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A NOTE ON IRREFLEXIVE, ASYMMETRIC AND INTRANSITIVE KRIPKE MODELS

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It is well known that certain schemata of modal logic correspond to certain conditions on the relation of the Kripke models. E.g., one can prove that all instances of $\Box A \supset \Box\Box A$ are valid in a model structure if and only if the relation of the structure is transitive.

In [1], S.K. Thomason mentions the problem of whether there is a schema of modal logic that corresponds in this sense to the relation being irreflexive. This problem is closely related to a result announced by Lemmon and Scott in [2], namely, that there is no schema Z such that a formula A is provable in the minimal (semi-)normal modal logic K with Z as added axiom schema if and only if A is valid in all irreflexive model structures. The proof was to be presented in later parts of [2]. They conjectured that the same is true of asymmetry and anti-symmetry.

We shall present here a simple proof that no schema corresponds to any of these properties, nor to a certain generalization of the notion of intransitivity, or any property entailed by it.

The notation is mainly taken from [3]. When U is a set, R a binary relation over U and V a function from propositional variables to subsets of U , $\mathcal{K} = \langle U_{\mathcal{K}}, R_{\mathcal{K}} \rangle$ is called a <u>model structure</u> (world system, frame) and $\mathcal{U} = \langle U_{\mathcal{U}}, R_{\mathcal{U}}, V_{\mathcal{U}} \rangle$ a <u>model</u>. When $U_{\mathcal{U}} = U_{\mathcal{K}}$, $R_{\mathcal{U}} = R_{\mathcal{K}}$, $\mathcal{U}$ is said to be a model <u>on</u> $\mathcal{K}$.

For $u \in U_{\mathcal{U}}$, $\models^{\mathcal{U}}_{u} A$ is defined in the usual way and reads "A is true at u in $\mathcal{U}$".
$\models^{\mathcal{U}} A$ means $\forall u \in U_{\mathcal{U}}$, $\models^{\mathcal{U}}_{u} A$, and reads "A is true in $\mathcal{U}$".

If $\not\models_u^{\mathcal{U}} A$ for some u, then $\mathcal{U}$ is said to be a countermodel for A. $\models^{\mathcal{K}} A$ means $\forall \mathcal{U}$ on $\mathcal{K}$, $\models^{\mathcal{U}} A$, and reads "A is valid in $\mathcal{K}$". When Z is a schema, $\models^{\mathcal{K}} Z$ means: For all instances A of Z, $\models^{\mathcal{K}} A$. For typographical reasons we sometimes write $u \models A$ for $\models_u^{\mathcal{U}} A$ and $u^* \models^* A$ for $\models_{u^*}^{\mathcal{U}^*} A$.

K is the system that has modus ponens and necessitation as rules and the axiom schema

$$\Box(A \supset B) \supset (\Box A \supset \Box B)$$

in addition to axioms from some axiomatization of propositional logic with modus ponens as the only rule.

The main point of this paper is the following lemma:

LEMMA 1. If a formula B has a countermodel, then it has an asymmetric (and hence irreflexive and anti-symmetric) and intransitive countermodel.

<u>Proof</u>. Let $\mathcal{U} = \langle U,R,V \rangle$ and $u_o \in U$ be such that $\not\models_{u_o}^{\mathcal{U}} B$. Construct a new countermodel $\mathcal{U}^* = \langle U^*,R^*,V^* \rangle$ as follows:

i) U^* consists of all finite sequences $\langle u_o,u_1,\ldots,u_n \rangle$ where $u_i \in U$ for $0 \leq i \leq n$ and $u_i R u_{i+1}$ for $0 \leq i \leq n-1$. (Call them R-sequences.)

ii) $\langle u_o,\ldots,u_m \rangle R^* \langle u_o,\ldots,u_n \rangle$ holds iff $\langle u_o,\ldots,u_n \rangle = \langle u_o,\ldots,u_m \rangle * \langle u_n \rangle$, where $*$ is the "clash" or concatenation operation of sequences.

iii) For atomic formulas A, $\langle u_o,\ldots,u_n \rangle \in V^*(A)$ iff $u_n \in V(A)$.

We shall prove that if $\langle u_o,\ldots,u_n \rangle \in U^*$, then $\langle u_o,\ldots,u_n \rangle \models^* A$ iff $u_n \models A$, or, when u' denotes the last element of

$u^* \in U^*$: $u^* \models^* A$ iff $u' \models A$, for all formulas A . The proof goes by induction:

at.) For atomic formulas A :

$u^* \models^* A \Longleftrightarrow u^* \in V^*(A) \Longleftrightarrow u' \in V(A) \Longleftrightarrow u' \models A$.

