, we are going to show that, locally for the étale topology on S, $f_*\mathcal{M}$ is a sub-module of a free module. This enables us to prove the following theorem:

THEOREM (3.1): Let A be a noetherian ring, $f : X \longrightarrow S = \mathrm{Spec}(A)$ a morphism of finite type, $\mathcal{M}$ a coherent $\mathcal{O}_X$-module which is A-flat and A-pure.
Suppose that for all $\mathfrak{p} \in \mathrm{Ass}(A)$, $\Gamma(X_{\mathfrak{p}}, \mathcal{M}_{\mathfrak{p}})$ is an $A_{\mathfrak{p}}$-module of finite type.
Then $M = \Gamma(X, \mathcal{M})$ is an A-module of finite type.

Proof: We will first show that for all points s in S, there is an affine elementary étale neighbourhood $(S' = \mathrm{Spec}(A'), s')$ of (S,s) such that if $X' = X \times_S S'$ and $\mathcal{M}' = \mathcal{M} \times_X X'$, then $\Gamma(X', \mathcal{M}')$ is a sub-module of a free A'-module. From this we deduce that $\Gamma(X', \mathcal{M}')$ is in fact an A'-module of finite type, and we conclude the proof by considering a finite covering of S by étale neighbourhoods of the above type.

So let s be a point of S. For each point x_α in the fibre of X at s there exists, by (2.1), a commutative diagram of pointed schemes

$$\begin{array}{ccc} (Y_\alpha , y_\alpha) & \xrightarrow{v_\alpha} & (X, x_\alpha) \\ \downarrow & & \downarrow \\ (S_\alpha , s_\alpha) & \longrightarrow & (S, s) \end{array}$$

where (Y_α, y_α) (resp. (S_α, s_α)) is an affine elementary étale neighbourhood of (X, x_α) (resp. (S,s)) and such that the inverse image of $\mathcal{M}$ on Y_α is a free $\mathcal{O}_{S_\alpha}$-module. $X \otimes k(s)$ is quasi-compact, so there exists a finite set I such that $X \otimes k(s)$ is contained in the union of the open sets $v_\alpha(Y_\alpha)$, for $\alpha \in I$.

Take an affine elementary étale neighbourhood (S', s') which domininates each of the (S_α, s_α) for $\alpha \in I$, and put

$$X' = X \times_S S' \qquad Y'_\alpha = Y_\alpha \times_S S' \qquad Y' = \coprod_{\alpha \in I} Y'_\alpha .$$

We obtain the following commutative diagram

$$(*)\qquad \begin{array}{ccc} Y' & & \\ & \searrow^{u'} \quad \searrow^{u} & \\ & X' \longrightarrow & X \\ & \downarrow & \downarrow \\ & S' \longrightarrow & S \end{array}$$

Observe the following:

(3.2) $(S's')$ being an elementary étale neighbourhood of (S,s) implies that the fibre $X'\otimes k(s')$ is isomorphic to $X\otimes k(s)$, and so is contained in the image U' - which is open - of Y' in X'.

(3.3) From the definition of purity, it follows that $\mathcal{M}' = \mathcal{M}\times_X X'$ is pure along $X'\otimes k(s')$ if and only if $\mathcal{M}$ is pure along $X\otimes k(s)$.

Let A' denote the ring of the affine scheme S'.

By construction, the A'-module $\Gamma(Y',u'^{*}\mathcal{M}')$ is free. We want to show that $\Gamma(X',\mathcal{M}')$ is, via the canonical homomorphism, isomorphic to a sub-module of this free module. It is at this point that we use the fact that $\mathcal{M}'$ is pure along $X'\otimes k(s')$.

We now need some lemmas concerning constructability and the property of pureness.

LEMMA (3.4): Let X be an algebraic scheme over a field k, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, U an open subscheme of X. Denote by $\mathbf{P}(X,\mathcal{F},U,k)$ the property that $\mathrm{Ass}(\mathcal{F})\subset U$. Then $\mathbf{P}$ is a constructible property (EGA,IV,9.2.1).

