AN INTRODUCTION TO THE FINE STRUCTURE OF
THE CONSTRUCTIBLE HIERARCHY
(Results of Ronald JENSEN)
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§ 0. Introduction

We shall work in Zermelo-Fraenkel set theory (including the
axiom of choice) throughout, and denote this theory by ZFC.
We shall adopt the usual, well-known, notations and conventions
of contemporary set theory (e.g. an ordinal is defined to be
the set of all smaller ordinals, cardinals are initial ordinals,
etc.)

The paper is entirely self-contained, but some familiarity with
the usual definition of the constructible universe, $\mathcal{L}$, in
terms of definability, and the proof that $\mathcal{L}$ is a model of
$\text{ZFC} + \text{GCH} + V = \mathcal{L}$, will be helpful.

The exposition is based, with permission, very strongly on a
set of notes written by Ronald Jensen and entitled "The Fine
Structure of the Constructible Hierarchy". Except where other-
wise stated, the results are entirely those of Professor Jensen.
It is convenient at this point for us to express our apprecia-
tion of several illuminating discussions with Professor Jensen
on his work in general.

Previously, Jensen worked, as did most other people, with the
usual "constructible hierarchy". Thus, one defines, inductively,
sets $L_\alpha$, $\alpha \in \text{OR}$, by setting $L_0 = \emptyset$, $L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha$ if $\lim(\lambda)$, and $L_{\alpha+1}$ is the set of all $x \subseteq L_\alpha$ such that for some $\in$-formula $\varphi$ and some $a_1,\ldots,a_n \in L_\alpha$, $x = \{z \in L_\alpha | L_\alpha \models \varphi[z,a_1,\ldots,a_n] \}$. One then defines the constructible universe as the class $L = \bigcup_{\alpha \in \text{OR}} L_\alpha$. Now, the important facts concerning this definition which one uses when studying $L$, are, firstly, that the construction is (in a strong way, to be made precise later) $\Sigma_1$-definable, and thus has certain absoluteness properties, and, secondly, that $L_{\alpha+1}$ contains all and only those subsets of $L_\alpha$ which are $L_\alpha$-definable (and which, therefore, must be in $L$ if $L$ is to be a model of ZFC). But if, indeed, these are the only conditions which we require (and loosely speaking they are), then it is clear that our above definition is unnecessarily restrictive. For instance, there are many simply definable functions or sets under which $L$ must be closed, but which increase rank - and these functions will lead out of the sets $L_\alpha$. For instance, unless $\lim(\alpha)$, $L_\alpha$ will not be closed under the formation of ordered pairs. Since this function plays a central role in even the most elementary parts of set theory, we see that this defect becomes quite important (though not unavoidable) when we try to study the fine structure of $L$ rather than $L$ itself. So, following Jensen, we define a new hierarchy of "constructible sets", which is sufficiently like the $L$-hierarchy to preserve the two properties mentioned above, but which has the extra property that each level in the hierarchy is closed under ordered pairs, etc. More precisely, we first define a certain class of set functions (called "rudimentary functions"), and then define a hierarchy $\langle J_\alpha | \alpha \in \text{OR} \rangle$ (the Jensen hierarchy) such that each $J_\alpha$ is closed under the rudi-
mentary functions, \( L = \bigcup_{\alpha \in \mathbb{R}} J_\alpha \), and the two properties above hold for this hierarchy. In most cases, \( J_\alpha \) will be a "constructively inessential" extension of \( L_\alpha \), and in fact, if \( \langle V_\alpha | \alpha \in \mathbb{R} \rangle \) denotes the familiar rank-hierarchy, the precise relationship between the \( J \)- and the \( L \)-hierarchies is easily seen to be \( J_0 = L_0 = \emptyset \) and \( L_{\omega + \alpha} = V_{\omega + \alpha} \cap J_{1 + \alpha} \) for all \( \alpha \).
Hence we have \( J_\alpha = L_\alpha \) iff \( \omega \alpha = \alpha \).

In § 1 we give some basic definitions. In § 2 we define the class of rudimentary functions and develop the elementary theory of this class. The reader may, if he wishes, safely skip all the proofs in this section without affecting the reading of the later parts. § 3 is devoted to a very brief discussion of the concept of an admissible set. In § 4 the Jensen hierarchy is defined and its elementary properties discussed. In § 5 we investigate the fine structure of the Jensen hierarchy. A corresponding theory may also be developed for the \( L \)-hierarchy, the only difference being that some awkward complications arise because of the above mentioned defects in this definition.

§ 1. Preliminaries

We shall be concerned with first-order structures of the form \( M = \langle M, \in, A \rangle \), where \( M \) is a non-empty set and \( A \subseteq M \). In general, we shall write \( \langle M, A \rangle \) for \( \langle M, \in, A \rangle \). The (first-order) language for such structures consists of the following:

(i) variables \( v_j, j \in \omega \) (generally denoted by \( v, w, x, y, z \), etc.) \( (\forall v) \)
(ii) predicates =, \( \in \), \( A \).
(iii) bounded quantifiers \( (\forall v_i \in v_j), (\exists v_i \in v_j), i, j \in \omega, i \neq j. \)
(iv) unbounded quantifiers \((\forall v_i), (\exists v_i)\), \(i \in w\).
(v) connectives \(\land, \lor, \land, \land\).

Finite strings of variables (or of elements of \(M\)) are denoted by \(\vec{v}, \vec{x}\), etc. We write \(\vec{a} \in X\) for \(a_1 \in X \land \ldots \land a_n \in X\), where we have \(\vec{a} = a_1, \ldots, a_n\). Similarly for \(\vec{a'}, \vec{v'}, \ldots\).

The notions of primitive formula (PFml), formula (Fml), free variable, statement, and satisfaction are assumed known.

A formula of this language is \(\Sigma_0\) (or \(\Pi_0\)) if it contains no unbounded quantifiers. Let \(n \geq 1\), and let \(Q_n\) denote \(\forall\) if \(n\) is even and \(\exists\) if \(n\) is odd. A formula is \(\Sigma_n(\Pi_n)\) if it is of the form \(\exists x_1 \forall x_2 \exists x_3 \ldots Q_n \vec{x}_n \phi(\forall x_1 \exists x_2 \forall x_3 \ldots Q_{n+1} \vec{x}_{n+1} \phi)\) where \(\phi\) is \(\Sigma_0\).

A formula in which the predicate \(A\) does not occur is called an \(\epsilon\)-formula.

\(|=|_M|\) denotes the satisfaction relation for \(M\). Thus, \(|=|_M|\) is the set of all \(<\varphi, \vec{z}>\) such that \(\varphi\) is a formula of the above language and \(\vec{z} \in M\) and \(\varphi\) holds in \(M\) at the point \(\vec{z}\). We generally write \(|=|_M|\varphi [\vec{z}]|\) for \(<\varphi, \vec{z}>\) \(\in|=|_M|\) \(\varphi\) denotes the set of all \(<\varphi, \vec{z}>\) \(\in|=|_M|\) such that \(\varphi\) is \(\Sigma_n\).

Let \(N \subseteq M\). A set \(R \subseteq M\) is \(\Sigma_n^M(N)\) (or \(\Pi_n^M(N)\)) iff there is a \(\Sigma_n(\Pi_n)\) formula \(\varphi(u, \vec{v})\) and elements \(\vec{a} \in N\) such that for all \(\vec{x} \in M\), \(R(\vec{x}) \iff |=|_M|\varphi[\vec{x}, \vec{a}]|\). The set of all such \(R\) is also denoted by \(\Sigma_n^M(N)\) (or \(\Pi_n^M(N)\)).

Set \(\Sigma_n^\sim(N) = \bigcup_{n \in w} \Sigma_n^M(N)\), \(\Lambda_n^\sim(N) = \Sigma_n^M(N) \cap \Pi_n^M(N)\).