$\neg$) $u^* \models^* \neg A \Longleftrightarrow \neg u^* \models^* A \Longleftrightarrow \neg u' \models A \Longleftrightarrow u' \models \neg A$.

$\wedge$) $u^* \models^* A \wedge B \Longleftrightarrow u^* \models^* A$ & $u^* \models^* B \Longleftrightarrow u' \models A$ & $u' \models B$ $\Longleftrightarrow u' \models A \wedge B$.

$\Box$) $u^* \models^* \Box A \Longleftrightarrow \forall\langle u_0,\dots,u_n\rangle \in U^*(u^*R^*\langle u_0,\dots,u_n\rangle \to \langle u_0,\dots,u_n\rangle \models^* A)$
$\Longleftrightarrow \forall\langle u_0,\dots,u_n\rangle \in U^*(\langle u_0,\dots,u_n\rangle = u^* * \langle u_n\rangle \to u_n \models A)$
$\Longleftrightarrow \forall u_n \in U(u'R\,u_n \to u_n \models A) \Longleftrightarrow u' \models \Box A$.

Hence we have $\langle u_0\rangle \not\models^* B$, i.e. $\not\models^{\mathcal{U}^*}_{\langle u_0\rangle} B$. And R^* is asymmetric and intransitive.

QED.

Instead of this induction we could have used the p-morphism theorem. Let 'xR^ny' mean $\exists x_1,\dots,x_{n-1} : xRx_1 \wedge x_1Rx_2 \wedge\dots\wedge x_{n-1}Ry$. Define $U_{u_0} = \{u \mid \exists n : u_0R^nu\}$. It is easy to check that the mapping from U^* to U_{u_0} which takes u^* into u' (the last element of u^*) is a p-morphism, reliable on every propositional letter. See [3].

The model constructed in the above proof has the form of a tree, and, of course, is not just asymmetric and intransitive. It has all properties which can be expressed by a sentence of the form

$(1)_{m,p}$ $\forall xy \neg (xR^my \wedge xR^py)$, for $m \neq p$,

or of the form

$(2)_m$ $\forall xyz_1,\dots,z_{m-1}u_1,\dots,u_{m-1}(xRz_1 \wedge z_1Rz_2 \wedge\dots\wedge z_{m-1}Ry \wedge$
$\wedge xRu_1 \wedge u_1Ru_2 \wedge\dots\wedge u_{m-1}Ry \to \bigwedge_{0<i<m}(z_i = u_i))$, for $2 \leq m$.

Let us call the property of satisfying $(1)_{m,p}$, for $m \neq p$, m,p-intransitivity, and the property expressed by $(2)_m$, for $2 \leq m$, m,m-intransitivity.

Then we see that:

irreflexivity is 1,0-intransitivity;

asymmetry is 2,0-intransitivity;

intransitivity is 2,1-intransitivity.

If we restrict ourselves to connected model structures, the class of model structures obtained from arbitrary model structures by the construction in the above proof is determined up to isomorphism by the $L_{\omega_1\omega}$-sentence

$$(3) \qquad \bigwedge_{m \neq p} (1)_{m,p} \wedge \bigwedge_{2 \leq m} (2)_m$$

i.e., we have:

LEMMA 2. $\mathcal{K}$ is connected (i.e., $U_{\mathcal{K}}$ is of the form U_{u_0}) and satisfies (3) as an ordinary relational structure for the first order language with one binary relation symbol $(\langle U_{\mathcal{K}}, R_{\mathcal{K}}\rangle \models (3))$ iff $\mathcal{K}$ is isomorphic to a model structure obtained by the construdtion in the proof of Lemma 1.

Proof. If $\mathcal{K}$ is isomorphic to a model structure obtained by the construction, then it certainly is connected and satisfies (3). If $\mathcal{K}$ is connected and satisfies (3), then use the construction on $\mathcal{K}$ itself, starting with some u_0 from which all points in the structure can be reached, and obtain $\mathcal{K}^*$. Then $\mathcal{K} \simeq \mathcal{K}^*$ is shown by the mapping $f : u_n \mapsto \langle u_0, \ldots, u_n \rangle$. For $\mathcal{K} \models (3)$ implies that if two $R_{\mathcal{K}}$-sequences, at least one of them of length greater than 2 , have the same last element, they are equal. If two $R_{\mathcal{K}}$-sequences have the same last element and neither of them has length greater that 2 , then they are trivially equal, since all $R_{\mathcal{K}}$-sequences begin with u_o and $\neg u_o R_{\mathcal{K}} u_o$.

This means that we can state the full strength of the proof of Lemma 1 as:

LEMMA 3. If (*) is any sentence implied in $L_{\omega_1\omega}$ by (3), then if a modal formula has a countermodel, it has a countermodel satisfying (*).