Proof: There are two things to be shown:

i) If k' is an extension of k, then $\mathrm{Ass}(\mathcal{F})\subset U$ if and only if $\mathrm{Ass}(\mathcal{F}\otimes_k k')\subset U\otimes_k k'$.

ii) Let S be an integral noetherian scheme with generic point η, $u:X\to S$ a morphism of finite type, $\mathcal{F}$ a

coherent $\mathcal{O}_X$-module, U an open set in X.
Denote by E the set of points s in S such that $\mathrm{Ass}(\mathcal{F}\otimes k(s)) \subset U\otimes k(s)$. Then one of the sets E, S-E contains a non-void open set.

By EGA,IV,4.2.7, the image of $\mathrm{Ass}(\mathcal{F}\otimes_k k')$ in $X\otimes_k k'$ is equal to $\mathrm{Ass}(\mathcal{F})$, so i) holds.

To show ii), we use another result from EGA, namely (EGA, IV,9.8.3):

Let $\{x_i\}_{i\in I} = \mathrm{Ass}(\mathcal{F}\otimes k(\eta))$ - this is a finite set. For each x_i, denote by Z_i its closure <u>in X</u>. For all $s\in S$ and all $i\in I$, let $\{x_{is\alpha} \mid \alpha\in J_{s,i}\}$ be the maximal points of $Z_i\otimes k(s)$.
There exists an open neighbourhood V of η in S such that, for all $s\in V$, the $x_{is\alpha}(i\in I, \alpha\in J_{s,i})$ are all distinct and

$$\mathrm{Ass}(\mathcal{F}\otimes k(s)) = \{x_{is\alpha} \mid i\in I, \alpha\in J_{s,i}\}.$$

Let us distinguish the two cases

a) $\eta\in E$

b) $\eta\in S-E$.

Case a): If $\eta\in E$, $\mathrm{Ass}(\mathcal{F}\otimes k(\eta))\subset U$. Let $i\in I$. $Z_i\otimes k(\eta)$ is irreducible, hence $U\otimes k(\eta)\cap Z_i\otimes k(\eta)$ is a dense open sub-scheme of $Z_i\otimes k(\eta)$. Put

$$F = \{t\in V \mid U\otimes k(t)\cap Z_i\otimes k(t) \text{ is dense in } Z_i\otimes k(t)\}.$$

F is a constructible set (EGA,IV,9.5.3). Since $\eta\in F$ and S is integral, this implies that F contains a non-void open set (EGA,IV,9.2.3). Denote this open set by V_i.

Thus $V' = \bigcap_{i\in I} V_i$ is an open neighbourhood of η contained in V.
Let $s\in V'$. If $x\in \mathrm{Ass}(\mathcal{F}\otimes k(s))$, there exist $i\in I$ and $\alpha\in J_{s,i}$ such that $x = x_{is\alpha}$. We know that $U\otimes k(s)\cap Z_i\otimes k(s)$ is dense in $Z_i\otimes k(s)$. Since $x_{is\alpha}$ is a maximal point of $Z_i\otimes k(s)$, this implies that

$$x = x_{is\alpha}\in U\otimes k(s)\cap Z_i\otimes k(s),$$

which then proves that V' is an open, non-void sub-set of E.

Case b): If $\eta \notin E$, there exists $x_i \in \mathrm{Ass}(\mathcal{F} \otimes k(\eta))$ such that $x_i \notin U$. Put

$$G = \{ t \in S \mid U \otimes k(t) \cap Z_i \otimes k(t) = \emptyset \};$$

G is a constructible set by EGA, IV, 9.5.3.

As above, one sees that G contains an open neighbourhood of η, which we denote by V'.

Let $s \in V'$. Then $U \otimes k(s) \cap Z_i \otimes k(s) = \emptyset$. Hence $x_{i,s,\alpha} \notin U \otimes k(s)$ for all $\alpha \in J_{s,i}$, consequently V' is a non-void open sub-set of S−E. This completes the proof of (3.4).