Write \(\Sigma_n^M\) for \(\Sigma_n^\sim(\emptyset)\) and \(\Sigma_n^M\) for \(\Sigma_n^\sim(M)\). Similarly for \(\Pi, \Delta\).
If $\phi$ is a formula, $\overline{\phi}$ denotes the relation $\{x \in M | M \models \phi(x)\}$. Similarly, and more generally, define $\phi^\mathfrak{a} = \{x \in M | M \models \phi(x, \mathfrak{a})\}$.

Let $F$ be a class of structures of the form $M = \langle M, A \rangle$. A relation $R$ is uniformly $\Sigma_n(M)$ for $M \in F$ iff there is a $\Sigma_n$ formula $\phi(u, v)$ and elements $\mathfrak{a} \in \cap\{M | M \in F\}$ such that whenever $M \in F$, $R \cap M = \phi^\mathfrak{a}$.

§ 3. Rudimentary Functions

A function $f : V^n \rightarrow V$ is rudimentary (rud) iff it is generated by the following schemata:

(i) $f(x_1, \ldots, x_n) = x_i$, $1 \leq i \leq n$.

(ii) $f(x_1, \ldots, x_n) = x_i \cdot x_j$, $1 \leq i, j \leq n$.

(iii) $f(x_1, \ldots, x_n) = \{x_i, x_j\}$, $1 \leq i, j \leq n$.

(iv) $f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n))$, where $g_1, \ldots, g_k, h$ are rud.

(v) $f(x_1, \ldots, x_n) = \bigcup_{y \in x_1} h(y, x_2, \ldots, x_n)$, where $h$ is rud.

For example, the following functions are clearly rud:

$f(\overline{x}) = U x_i$

$f(\overline{x}) = x_i \cup x_j$ ( = $U\{x_i, x_j\}$)

$f(\overline{x}) = \{\overline{x}\}$

$f(\overline{x}) = \langle \overline{x} \rangle = \{[x_1], [x_1, \langle x_2, \ldots, x_n \rangle]\}$.

And if $f(y, \overline{x})$ is rud, so is $g(y, \overline{x}) = \langle f(z, \overline{x}) | z \in y \rangle$ ( = $U_{z \in y}\{\langle f(z, \overline{x}), z \rangle\}$).
We say that \( R \subseteq V^n \) is **rudimentary** (rud) iff there is a rud function \( r : V^n \to V \) such that \( R = \{ \langle \vec{x} \rangle | r(\vec{x}) \neq \emptyset \} \).

For example, \( \notin \) is rud, since \( y \notin x \iff \{y\} \subseteq x \neq \emptyset \).

We list some basic properties of rudimentary functions and relations.

1. If \( f,R \) are rud, so is \( g(x) = \begin{cases} f(x), & \text{if } R(x) \\ \emptyset, & \text{if } \neg R(x). \end{cases} \)

**Proof:** By definition, there is a rud \( r \) such that \( R(x) \iff r(x) \neq \emptyset \). Then \( g(x) = \bigcup_{y \in r(x)} f(x) \).

2. Let \( \chi_R \) be the characteristic function of \( R \). \( R \) is rud if \( \chi_R \) is rud.

**Proof:** By (1).

3. \( R \) is rud iff \( \neg R \) is rud.

**Proof:** By (2), since \( \chi_{\neg R}(\vec{x}) = 1 - \chi_R(\vec{x}) \).

4. Let \( f_i : V^n \to V \) be rud, \( i = 1, \ldots, n \). Let \( R_i \subseteq V^n \) be rud and mutually disjoint, \( i = 1, \ldots, m \), and such that \( \bigcup_{i=1}^m R_i = V^n \). Define \( f : V^n \to V \) by \( f(x) = f_i(x) \) iff \( R_i(\vec{x}) \). Then \( f \) is rud.

**Proof:** Set \( f_i(x) = \begin{cases} f_i(x), & \text{if } R_i(x) \\ \emptyset, & \text{if } \neg R_i(x), \quad i = 1, \ldots, m. \end{cases} \)

By (1), \( f_i \) is rud. But \( f(x) = \bigcup_{i=1}^m f_i(x) \).

5. If \( R(y,\vec{x}) \) is rud, so is \( f(y,\vec{x}) = y \cap \{ z \cap R(z,\vec{x}) \} \)

**Proof:** Set \( h(y,\vec{x}) = \begin{cases} \{y\}, & \text{if } R(y,\vec{x}) \\ \emptyset, & \text{otherwise.} \end{cases} \)

Then \( h \) is rud. But \( f(y,\vec{x}) = \bigcup_{z \in y} h(z,\vec{x}) \).
(6) Suppose \( R(y,\bar{x}) \) is rud and \( (\forall \bar{x})(\exists ! y)R(y,\bar{x}) \). Set
\[
f(y,\bar{x}) = \begin{cases} 
\text{the unique } z \in y \text{ such that } R(z,\bar{x}), & \text{if such a } z \text{ exists.} \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

Then \( f \) is rud.

**Proof:** \( f(y,\bar{x}) = \bigcup\{y \cap \{z \mid R(z,\bar{x})\} \} \)

(7) If \( R(y,\bar{x}) \) is rud, so is \( (\exists z \in y)R(z,\bar{x}) \).

**Proof:** Take \( r \) rud so that \( R(y,\bar{x}) \iff r(y,\bar{x}) \not\iff 0 \). Then
\[
(\exists z \in y)R(z,\bar{x}) \iff \bigcup_{z \in y} r(z,\bar{x}) \not\iff 0.
\]

(8) If \( R_i(\bar{x}) \) are rud, \( i = 1, \ldots, m \), then so are
\[
\bigcup_{i=1}^{m} R_i, \quad \bigcap_{i=1}^{m} R_i, \quad \text{(Trivial)}.
\]

(9) Let \( (-)_o, (-)_1 \), denote the inverse functions to \( \langle -, - \rangle \).

Then \( (-)_o, (-)_1 \) are rud. More generally, let \( (-)_o^n, \ldots, (-)_1^n \) denote the inverse functions to \( \langle x_1, \ldots, x_n \rangle \).

Then \( (-)_o^n, \ldots, (-)_1^n \) are rud.

**Proof:** \( (x)_o = \begin{cases} 
\text{the unique } z \in Ux \text{ such that} \\
(\exists u_1, u_2 \in Ux)(x = \langle u_1, u_2 \rangle \land u_1 = z) \\
\emptyset, \text{if no such } z \text{ exists.}
\end{cases} \)

etc.

(10) The function \( f(x,y) = x(y) = \begin{cases} 
\text{the unique } z \in Ux \text{ such that} \\
\langle z, y \rangle \in x \\
\emptyset, \text{if no such } z \text{ exists.}
\end{cases} \)

is rud (By definition.)

(11) \( \text{dom and ran are rud.} \)

**Proof:** \( \text{dom}(x) = \{z \in Ux \mid (\exists w \in Ux)(\langle w, z \rangle \in x)\} \)
\[
\text{ran}(x) = \{z \in Ux \mid (\exists w \in Ux)(\langle z, w \rangle \in x)\}.
\]
(12) \( f(x,y) = x \times y = \bigcup_{u \in x} \bigcup_{v \in y} \{ <u,v> \} \) is rud.

(13) \( f(x,y) = x \uparrow y = x \cap (\text{ran}(x) \times y) \) is rud.

(14) \( f(x,y) = x \downarrow y = \text{ran}(x \downarrow y) \) is rud.

(15) \( f(x) = x^{-1} \) is rud.

**Proof:** Set \( h(z) = \langle (z)_1, (z)_0 \rangle \). Then \( h \) is rud. But clearly, \( f(x) = x^{-1} = h''(x \cap (\text{ran}(x) \times \text{dom}(x))) \).