A class of model structures $\mathcal{C}$ is said to <u>determine</u> a modal logic S if for all formulas A $\vdash_S A \Longleftrightarrow \forall\mathcal{K}(\mathcal{K}\in\mathcal{C}\rightarrow\models^{\mathcal{K}}A)$. A property of model structures is said to determine a logic if the class of model structures having the property determines it.

THEOREM 1. If (*) is any sentence implied by (3), K is determined by the property of satisfying (*) ("determined by (*)").

<u>Proof</u>. We have $\vdash_K A \Longleftrightarrow \forall\mathcal{K}\models^{\mathcal{K}}A$. Hence, one way is immediate. If $\nvdash_K A$, then A has a countermodel, and hence, by Lemma 3, a countermodel satisfying (*).

COROLLARY. There is no schema Z such that $K \subsetneq KZ$ and KZ is determined by (*), for any (*) implied by (3). In particular we can take as (*) the sentence expressing irreflexivity, or asymmetry, anti-symmetry or intransitivity.

COROLLARY. Every class of model structures between the class of all $\mathcal{K}$ and the class of $\mathcal{K}$ satisfying (3) determines the logic K.

A schema Z is said to correspond to a property if $\models^{\mathcal{K}} Z$ iff $\mathcal{K}$ has the property.

THEOREM 2. There is no schema Z corresponding to any property which can be expressed by a sentence (*) implied by (3), except the trivial property of just being a model structure.

Proof. Suppose there was such a schema Z . There exists a model not satisfying (*). It must be a countermodel for some instance of Z . Hence, by Lemma 3, this instance has a countermodel satisfying (*), which contradicts the assumption.

We now admit more than one relation in the model structures and more than one necessity operator. The necessity operator belonging to a relation R is denoted $\boxed{R}$, etc. K^n is the n-modal system (system with n necessity operators) where we have the K-axiom and necessitation for all operators. Below, $\mathcal{K}$ is supposed to range over model structures with the appropriate number of relations, $R_{\mathcal{K}}$ being one of them.

THEOREM 3. For any sentence (*) implied by (3), the property that $R_{\mathcal{K}}$ satisfies (*) determines K^n . For any (*) implied by (3), except the trivial sentence, no schema in the language of K^n corresponds to the property that $R_{\mathcal{K}}$ satisfies (*).

Proof. We have to generalize our Lemma 3. Given a model $\mathcal{U} = \langle U, R, \ldots, V \rangle$, let T be the union of the n relations of $\mathcal{U}$. Let U* be the set of all (finite) T-sequences beginning with u_0.

For all relations S of $\mathcal{U}$, define S^* by

$$u^*_1 \; S^* \; u^*_2 \quad \text{iff} \quad u^*_2 = u^*_1 * \langle u'_2 \rangle \text{ and } u'_1 \; S \; u'_2 .$$

For atomic A, let $u^* \in V^*(A)$ iff $u' \in V(A)$.
Then for all S of $\mathcal{U}$, the inductive clause $\boxed{S}$) is proved as $\Box$) is proved in Lemma 1.

Could we add an axiom schema to some system other than K, a system which is determined by some property, and get a new system which is determined by, for example, irreflexivity and the given property? Is there a schema corresponding to irreflexivity together with some other property to which a conjunct of the schema corresponds?

We do not try to answer the general question but restrict ourselves to systems obtained by adding to K some of the axiom schemas

$G'_{m,n,p,q}$ $\quad \Diamond^m \Box^n A \supset \Box^p \Diamond^q A$

In [2] there are given completeness results for all these systems, with respect to the properties expressed by

$g'_{m,n,p,q}$ $\quad \forall u,v,w(uR^m v \wedge uR^p w \longrightarrow (\exists t_o)(vR^n t_o \wedge wR^q t_o))$

We may argue roughly as follows (details of the argument is given below):

We have the system $KG'_{m,n,p,q}$. (Or a system with several G'-schemas added to K.) All theorems in the system are valid in all model structures satisfying $g'_{m,n,p,q}$ (in the following we omit the parameters m,n,p,q), hence in all model structures satisfying g' and conjuncts of (3) which are consistent with g'.

Conversely, given a non-theorem of the theory, it has a countermodel $\mathcal{U}$ satisfying g'. "Unravel" $\mathcal{U}$ as in the proof of Lemma 1, getting $\mathcal{U}^*$. Take the closure of $\mathcal{U}^*$ with respect to g'. The new model $\tilde{\mathcal{U}}$ behaves like $\mathcal{U}^*$, and satisfies g' and some of the conjuncts of (3). The conjuncts of (3) (or more generally, the sentences implied by (3)) which are "saved" by this construction, no matter what countermodel we started with, express properties which cannot be matched by any schemas added to the system.