LEMMA (3.5): Let S be a noetherian scheme, X an S-scheme of finite type, $\mathcal{M}$ a coherent $\mathcal{O}_X$-module. Let s be a point of S, U an open set in X which contains $X \otimes k(s)$. If $\mathcal{M}$ is pure along $X \otimes k(s)$, then U "contains $\mathrm{Ass}(\mathcal{M}/S)$ over an open neighbourhood of s", i.e. there exists an open neighbourhood V of s in S such that $\mathrm{Ass}(\mathcal{M} \otimes k(t)) \subset U$ for all $t \in V$.

Proof: Let $E = \{ t \in S \mid \mathrm{Ass}(\mathcal{M} \otimes k(t)) \subset U \}$. By (3.4), E is a locally constructible set in S (EGA, IV, 9.2.3); it is even constructible, since S is noetherian.

In order to prove the lemma, we must show that E contains an open neighbourhood of the given point s. Since E is constructible, it will be enough to show that every generisation of s is in E (EGA, 0_{III}, 9.2.5).

So let $g : (\tilde{S}, \tilde{s}) \to (S, s)$ be a henselisation of S at s, and consider the cartesian diagram

$$\begin{array}{ccc} X \times_{x} \tilde{S} = \tilde{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \tilde{S} & \xrightarrow{g} & S \end{array}$$

Let $\tilde{\mathcal{M}} = \mathcal{M} \times_X \tilde{X}$. Since $\mathcal{M}$ is pure along $X \otimes k(s)$, any open set of $\tilde{X}$ containing $\tilde{X} \otimes k(\tilde{s})$, contains $\mathrm{Ass}(\tilde{\mathcal{M}}/\tilde{S})$, cfr. (2.3). Hence $U \times_S \tilde{S}$ contains $\mathrm{Ass}(\tilde{\mathcal{M}}/\tilde{S})$. Since the property $\mathbf{P}$ considered in (3.4) is constructible, it results from this that $g(\tilde{S}) \subset E$ (EGA, IV, 9.2.2(iv)). Now all generisations of s are contained in $g(\tilde{S})$, so the proof of the lemma is completed.

We may now return to the proof of the theorem, that is, to the situation of the diagram (✱).

If we consider S', X', $\mathcal{M}'$ and the (open) image U' of Y' in X', it follows from (3.5) that we may assume - except for replacing, if necessary, S' by an open affine neighbourhood of S' - that in the diagram (✱) we have

$$\mathrm{Ass}(\mathcal{M}'/S') \subset U' = u'(Y').$$

À fortiori we then have $\mathrm{Ass}(\mathcal{M}') \subset U'$, so that the restriction map

$$\Gamma(X', \mathcal{M}') \longrightarrow \Gamma(U', \mathcal{M}')$$

is injective (EGA, IV, 3.1.8).

The morphism $u' : Y' \longrightarrow U'$ is faithfully flat, so we also get an injection (EGA, IV, 2.2.8)

$$\Gamma(U', \mathcal{M}') \longrightarrow \Gamma(Y', u'^{*}\mathcal{M}') = \Gamma(Y', u^{*}\mathcal{M}).$$

Let A' be the ring of S'. We have now shown that $\Gamma(X', \mathcal{M}')$ can be viewed as a sub-module of the free A'-module $\Gamma(Y', u^{*}\mathcal{M})$ - this was the first step of the theorem.

Furthermore, A' being flat over A implies that (EGA, III, 1.4.15)

$$\Gamma(X, \mathcal{M}) \otimes_A A' \cong \Gamma(X', \mathcal{M}').$$

Let $M = \Gamma(X, \mathcal{M})$ and $M' = M \otimes_A A'$. We are going to show that the hypothesis of "generic finiteness" made on $\mathcal{M}$ now implies that M' is of finite type over A'.

Let $A \longrightarrow \prod_{\wp \in \mathrm{Ass}\, A} A_\wp = B$ be the canonical homomorphism, which is injective. By hypothesis,

$$\prod_{\wp \in \mathrm{Ass}\, A} \Gamma(X_\wp, \mathcal{M}_\wp) = \prod_{\wp \in \mathrm{Ass}\, A} M_\wp = M \otimes_A B$$

is a B-module of finite type.