Recalling our preliminary discussion (§ 0), we observe that though rud functions increase rank, they only do so by a finite amount. More precisely, by induction on the rud definition* of a given rud function \( f \), we see that there is a \( p \in \omega \) such that for all \( x_1, \ldots, x_n \), \( \text{rank}(f(x_1, \ldots, x_n)) < \max\{\text{rank}(x_1), \ldots, \text{rank}(x_n)\} + p. \)

* **Note:** In future, we shall often refer to "the rud definition of \( f \)" or simply "the definition of \( f \". We mean an arbitrary such definition, the actual choice being irrelevant, and hence assumed made once and for all.

We now prove that the rud functions do in fact encompass all of the "simply definable" functions we spoke about in § 0. First, let us call a function \( f : V^n \rightarrow V \) simple iff whenever \( \varphi(z,\overrightarrow{y}) \) is a \( \Sigma_0 \) \( \in \)-formula, there is a \( \Sigma_0 \) \( \in \)-formula \( \psi \) such that \( \models \varphi(f(\overrightarrow{x}),\overrightarrow{y}) \rightarrow \psi(\overrightarrow{x},\overrightarrow{y}). \) A useful characterisation of this concept is given by the following:

**Proposition**
A function \( f : V^n \rightarrow V \) is simple iff

(i) the predicate \( x \in f(\overrightarrow{y}) \) is \( \Sigma_0^V \); and

(ii) whenever \( A(x) \) is \( \Sigma_0^V \), so is \( (\forall x \in f(\overrightarrow{y}))A(x) \).
Proof: \((-\)\) By definition.

(-) Let \(f\) satisfy \((i)\) and \((ii)\), and let \(\varphi(x, y)\) be a \(\Sigma_0\) \(\mathcal{L}\)-formula. An easy induction on the length of \(\varphi\) shows that \(\varphi(f(x), y)\) is equivalent to a \(\Sigma_0\) \(\mathcal{L}\)-formula; so \(f\) is simple.

Using this proposition, and easy induction on the definition of \(f\) yields:

Lemma 1

If \(f\) is rud then \(f\) is simple.

Now, since there are \(\Sigma^V_0\) functions which increase rank by an infinite amount, it is clear that the converse to the above lemma is false. However, we do have:

Lemma 2

\(R \subseteq V^n\) is \(\Sigma^V_0\) iff \(R\) is rud.

Proof: \((-\)\) Let \(R\) be \(\Sigma^V_0\). By \((3), (7),\) and \((8)\) above, an easy induction on the \(\Sigma^V_0\) definition of \(R\) shows that \(R\) is rud.

(-) Let \(R\) be rud. Then \(\chi_R\) is rud. So by lemma 1, \(\chi_R\) is simple. Using our above proposition, an easy induction on the rud definition of \(\chi_R\) shows that \(\chi_R\), and hence \(R\), is \(\Sigma^V_0\).

We require some generalisations of these concepts.

Let \(A \subseteq V\). We say that a function \(f\) is rud in \(A\) iff \(f\) is generated by the schemata for rud functions and the function \(\chi_A\).

Let \(p \in V\). We say that a function \(f\) is rud in parameter \(p\) iff \(f\) is generated by the schemata for rud functions and the
constant function $h(x) = p$.

**Lemma 3**
If $f$ is rud in $A \subset V$, there are rud functions $g_1, \ldots, g_n$ such that $f$ is expressible (in a uniform way with respect to the rud definition of $f$) as a composition of $g_1, \ldots, g_n$ and the function $h(x) = A \cap x$.

**Proof:** By induction on the (rud) definition of $f$.

A set $X$ is said to be **rud closed** if for all rud functions $f, f^n X \subset X$.

A structure $\mathcal{M} = \langle M, A \rangle$ is said to be **rud closed** if for all functions $f$ which are rud in $A$, $f^\mathcal{M} \subset M$.

We say a structure $\mathcal{M} = \langle M, A \rangle$ is **amenable** if $u \in M \rightarrow A \cap u \in M$.

**Lemma 4**
A structure $\mathcal{M} = \langle M, A \rangle$ is rud closed iff the set $M$ is rud closed and $\mathcal{M}$ is amenable.

**Proof:** By lemma 3.

**Lemma 5**
Let $A \subset V$. If $f$ is rud in $A$, then $f \upharpoonright M^\mathcal{N}$ is uniformly $\Sigma_0(\langle M, A \cap M \rangle)$ for all transitive, rud closed $\mathcal{M} = \langle M, A \cap M \rangle$.

**Proof:** By lemmas 2 and 3.

The next result will be of considerable use to us later on.
Lemma 6

Every rud function is a composition of some of the following rud functions:

\[
\begin{align*}
F_0(x,y) &= \{x,y\} \\
F_1(x,y) &= x - y \\
F_2(x,y) &= x \times y \\
F_3(x,y) &= \{\langle u,z,v \rangle \mid z \in x \land \langle u,v \rangle \in y\} \\
F_4(x,y) &= \{\langle u,v,z \rangle \mid z \in x \land \langle u,v \rangle \in y\} \\
F_5(x,y) &= \cup x \\
F_6(x,y) &= \text{dom}(x) \\
F_7(x,y) &= x \cap x^2 \\
F_8(x,y) &= \{x^n[z] \mid z \in y\}.
\end{align*}
\]

Proof: Let \( \mathcal{C} \) denote the class of all functions obtainable from \( F_0, \ldots, F_8 \) by composition. We must show that \( f \in \text{rud} \rightarrow f \in \mathcal{C}. \)

For each \( \varepsilon \)-formula \( \varphi(x_1, \ldots, x_n) \), set

\[
t_\varphi(u) = \{\langle x_1, \ldots, x_n \rangle \mid x_1, \ldots, x_n \in u \land \langle u, e \rangle \varphi[x_1, \ldots, x_n]\}.
\]

By induction on \( \varphi \), we show that for all \( \varphi \), \( t_\varphi \in \mathcal{C} \). (The required result will then be proved using this fact.)

(a) \( \varphi(\overline{x}) \equiv x_1 \in x_j \; , \; 1 \leq i < j \leq n \).

Write \( F_x(y) \) for \( F_3(x,y) \). Then \( t_\varphi(u) = u^{i-1} \times F^{j-1}_u(F_4(\varepsilon \cap u^2, u^{n-j})) \) so \( t_\varphi \in \mathcal{C} \).

(b) Let \( \varphi_1(\overline{x}), \ldots, \varphi_p(\overline{x}) \) be such that \( t_{\varphi_1}, \ldots, t_{\varphi_p} \in \mathcal{C} \).

Let \( \varphi(\overline{x}) \) be any propositional combination of \( \varphi_1, \ldots, \varphi_p \).

Since \( x - y, x \cup y (= U\{x,y\}), x \cap y (= x - (x-y)) \in \mathcal{C} \), we clearly have \( t_\varphi \in \mathcal{C} \).
(c) Let \( \bar{\varphi}(y, \bar{x}) \) be such that \( t_\varphi \in \mathcal{C} \). Let \( \varphi(\bar{x}) = \forall y \varphi(y, \bar{x}) \) or \( \exists y \varphi(y, \bar{x}) \).

Clearly, \( t_\forall \varphi(u) = \text{dom}(t_\varphi(u)) \) and \( t_\exists \varphi(u) = u^n - \text{dom}(u^n - t_\varphi(u)) \).

So in either case, \( t_\varphi \in \mathcal{C} \).

(d) \( \varphi(\bar{x}) = x_i = x_j \).

Let \( \theta(y, \bar{x}) = y \in x_i \rightarrow y \in x_j \). By (a), (b), \( t_\theta \in \mathcal{C} \). But look, \( \{u, \bar{x} : \varphi[\bar{x}] \} \iff (\forall y \in Uu)[ \{u \cap (u \cap u) \}, \bar{y}(y, \bar{x})] \).