Examples show that not all conjuncts of (3) which are consistent with g' are saved. The difficult problem then is to determine what conjuncts of (3) are saved.

We have to be more precise about the phrase "the closure of $\mathcal{U}^*$ with respect to g' ".

Let $\mathcal{U}_{u_o} = \langle U_{u_o}, R \restriction U_{u_o} \rangle$, $l(u^*)$ be the length of u^*, u_i be the (i+1)st element of u^*, u' be the last element of u^* as before, and $u^*|j = \langle u_o, \ldots, u_{j-1} \rangle$

If $\mathcal{U}$ satisfies $g'_{m,n,p,q}$ and $u_o \in U$, then $\mathcal{U}_{u_o}$ also satisfies $g'_{m,n,p,q}$. Unravel $\mathcal{U}$ from u_o and get $\mathcal{U}^*$.

For all t_o that we can "find" by the existential quantifier in $g'_{m,n,p,q}$ given any u,v,w, and for all $u^*, v^* \in U^*$, if

$$\exists j \geq 0 : u^*|j = v^*|j$$

$$l(u^*) = m+n+j, \; l(v^*) = p+q+j,$$

$$\text{and} \quad u' = v' = t_o$$

then identify u^* and v^* and the trees above them (i.e. delete

v^* and the tree over it from $\mathcal{U}^*$ and substitute "arrows" pointing to v^* with "arrows" pointing to u^*). We get $\tilde{\mathcal{U}}$.
Since u^* and v^* are copies in $\mathcal{U}^*$ of the same point, t_o , in $\mathcal{U}$, the trees over u^* and v^* in $\mathcal{U}^*$ are isomorphic and the valuation over the trees is the same. Hence, for all formulas A ,

$$\langle u_o,\ldots,u_n\rangle \tilde{\models} A \Longleftrightarrow \langle u_o,\ldots,u_n\rangle \overset{*}{\models} A \Longleftrightarrow u_n \models A ,$$

if $\langle u_o,\ldots,u_n\rangle \in \tilde{U}$, i.e., $\tilde{\mathcal{U}}$, $\mathcal{U}^*$ and $\mathcal{U}_{u_o}$ are equivalent.

We shall prove that $\tilde{\mathcal{U}}$ satisfies $g'_{m,n,p,q}$. Let $u^*,v^*,w^* \in \tilde{U}$, $u^* \tilde{R}^m v^*$ and $u^* \tilde{R}^p w^*$ .If $l(v^*) = k_1$, $l(w^*) = k_2$, we then have

$$v^* = u^* * \langle v_{k_1-m},\ldots,v_{k_1-1}\rangle$$

$$w^* = u^* * \langle w_{k_2-p},\ldots,w_{k_2-1}\rangle ,$$

so we have $v_{k_1-m-1} R^m v_{k_1-1}$ and $w_{k_2-p-1} R^p w_{k_2-1}$, and $v_{k_1-m-1} = w_{k_2-p-1} = u'$.

Since $g'_{m,n,p,q}$ is satisfied by $\mathcal{U}$ there is a $t_o \in U$ such that $v_{k_1-1} R^n t_o$ and $w_{k_2-1} R^q t_o$. I.e., there exist in U $x_1,\ldots,x_{n-1},y_1,\ldots,y_{q-1}$ such that

$$v^* * \langle x_1,\ldots,x_{n-1},t_o\rangle \in U^* \quad \text{and} \quad w^* * \langle y_1,\ldots,y_{q-1},t_o\rangle \in U^* .$$

But these two $R_{\mathcal{U}}$- sequences must have been identified in the construction of $\tilde{\mathcal{U}}$. Hence there is a $t_o^* \in \tilde{U}$ such that $v^* \tilde{R}^n t_o^*$ and $w^* \tilde{R}^q t_o^*$.

We may now get theorems of the following kind:

K + the schema $G'_{m,1,o,o}$ $(\Diamond^m \Box A \supset A)$ is determined not only by the property expressed by $g'_{m,1,o,o}$ but also by the property expressed by $g'_{m,1,o,o} \wedge \bigwedge_{o<m'\leq m} (1)_{m',o}$.

K + the schema $G'_{o,1,p,o}$ $(\Box A \supset \Box^{p} A)$ is determined not only by the property expressed by $g'_{o,1,p,o}$ but also by the property expressed by $g'_{o,1,p,o} \wedge \bigwedge_{o<m'} (1)_{m',o}$.

But these theorems are rather limited. More general cases are either difficult to analyse or are such that every conjunct of (3) is violated by construction from some countermodel.

If we knew enough about the different systems to be able to choose our countermodels suitably, the matter might be considerably simplified.

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