$A \rightarrow A'$ being flat implies that the homomorphism $A' \longrightarrow B' = B \otimes_A A'$ is injective.

$M' \otimes_{A'} B'$ is a B'-module of finite type, and M' is a sub-module of a free A'-module.

The fact that M' is of finite type over A' is a consequence of the following lemma:

LEMMA (3.6): Let R be a noetherian ring, $\varphi : R \longrightarrow R'$ an injective ring homomorphism, P a sub-module of a free R-module.
If $P \otimes_R R'$ is an R'-module of finite type, then P is of finite type over R.

Proof: We have a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\;j\;} & R^{(J)} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \varphi^{(J)}} \\ P \otimes_R R' & \xrightarrow{\;j'\;} & R'^{(J)} \end{array}$$

where p is the canonical homomorphism and $j' = j \otimes_R R'$. Since $P \otimes_R R'$ is of finite type over R', there exists a finite subset I of J such that

$$j'(P \otimes_R R') \subset R'^{(I)}.$$

This implies also that

$$\varphi^{(J)}(j(P)) \subset R'^{(I)}.$$

$R \longrightarrow R'$ is injective, so

$$\varphi^{(J)^{-1}}\{\varphi^{(J)}(j(P))\} = j(P) \subset \varphi^{(J)^{-1}}(R'^{(I)}) = R^{(I)}.$$

Hence P is a sub-module of an R-module of finite type. so it is itself of finite type over R.

End of the proof of (3.1):

By taking a finite covering of $S = Spec(A)$ by affine elementary étale neighbourhoods $S' = Spec(A')$ such that $M \otimes_A A' = \Gamma(X, \mathcal{M}) \otimes_A A'$ is of finite type over A', we conclude - by faithfully flat descente - that M is an A-module of finite type. Q.E.D.

4. Applications.

Firstly, we will give some propositions concerning the purity concept, in order to illuminate this concept (cfr. [1], 3.3.4).

PROPOSITION (4.1): Let $f : X \to S$ be a morphism locally of finite type.
If f is universally closed - à fortiori if f is proper - then every quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ of finite type is S-pure.

Proof: Let $s \in S$, $(\tilde{S}, \tilde{s})$ a henselisation of (S,s), $\tilde{\mathcal{M}} = \mathcal{M} \times_S \tilde{S}$. x a point of $Ass(\tilde{\mathcal{M}}/\tilde{S})$. If $\overline{\{x\}}$ denotes the closure of x in $X \times_S \tilde{S}$, the image of $\overline{\{x\}}$ in $\tilde{S}$ is closed, hence $\tilde{s}$ is contained in this image.

PROPOSITION (4.2): Let $f : X \to S$ be a morphism locally of finite type.
If f is flat with geometrically integral fibres (EGA, IV, 4.6.2), then X is S-pure (i.e. $\mathcal{O}_X$ is S-pure).

Proof: Let $s \in S$, $(\tilde{S},\tilde{s})$ a henselisation of (S,s), $\tilde{X} = X \times_S \tilde{S}$. Suppose that there exist a point t in $\tilde{S}$ and $x \in \mathrm{Ass}(\tilde{X} \otimes k(t))$ such that

$$\overline{\{x\}} \cap \tilde{X} \otimes k(\tilde{s}) = \emptyset .$$

Since $\tilde{X} \otimes k(t)$ is integral, $\tilde{X} \otimes k(t) \subset \overline{\{x\}}$.
Let $U = \tilde{X} - \overline{\{x\}}$; this is an open set containing $\tilde{X} \otimes k(\tilde{s})$. $\tilde{f} = f \times_S \tilde{S} \ : \tilde{X} \longrightarrow \tilde{S}$ being flat and of finite type, is open, so that $\tilde{f}(U)$ is open in $\tilde{S}$, and contains $\tilde{s}$.

But $\tilde{S}$ is local with closed point $\tilde{s}$ - consequently we must have $\tilde{f}(U) = \tilde{S}$, and we have thus reached a contradiction.

PROPOSITION (4.3): Let $f : X \rightarrow S$ be a separated and quasi-finite (EGA,II,6.2.3) morphism of finite presentation.
Then X is S-pure if and only if f is finite.