Hence \( t_\varphi(u) = u^n \cap t_\exists \theta(uuu (Uu)), \) so \( t_\varphi \in \mathcal{C} \), by (c).

(e) \( \varphi(\bar{x}) = x_i \in x_j \), \( 1 \leq j < i \leq n \).

Let \( \psi(y, z, \bar{x}) = y \in z \wedge y = x_i \wedge z = x_j \). By (a), (b), (d), \( t_\psi \in \mathcal{C} \). But \( \varphi(\bar{x}) \rightarrow \exists y \exists z \psi(y, z, \bar{x}) \), so by (c), \( t_\psi \in \mathcal{C} \).

Hence, for any \( \epsilon \)-formula \( \varphi \), \( t_\varphi \in \mathcal{C} \).

If \( f : V^n \rightarrow V \), define \( f^* : V \rightarrow V \) by \( f^*(u) = f^"u" \). Using our above result, we prove by induction on the rud definition of \( f \), that \( f \) rud \( - f^* \in \mathcal{C} \). This easily implies the required result.

(a) \( f(\bar{x}) = x_i \).

\( f^*(u) = f^"u^n = u = u - (u-u) \in \mathcal{C} \).

(b) \( f(\bar{x}) = x_1 - x_j \).

\( f^*(u) = f^"u^n = \{x-y \mid x, y \in u\} \). Let \( \varphi(z, x, y) = z \in x - y \).

Let \( F(u) = t_\varphi(uuu (Uu)) \cap (u \cap u^2) = \{\langle z, x, y \rangle \mid x, y \in u \wedge z \in x - y \} \).

Then \( f^*(u) = F_\varphi(F(u), u^2) \in \mathcal{C} \), since \( t_\varphi \in \mathcal{C} \).

(c) \( f(\bar{x}) = [x_i, x_j] \).

\( f^*(u) = f^"u^n = \{[x, y] \mid x, y \in u\} = uuu^2 \in \mathcal{C} \).
(d) \( f(\overline{x}) = h(y_1(\overline{x}),\ldots,g_k(\overline{x})) \).

Let \( G(u) = \bigcup_{i=1}^{k} g_i^*(u) \), \( H(u) = h^*(\bigcup_{i=1}^{k} g_i^*(u)) = h^*(G(u)) \), and \( K(u) = u^N \cup G(u) \cup H(u) \). By hypothesis, \( G, H, K \in \mathcal{C} \). By lemma 1, let \( \varphi(y,\overline{x}) \) be an \( \xi \)-formula equivalent to the formula

\[ \exists z_1 \ldots \exists z_k (z_1 = g_1(\overline{x}) \land \ldots \land z_k = g_k(\overline{x}) \land y = h(z_1,\ldots,z_k)) \]

Clearly, \( f^*(u) = \mathcal{F}(\varphi(K(u))) \cap [H(u) \times u^N] \), \( u^N \in \mathcal{C} \).

(e) \( f(y,\overline{x}) = \bigcup_{z \in y} g(z,\overline{x}) \).

Let \( G(u) = \{ \langle z,\overline{y},\overline{x} \rangle \mid (\forall v \in y)[z \in g(v,\overline{x})] \land z \in u \} \). As above \( G \in \mathcal{C} \). But \( f^*(u) = \mathcal{F}(G(u), u^{n+1}) \in \mathcal{C} \).

Hence \( f \text{ rud } \rightarrow f^* \in \mathcal{C} \), for all \( f \).

Finally, let \( f \) be rud. We show that \( f \in \mathcal{C} \).

Set \( \overline{f}(\overline{z}) = f(\overline{z}) \), \( \overline{f}(y) = 0 \) in all other cases. Thus \( \overline{f} \) is rud. So by the above, \( \overline{f}^* \in \mathcal{C} \). Let \( P(\overline{x}) = \{ \langle \overline{x} \rangle \} \). Thus \( P \in \mathcal{C} \).

But look, \( f(\overline{x}) = \bigcup \{ f(\overline{x}) \} = \bigcup \overline{f}(\overline{z}) = \bigcup \mathcal{F}(\overline{f}^*(P(\overline{x})), P(\overline{x})) \in \mathcal{C} \).

As an immediate corollary of lemmas 3 and 6 we have:

**Lemma 7**

Let \( A \subseteq V \) and define \( F_9 \) by \( F_9(x, y) = A \cap x \). Every function rud in \( A \) may be expressed as a composition of some of the (rud in \( A \)) functions \( F_0,\ldots,F_9 \).

We shall make immediate use of lemma 7 in investigating the logical complexity of the predicates \( \Sigma_m^n \) for suitable \( M \). We assume, once and for all, that we have a fixed arithmetisation of our language.
Lemma 8

$\Sigma_0^M$ is uniformly $\Sigma_1^M$ for transitive, rud closed $M = \langle M, A \rangle$.

Proof: Let $L$ be the language consisting of:

(i) variables $w_i$, $i \in \omega$.

(ii) function symbols (binary) $f_0, \ldots, f_9$.

We shall assume we have a fixed arithmetisation of $L$. We also assume that the reader understands what is meant by a "term" of $L$. Henceforth, let $M = \langle M, A \rangle$ be arbitrary, transitive, and rud closed.

We first define precisely how $L$ is to be interpreted in $M$.

Let $Q$ be the set of functions $\rho$ mapping a finite subset of $\{w_i | i \in \omega\}$ into $M$. We may clearly assume $Q$ is rud. Let $C$ be the (rud) function which to each term $\tau$ of $L$ assigns the set of all component terms of $\tau$, including variables. Let $\text{Vb}_L$ be the rud predicate defining the set $\{w_i | i \in \omega\}$.

Let $P$ be the predicate

$P(u,g,v) \iff \text{dom}(g) = u \land (\forall x \in u)[[x \in \text{Vb}_L \rightarrow x \in \text{dom}(v) \land g(x) = v(x)]$

$\land \land_{1 \leq 0} (\forall t_0, t_1 \in u)[x = f_1(t_0, t_1) \rightarrow g(x) = F_1(g(t_0), g(t_1))]]$.  

Thus $P$ is rud in $A$.

We may now define the interpretation of a term $\tau$ of $L$ at a "point" $\rho \in Q$ by:

$y = \tau^M_{\rho} \iff \tau$ is an $L$-term" $\land \rho \in C \land \exists g[P(C(\tau), g, \rho) \land g(\tau) = y]$

Hence the function $f(\tau, \rho) = \begin{cases} M_{\rho}, & \text{if } \tau \text{ is an } L\text{-term and } \rho \in Q \\ \emptyset, & \text{otherwise} \end{cases}$

is (uniformly) $\Sigma_1^M$ (for transitive, rud closed $M$).
Since $M$ is rud closed we can use the above result to define $\vdash^\Sigma_0$ as an $M$-predicate.

Let $\varphi \in \text{Fml}^{\Sigma_0}$. By lemma 2, $\varphi^M$ is rud in $A$. Hence the function $\Gamma$ defined by

$$\Gamma(\vec{x}) = \begin{cases} 1, & \text{if } \varphi^M[\vec{x}] \\ 0, & \text{otherwise} \end{cases}$$

is rud in $A$. So, by lemma 7, we may assume $\Gamma = \tau^M$, where $\tau$ is a term of $\mathfrak{L}$, under the above interpretation (i.e., with $F_i$ interpreting $f_i$ for each $i$). In fact, we may clearly pick a recursive function $\sigma$ mapping $\text{Fml}^{\Sigma_0}$ into the terms of $\mathfrak{L}$ so that whenever $\varphi \in \text{Fml}^{\Sigma_0}$, $\varphi^M[\vec{x}] \rightarrow [\sigma(\varphi)][\vec{x}] = 1$. But by our above result, this implies that $\vdash^\Sigma_0$ is (uniformly) $\Sigma_1^M$ (for transitive, rud closed $M$).

As an immediate consequence of this result, we have

**Lemma 9**

Let $n \geq 1$. Then $\vdash^\Sigma_n$ is uniformly $\Sigma_n^M$ for transitive, rud closed $M = (M,A)$.

We conclude this section with a few miscellaneous results of use later. The first two are technical, and will often be used without mention.

**Lemma 10**

Let $M = (M,A)$ be rud closed. If $R \subset M$ is $\Sigma_n(M)$, there is a $\Sigma_0(M)$ relation $P$ such that $R(x) \rightarrow \forall x_1 \forall x_2 \forall x_3 \ldots \forall x_n P(x,x_1,\ldots,x_n)$.

**Proof:** Suppose $R(x) \rightarrow \vdash^M \forall \vec{v}_1 \forall \vec{v}_2 \forall \vec{v}_3 \ldots \forall \vec{v}_n \varphi(\vec{v},\vec{v}_1,\ldots,\vec{v}_n)[x]$, where $\varphi$ is a $\Sigma_0$-formula. Using the rud functions $\langle -,\ldots,- \rangle$, $\langle - \rangle_o,\ldots,\langle - \rangle_{m-1}^m$, we can easily obtain, via
Lemma 11
Let \( \mathfrak{M} = (M, A) \) be rud closed. If \( R \subseteq M \) is \( \Sigma_n(M) \), there is a single element \( p \in M \) such that \( R \) is \( \Sigma_n(p) \).

Proof: If \( R \) is \( \Sigma_n([p_1, \ldots, p_n]) \), then \( R \) is also \( \Sigma_n([p_1, \ldots, p_n]) \).

Let \( \mathfrak{M} = (M, A) \) \( n \geq 0 \). Write \( X \underline{\subseteq} \Sigma_n \mathfrak{M} \) iff \( X \subseteq M \) and for every \( \Sigma_n \) formula \( \varphi \) and every \( \bar{x} \in X \),

\[
\vDash_{\Sigma_n \mathfrak{M}} \varphi(\bar{x}) \iff \vDash_{\mathfrak{M}} \varphi(\bar{x}).
\]

Clearly, if \( X, M \) are transitive and \( X \subseteq M \), we always have \( X \underline{\subseteq} \Sigma_0 \mathfrak{M} \). And for \( n > 0 \), we have \( X \underline{\subseteq} \Sigma_n \mathfrak{M} \) iff \( X \subseteq M \) and for every \( P \in \Sigma_n^M(X) \), \( P \vDash \varnothing \iff P \cap X \vDash \varnothing. \)

Recall that if \( \langle X, \epsilon \rangle \) satisfies the axiom of extensionality, there is a unique isomorphism \( \pi : \langle X, \epsilon \rangle \cong \langle W, \epsilon \rangle \), where \( W \) is a unique transitive set. Furthermore, if \( Z \subseteq X \) is transitive, then \( \pi|Z = \text{id}|Z \). In fact, \( \pi \) is defined by \( \epsilon \)-induction thus:

\[
\pi(x) = \{ \pi(y) | y \in x \cap X \} \text{ for each } x \in X.
\]

The next result is of considerable importance.

Lemma 12

Let \( \mathfrak{M} \) be transitive and rud closed. Let \( X \underline{\subseteq} \Sigma_1 \mathfrak{M} \). Then \( \langle X, \text{A} \cap X \rangle \) satisfies the axiom of extensionality and is rud closed. Let \( \pi : \langle X, \text{A} \cap X \rangle \cong \langle W, B \rangle \), where \( W \) is transitive. Let \( f : M \to M \)
be rud in A. Then for all \( \overline{z} \in X \), \( \pi(f(\overline{z})) = f(\pi(\overline{z})) \).

**Proof:** Since \( \mathcal{M} \) is transitive, \( \mathcal{M} \) satisfies the axiom of extensionality. Hence as \( X \subset \Sigma^1_1 \mathcal{M} \), so does \( \langle X, A \cap X \rangle \). Similarly, by lemma 5, \( \langle X, A \cap X \rangle \) is rud closed. Hence, in particular, \( \overline{z} \in X \rightarrow f(\overline{z}) \in X \) for \( f : \mathcal{M} \rightarrow \mathcal{M} \) rud in A. By induction on the (rud in A) definition of \( f \), \( \pi(f(\overline{z})) = f(\pi(\overline{z})) \) for each \( \overline{z} \in X \).

§ 3. Admissible Sets.

Let \( \mathcal{M} = \langle M, A \rangle \) be non-empty and transitive. We say \( \mathcal{M} \) is admissible iff \( \mathcal{M} \) is rud closed and satisfies the \( \Sigma_0 \)-Replacement Axiom: for all \( \Sigma_0 \) formulas \( \varphi \) and all \( \overline{a} \in M \),

\[
\models \mathcal{M}[\forall x \exists y \varphi \rightarrow \forall u \exists v (\exists x \in u)(\exists y \in v)\varphi][\overline{a}].
\]

In case \( A = \varnothing \) in the above, we call \( M \) an admissible set.

More generally, \( \mathcal{M} \) is \( \Sigma_n \)-admissible iff \( \mathcal{M} \) is rud closed and satisfies the (analogous) \( \Sigma_n \)-Replacement Axiom. Likewise a \( \Sigma_n \)-admissible set. We prove below that \( \mathcal{M} \) is admissible iff \( \mathcal{M} \) is \( \Sigma_1 \)-admissible. All our results extend trivially from admissibility to \( \Sigma_n \)-admissibility, with "\( \Sigma_n \)" everywhere replacing \( \Sigma_1 \), etc.

Roughly speaking, an admissible set (or structure) behaves like the universe as far as \( \Sigma_1 \) concepts are concerned. We give a few elementary results which set the tone for the rest of this exposition.

**Convention:** For the whole of this paper, we shall adopt the following abuse of notation. Suppose \( \mathcal{M} \) is a structure, \( \varphi(\overline{v}) \)
is a formula, and $\vec{x} \in M$. We shall write $\models_M \varphi(\vec{x})$ rather than $\models_M \varphi(\vec{v}) [\vec{x}]$. Clearly, this is purely a notational convenience.

Firstly, we give the promised "stronger" form of the admissibility definition.

**Lemma 13 (Σ₁-Replacement)**

Let $\bar{M}$ be admissible, and let $\varphi$ be a $\Sigma_1$-formula, $\vec{a} \in M$. Then

$$\models_{\bar{M}} \forall x \exists y \varphi(x, y, \vec{a}) \rightarrow \forall u \exists v (\forall x \in u \exists y \in v) \varphi(x, y, \vec{a})$$

**Proof:** Let $\psi$ be a $\Sigma_0$-formula such that

$$\models_{\bar{M}} \varphi(x, y, \vec{a}) \leftrightarrow \exists z \psi(x, y, \vec{a})$$

Then

$$\models_{\bar{M}} \forall x \exists y \varphi(x, y, \vec{a}) \rightarrow \forall x \exists y \exists z \psi(x, y, z, \vec{a})$$

$$\rightarrow \forall x \exists w \psi(x, (w)_0, (w)_1, \vec{a})$$

$$\rightarrow \forall u \exists v (\forall x \in u) (\exists w \in v) \psi(x, (w)_0, (w)_1, \vec{a}),$$

by $\Sigma_0$-Replacement

$$\rightarrow \forall u \exists v (\forall x \in u) (\exists y \in v) \varphi(x, y, \vec{a})$$

**Convention:** The essentially superfluous role played by $\vec{a}$ in the above theorem leads us to extend our previous convention slightly by allowing formulas to contain members of $\bar{M}$ as parameters. Again, this is clearly an avoidable convenience.

**Lemma 14**

Let $\bar{M}$ be admissible. If $R(\vec{x}, y)$ is $\Sigma_1(\bar{M})$, so is $(\forall y \in z)R(\vec{x}, y)$.