Proof: If f is finite, f is universally closed; so X is S-pure by (4.1).

Suppose then that f is separated, quasi-finite and of finite presentation, and that X is S-pure. It is enough to show that for every point s in S, there is an open affine neighbourhood U of s such that $f|_{f^{-1}(U)}$ is a finite morphism.

We may thus assume that S is affine - let $S = \mathrm{Spec}(A)$. By one of the versions of Zariskis Main Theorem (EGA,IV,8.12.6), there exists a commutative diagram

$$(**) \qquad \begin{array}{ccc} X & \xrightarrow{\ j\ } & \mathrm{Spec}(B) \\ {\scriptstyle f}\downarrow & \swarrow & \\ \mathrm{Spec}(A) & & \end{array}$$

where j is an open embedding, and B is a finite A-algebra. Let $\mathfrak{p} \in \mathrm{Spec}(A)$, and let $(\tilde{A}_{\mathfrak{p}}, \tilde{\mathfrak{p}})$ be a henselisation of $(A, \mathfrak{p})$. By base change $A \rightarrow \tilde{A}_{\mathfrak{p}}$, we obtain from (**), if we put $\tilde{X} = X \otimes_A \tilde{A}_{\mathfrak{p}}$ and $\tilde{B} = B \otimes_A \tilde{A}_{\mathfrak{p}}$,

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \operatorname{Spec}(\tilde{B}) \\ \tilde{f} \downarrow & \swarrow & \\ \operatorname{Spec}(\tilde{A}_{\mathfrak{p}}) & & \end{array}$$

Now $\tilde{B}$ is finite over $\tilde{A}_{\mathfrak{p}}$ which is local, henselian - so $\tilde{B}$ "splits into its local components", i.e. $\tilde{B} = \prod_{i=1}^{n} \tilde{B}_{\eta_i}$ where $\{\eta_i\}_1^n$ are the maximal ideals of the semi-local ring $\tilde{B}$.

The fibres of $\tilde{f}$ are finite, hence the hypothesis that X is S-pure implies the following:

Let $x \in \tilde{X}$, and denote by $\overline{\{x\}}$ its closure in $\tilde{X}$. Then

$$(\S) \qquad \overline{\{x\}} \cap \tilde{X} \otimes k(\tilde{\mathfrak{p}}) \neq \emptyset .$$

Denote by i the integer such that $x \in \operatorname{Spec}(\tilde{B}_{\eta_i})$. By (§) the closed point η_i of this local component is contained in $\tilde{X} \cap \operatorname{Spec}(\tilde{B}_{\eta_i})$. But this latter set is open in $\operatorname{Spec}(\tilde{B}_{\eta_i})$, hence $\tilde{X} \cap \operatorname{Spec}(\tilde{B}_{\eta_i}) = \operatorname{Spec}(\tilde{B}_{\eta_i})$.

It follows that $\tilde{X}$ is equal to $\coprod_{j \in J} \operatorname{Spec}(\tilde{B}_{\eta_j})$, where $J \subseteq \{1,\dots,n\}$. so $\tilde{X}$ is finite over $\tilde{A}_{\mathfrak{p}}$.

Since $A_{\mathfrak{p}} \longrightarrow \tilde{A}_{\mathfrak{p}}$ is faithfully flat, we conclude that $X \otimes_A A_{\mathfrak{p}}$ is finite over $A_{\mathfrak{p}}$ (EGA,IV,2.7.1). Then there exists a $g \in A - \mathfrak{p}$ such that $X \otimes_A A_g$ is finite over A_g (EGA,IV,8.10.5). We have thus proved what we wanted to do.

(4.2) implies that the theorem (3.1) can be applied to the following situation:

COROLLARY (4.4): Let A be a noetherian integral domain, K its fraction field, $f : X \to S = \operatorname{Spec}(A)$ a flat morphism of finite type, with geometrically integral fibres. Suppose that $X \otimes_A K$ is a proper K-scheme. Then $\Gamma(X, \mathcal{O}_X)$ is an A-module of finite type.

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