**Proof:** Let $\varphi$ be a $\Sigma_0$-formula with parameters from ("w.p.f."")$M$ such that

$$R(\vec{x}, y) \rightarrow \models_{\bar{M}} \forall w \varphi(\vec{x}, y).$$

Then

$$(\forall y \in z)R(\vec{x}, y) \rightarrow \models_{\bar{M}} (\forall y \in z) \exists w \varphi(\vec{x}, y)$$

$$\rightarrow \models_{\bar{M}} \forall y \exists w [(y \in z \wedge \varphi(\vec{x}, y)) \vee (y \notin z)].$$
So by Σ₀- Replacement,
\[(\forall y \in z)R(x, y) \iff \models M \exists v(\forall y \in z)(\exists w \in v)\varphi(x, y),\]
which is \(\Sigma_1(M)\).

**Lemma 15** \((\Delta_1\text{-comprehension})\)

Let \(M\) be admissible, \(P \in \Delta_1(M)\). Then \(u \in M \to P \cap u \in M\).

**Proof:** Let \(\psi, \varphi\) be \(\Sigma_0\)-formulas w.p.f. \(M\) such that
\[P(z) \iff \models M \forall x \varphi(x, z) \iff \models M \exists y \psi(y, z).\]

Then,
\[\models M \forall w_1 \exists w_2 [[w_1 \in u \land (\exists y (\psi \land w_2 = y)) \lor \exists x (\varphi \land w_2 = x))] \lor [w_1 \notin u \land w_2 = \emptyset].\]

So by \(\Sigma_1\)-Replacement there is \(v \in M\) such that
\[\models M (\forall w_1 \in u)(\exists w_2 \in v)[\exists y (\psi \land w_2 = y) \lor \exists x (\varphi \land w_2 = x)].\]

So,
\[P \cap u = \{z \mid \models M \forall x \varphi(x, z)\} \cap u = \{z \mid \models M (\exists y \in v)\psi(y, z)\} \cap u.\]

But \(M\) is closed (so satisfies what might be called the \(\Sigma_0\)-comprehension axiom), and therefore we conclude that
\[P \cap u = \{z \in u \mid \models M (\exists y \in v)\psi(y, z)\} \in M.\]

The next result has nothing specifically to do with admissibility, but is of considerable value. Let \(f : \subset M \to M\) mean that
\(f : X \to M\) for some \(X \subset M\).

**Lemma 16**

Let \(M\) be arbitrary, \(f : \subset M \to M\) be \(\Sigma_1(M)\). If \(\text{dom}(f)\) is \(\Pi_1(M)\), then in fact \(f\) and \(\text{dom}(f)\) are \(\Delta_1(M)\).
Proof: (a) \[ \frac{f(x) = y}{\Sigma_1(M)} \quad \frac{x \in \text{dom}(f) \land \forall z(z \neq y \rightarrow f(x) \neq z)}{\Pi_1(M)} \]

(b) \[ \frac{x \in \text{dom}(f)}{\Pi_1(M)} \quad \frac{\exists y(f(x) = y)}{\Sigma_1(M)} \]

It was necessary to state the above result explicitly because we shall frequently have to deal with functions which, though definable, are not total functions. A particular case of the above theorem would of course occur when \( \text{dom}(f) \in M \), (when \( \text{dom}(f) \) is \( \Sigma_0(M) \)).

As usual, we shall use the notation \( g(x) \sim y(x) \) for partial functions, with its usual meaning (i.e. \( f(x) \) is defined iff \( g(x) \) is defined, in which case \( f(x) = g(x) \)).

Lemma 17

Let \( M \) be admissible, \( f : \subset M \rightarrow M \) be \( \Sigma_1(M) \). If \( u \in M \) and \( u \subset \text{dom}(f) \), then \( f''u \in M \).

Proof: Since \( M \) is rud closed and \( f''u = \text{ran}(f\lfloor u) \), it suffices to prove that \( f\lfloor u \in M \).

Now, as \( u \in M \), \( f\lfloor u \) is \( \Delta_1(M) \) by lemma 16.

Let \( \varphi(x,y) \) be a \( \Sigma_1 \)-formula w.p.f. \( M \) such that \( f(x) = y \rightarrow f\lfloor u = (f\lfloor u) \cap (v \times u) \in M \).

Theorem 18 (Recursion Theorem)

Let \( M \) be admissible. Let \( h : M^{n+1} \rightarrow M \) be a \( \Sigma_1(M) \) function such that for all \( \bar{x} \in M \), \( \{ \langle z\bar{y} \rangle | z \in h(y, \bar{x}) \} \) is well-founded.
Let $G = M^{n+2} \rightarrow M$ be $\Sigma_1(M)$. Then there is a unique $\Sigma_1(M)$ function $F$ such that

(i) $\langle y, x \rangle \in \text{dom}(F) \iff \{ \langle z, x \rangle \mid z \in h(y, x) \} \subseteq \text{dom}(F)$

(ii) $F(y, x) \equiv G(y, x, \langle F(z, x) \mid z \in h(y, x) \rangle)$.

Proof: Let $\phi$ be the predicate

$\phi(f, x) \equiv "f$ is a function" \land (\forall y \in \text{dom}(f))(\forall z \in h(y, x))(z \in \text{dom}(f))$ \land $(\forall y \in \text{dom}(f))(f \upharpoonright y) = G(y, x, f \upharpoonright h(y, x))$.

By lemma 16, $h, G$ are $\Delta_1(M)$, so $\phi$ is $\Delta_1(M)$.

Let $\varphi$ be a $\Sigma_1$-formula w.r.t. $M$ such that $\phi(f, x) \equiv \models_M \varphi(f, x)$.

Define a $\Sigma_1(M)$ predicate $F$ by (using notation which will later be justified)

$F(y, x) = u \iff \exists f[\phi(f, x) \land f(y) = u]$.

We verify (i) for this $F$. Suppose first that $\langle y, x \rangle \in \text{dom}(F)$.

Then, by definition, $\exists f[\phi(f, x) \land y \in \text{dom}(f)]$. By definition of $\phi$, for such an $f$ we must have $(\forall z \in h(y, x))(z \in \text{dom}(f))$. Hence $z \in h(y, x) \rightarrow \langle z, x \rangle \in \text{dom}(F)$. Now suppose that $z \in h(y, x) \rightarrow \langle z, x \rangle \in \text{dom}(F)$.

Note that as $M$ is transitive, $h(y, x) \subseteq M$.

By our supposition,

$\vdash_M (\forall z) \exists f[(z \in h(y, x) \land z \in \text{dom}(f) \land \varphi(f, x)) \lor (z \notin h(y, x) \land f = \varnothing)]$.

so by $\Sigma_1$-Replacement,

$\vdash_M \forall v(\forall z \in h(y, x))(\exists f \in v)[z \in \text{dom}(f) \land \varphi(f, x)]$.

pick such a $v$. As $\phi$ is $\Delta_1(M)$, by $\Delta_1$-Comprehension we see that $w = v \cap \{ f \mid \phi(f, x) \} \in M$. Hence $\models_M w \subseteq M$. It is easily seen
that \( \xi(Uw,\bar{x}) \). Noting that \( h(y,\bar{x}) \subseteq \text{dom}(Uw) \), note that \( Uw \models h(y,\bar{x}) \in M \). Set \( f = Uw \models h(y,\bar{x}) \cup \{ \langle G(y,\bar{x}, Uw \models h(y,\bar{x})), y \rangle \} \). Clearly, \( \xi(f,\bar{x}) \), so \( \langle y,\bar{x} \rangle \in \text{dom}(F) \). Hence (i) holds for this \( F \).

We now show that \( F \) is a function and is unique. By (i), \( \text{dom}(F) \) is already uniquely determined, so for both of these it suffices to prove the following:

\[
\xi(f,\bar{x}) \land \xi(f',\bar{x}) \land y \in \text{dom}(f) \cap \text{dom}(f') \rightarrow f(y) = f'(y)
\]

To this end, suppose not. Then \( P = \{ y \mid y \in \text{dom}(f) \cap \text{dom}(f') \land f(y) \neq f'(y) \} \neq \emptyset \). Let \( y_0 \) be an \( h \)-minimal element of \( P \). Since \( y_0 \in P \), \( f(y_0) \neq f'(y_0) \). But \( \xi(f,\bar{x}), \xi(f',\bar{x}) \), so clearly \( f(y_0) = f'(y_0) \) by the \( h \)-minimality of \( y_0 \in P \). This contradiction suffices (and thus justifies our notation somewhat).

Finally, it is trivial to note that (ii) must hold, virtually by definition.

In view of the many set theoretic concepts defined by a recursion of the above type, it is clear that admissible sets play an important role in set theory.

Say \( \mathfrak{M} \) is strongly admissible iff \( \mathfrak{M} \) is non-empty, transitive, rud closed, and satisfies the Strong \( \Sigma_0 \)-Replacement Axiom: for all \( \Sigma_0 \) formulas \( \varphi \) w.p.f. \( M \), \( \models M \forall u \exists v (\forall x \in u)[\exists y \varphi(x,y) \rightarrow (\forall y \in u) \varphi(x,y)] \). (Clearly, such an \( \mathfrak{M} \) will also satisfy the "Strong \( \Sigma_1 \)-Replacement Axiom".)

Strongly admissible structures \( \mathfrak{M} \) are (for reasons to be indicated later) also called non-projectible admissible structures. The difference between admissibility and strong admissibility is
closely connected with the difference between $\Sigma_n$ predicates and $\Delta_n$ predicates, which is in turn closely connected with the difference between a function being partial and total. We shall have more to say on this matter later.

§ 4. The Jensen Hierarchy

Let $X$ be a set. The **rudimentary closure** of $X$ is the smallest set $Y \supset X$ such that $Y$ is rud closed.

**Lemma 19**

If $U$ is transitive, so is its rud closure.

**Proof:** Let $W$ be the rud closure of $U$. Since rud functions are closed under composition, we clearly have

$$W = \{ f(x) | x \in U \land f \text{ is rud} \}.$$  

An easy induction on the rud definition of any rud $f$ shows that $x \in U - \text{TC}(f(x)) \subset W$. Hence $W$ is transitive.

For $U$ transitive, let rud$(U) =$ the rud closure of $U \cup \{U\}$.

Of crucial importance is:

**Lemma 20**

Let $U$ be transitive. Then $\wp(U) \cap \text{rud}(U) = \Sigma_w(U)$.

**Proof:** Clearly, $\wp(U) \cap \Sigma_o(U \cup \{U\}) = \Sigma_w(U)$, so it suffices to show that $\wp(U) \cap \Sigma_o(U \cup \{U\}) = \wp(U) \cap \text{rud}(U)$.

Let $X \in \wp(U) \cap \Sigma_o(U \cup \{U\})$. Then, exactly as in the proof of lemma 2, $X \in \text{rud}(U)$ (by induction on the $\Sigma_o$ definition of $X$). Now let $X \in \wp(U) \cap \text{rud}(U)$. Then $X$ is a $\Sigma_o(\text{rud}(U))$ subset of $U$. By lemma 1, we may in fact
assume that $X$ is $\Sigma^0_rud(U)(UU[U])$. But $X \subseteq UU[U] \subseteq rud(U)$ and
$UU[U]$, $rud(U)$ are transitive, so $X$ is actually $\Sigma^0_rud(U)(UU[U]) = \Sigma^0(UU[U])$.

Also very relevant is:

**Lemma 21**
There is a rud function $S$ such that whenever $U$ is transitive, $S(U)$ is transitive, $UU[U] \subseteq S(U)$ and $\bigcup_{n \in \omega} S^n(U) = rud(U)$.

**Proof:** Set $S(U) = (UU[U]) U \bigcup_{i=0}^{\omega} (U \bigcup_{i=0}^{\omega} F^i(UU[U]))^2$. The result follows
by lemma 6.

**Lemma 22**
There is a rud function $\omega_0$ such that whenever $r$ is a well-ordering of $u$, $\omega_0(r,u)$ is an end-extension of $r$ which well-orders $S(u)$.

**Proof:** Define $i^u_1, i^u_2, j^u_1, j^u_2$ by:-

- $i^u(x) =$ the least $i \leq 8$ such that $(\exists x_1, x_2 \in u)[F_1(x_1, x_2) = x]$
- $j^u_1(x) =$ the $r$-least $x_1 \in u$ such that $(\exists x_2 \in u)[F_1u(x_1, x_2) = x]$
- $j^u_2(x) =$ the $r$-least $x_2 \in u$ such that $F_1u(x_2) = x$.

Clearly, $i^u_1, i^u_2, j^u_1, j^u_2$ are rud functions of $u, x$.

Define $\omega_0(r,u) = \{\langle x, y \rangle | x, y \in u \wedge xry\}$

- $\bigcup \{\langle x, y \rangle | x \notin u \wedge y \notin u\}$
- $\bigcup \{\langle x, y \rangle | x \notin u \wedge y \notin u \wedge [i^u(x) < i^u(y)] \wedge [j^u_1(x) < j^u_1(y)] \wedge [j^u_2(x) < j^u_2(y)]\}$

The Jensen hierarchy, $\langle J_\alpha | \alpha \in OR \rangle$, is defined as follows:

- $J_0 = \emptyset$
- $J_{\alpha + 1} = rud(J_\alpha)$
- $J_\lambda = \bigcup_{\alpha < \lambda} J_\alpha$, if $\text{lim}(\lambda)$. 

\begin{itemize}
  \item [\textbf{Proof:}] Set $S(U) = (UU[U]) U \bigcup_{i=0}^{\omega} (U \bigcup_{i=0}^{\omega} F^i(UU[U]))^2$. The result follows
by lemma 6.
\end{itemize}
Lemma 23
(i) Each $J_\alpha$ is transitive.
(ii) $\alpha \leq \beta \rightarrow J_\alpha \subset J_\beta$
(iii) $\text{rank}(J_\alpha) = \text{OR} \cap J_\alpha = \omega$.  

Proof: (i) By lemma 19.
(ii) Immediate.
(iii) By induction: $\text{rank}(J_{\alpha+1}) = \text{rank}(\text{rud}(J_\alpha)) = \text{rank}(J_\alpha) + \omega$
(by an earlier remark, this last step is easily verified.)

To facilitate our handling of the hierarchy, we "stratify" the $J_\alpha$'s by defining an auxiliary hierarchy $\langle S_\alpha | \alpha \in \text{OR} \rangle$ as follows:

$S_0 = \emptyset$

$S_{\alpha+1} = S(S_\alpha)$

$S_\lambda = \bigcup_{\alpha < \lambda} S_\alpha$, if $\lim(\lambda)$.

Clearly, the $J_\alpha$'s are just the limit points of this sequence.

In fact:

Lemma 24
(i) Each $S_\alpha$ is transitive
(ii) $\alpha \leq \beta \rightarrow S_\alpha \subset S_\beta$
(iii) $J_\alpha = \bigcup_{\nu < \omega \alpha} S_\nu = S_\omega \alpha$.

Proof: (i) By lemma 21.
(ii) Immediate.
(iii) By induction: $J_{\alpha+1} = \text{rud}(J_\alpha) = \bigcup_{n \in \omega} S^n(J_\alpha) = \bigcup_{n \in \omega} S^n(S_\omega \alpha) = \bigcup_{n \in \omega} S_{\omega \alpha + n} = S_{\omega \alpha + n} = S_{\omega(\alpha + 1)}$.

Lemma 25
$\langle S_\nu | \nu < \omega \alpha \rangle$ is uniformly $\Sigma^1_1$ for all $\alpha$.  

Proof: Set $\xi(f) = "f \text{ is a function}" \land \text{dom}(f) \in \text{OR} \land f(0) = \emptyset \land
(\forall v \in \text{dom}(f))[\text{succ}(v) \to f(v) = \xi(f)(v-1)] \land
[\lim(v) \to f(v) = \bigcup_{\alpha \in v} f(\alpha)]$.

Clearly, $\xi$ is uniformly $\Sigma^J_\alpha$. And by definition,
$y = S_v \leftrightarrow \exists f(\xi(f) \land y = f(v))$. Thus it suffices to show that for any $\alpha, v < \omega_\alpha$, the existential quantifier here can be restricted to $J_\alpha$. In other words, we must show that whenever $\tau < \omega_\alpha$, then
$\langle S_v \mid v < \tau \rangle \in J_\alpha$. This is proved by induction on $\alpha$. For $\alpha = 0$ it is trivial. For limit $\alpha$ the induction step is immediate.

So assume $\alpha = \beta + 1$ and that $\tau < \omega_\beta \to \langle S_v \mid v < \tau \rangle \in J_\beta$. Then, by our above remarks, it is clear that $\langle S_v \mid v < \omega_\beta \rangle$ is $\Sigma^J_\beta$.

So by lemma 20, $\langle S_v \mid v < \omega_\beta \rangle \in J_\alpha$. Thus for all $n < \omega$,
$\langle S_v \mid v < \omega_\beta + n \rangle = \langle S_v \mid v < \omega_\beta \rangle \cup \{(\xi^{m}(J_\beta),\omega_\beta + m) \mid m < n \} \in J_\alpha$ as $J_\alpha$ is rud closed.

Lemma 26
$\langle J_v \mid v < \alpha \rangle$ is uniformly $\Sigma^J_1$ for all $\alpha$.

Proof: By an easy induction, $\langle \omega_v \mid v < \alpha \rangle$ is uniformly $\Sigma^J_1$ for all $\alpha$. Since $J_v = S_{\omega_v}$, the result follows by lemma 25.

Lemma 27
There are well-orderings $<_v$ of the $S_v$ such that:
(i) $\nu_1 < \nu_2 \to \nu_1 \subset \nu_2$;
(ii) $<_{\nu+1}$ is an end-extension of $<_v$;
(iii) $\langle <_v \mid v < \omega_\alpha \rangle$ is uniformly $\Sigma^J_1$ for all $\alpha$.

Proof: We use lemma 22. Set $<_\circ = \emptyset$, and by induction:
$<_{\nu+1} = \omega_\circ (<_v, S_v)$
$<_{\lambda} = \bigcup_{\nu \in \lambda} <_v$, if $\lim(\lambda)$.

(i) and (ii) are immediate and (iii) is proved like lemma 25.
Lemma 28
There we well-orderings \(<_J^\alpha\) of the \(J^\alpha\) such that:

(i) \(\alpha_1 < \alpha_2 \Rightarrow <_J^\alpha \subseteq <_J^\alpha_2\);

(ii) \(<_J^\alpha_{+1}\) is an end-extension of \(<_J^\alpha\);

(iii) \(\langle <_J^\beta|\beta < \alpha \rangle\) is uniformly \(\Sigma_1^\alpha\);

(iv) \(<_J^\alpha\) is uniformly \(\Sigma_1^\alpha\);

(v) the function \(pr_\alpha(x) = \{z|z < J^\alpha x\}\) is uniformly \(\Sigma_1^\alpha\).

("pr" stands for "predecessors" of course.)

Proof: Set \(<_J^\omega = <_{\omega\alpha}\) (i)-(iii) are immediate by lemma 27.

For (iv), note simply that \(x <_J y \Leftrightarrow \exists v(x <_v y)\). Finally, for (v), note that \(y = pr_\alpha(x) = \forall v[x \in S_v \wedge y = \{z|z <_v x\}]\) (and that \(<_v \in J^\alpha\), and use lemma 27.

Lemmas 12 and 26 enable us to prove the following extremely powerful result (due in its original form to Gödel, the present version being Jensen's.):

Theorem 29 (Condensation Lemma)
Let \(X <_{\Sigma_1} J^\alpha\). Then for some \(\beta \leq \alpha\), \(X \cong J^\beta\).

Proof: Let \(X <_{\Sigma_1} J^\alpha\). Then by lemma 12, let \(\pi: X \cong W\), where \(W\) is transitive. We prove by induction on \(\alpha\) that \(W = J^\beta\) for \(\beta = \pi''(X \cap \alpha)\).

Assume, therefore, that whenever \(\nu < \alpha\) and \(X^\nu <_{\Sigma_1} J^\nu\), the unique isomorphism \(\pi^\nu\) of \(X^\nu\) onto a transitive set \(W^\nu\) yields \(W^\nu = J^\beta\) for \(\beta = \pi''(X \cap \alpha)\).

Note that, as \(\langle J^\nu|\nu < \alpha \rangle\) is \(\Sigma_1^\alpha\), \(\nu \in X \cap \alpha \Rightarrow J^\nu \in X\).
Claim 1: For all \( v \in X \cap \alpha \), \( \pi(J_v) = J_{\pi(v)} \).

To see this, note first that for \( v \in X \cap \alpha \), \( X \cap J_v \subseteq J_v \). [For, let \( A \in \Sigma_1^v(X \cap J_v) \). Since \( J_v \in X \), \( A \in \Sigma_1^\alpha(X) \). So, if \( A \not\subseteq \emptyset \), then as \( X \not\subseteq \Sigma_1^\alpha \), \( A \cap X \not\subseteq \emptyset \). But \( A \subseteq J_v \), so \( A \cap (X \cap J_v) \not\subseteq \emptyset \).] Hence by induction hypothesis, \( \pi' : X \cap J_v \cong J_{\pi''(X \cap J_v \cap \alpha)} \) for some unique \( \pi' \). But look, \( J_\alpha \) is an \( \epsilon \)-end extension of \( J_v \), so \( \pi \) maps \( X \cap J_v \) isomorphically onto a transitive set also. In other words, \( \pi' = \pi \upharpoonright X \cap J_v \), and \( \pi'' X \cap J_v = J_{\pi''(X \cap J_v \cap \alpha)} \). So, \( \pi(J_v) = \pi''(X \cap J_v) = J_{\pi''(X \cap J_v \cap \alpha)} = J_{\pi(v)} \), by the definition of \( \pi \), as claimed.

For \( v < \alpha \), define \( \text{rud}_X(J_v) = \text{the rud closure of } X \cap (J_v \cup \{J_v\}) \).

Claim 2: \( X = \bigcup_{v \in X \cap \alpha} \text{rud}_X(J_v) \).

To establish this claim, note that as \( X \subseteq \Sigma_1^\alpha \), \( X \) is rud closed, so \( \supseteq \) is obvious. For the converse, let \( x \in X \). Then \( x \in J_\alpha = \bigcup_{v < \alpha} \text{rud}(J_v) \), so for some rud function \( f \), \( f \upharpoonright J_v \)(\( \forall p \in J_v \)(x = f(p, J_v))). But \( X \subseteq \Sigma_1^\alpha \), so \( (\exists v \in X \cap \alpha)(\forall p \in J_v \cap X)(x = f(p, J_v)) \). In other words, \( x \in \bigcup_{v \in X \cap \alpha} \text{rud}_X(J_v) \). Hence claim 2.

Claim 3: For \( v \in X \cap \alpha \), \( \pi'' \text{rud}_X(J_v) = \text{rud}(J_{\pi(v)}) \).

To see this, let \( v \in X \cap \alpha \). Suppose first that \( x \in \text{rud}_X(J_v) \). Then for some rud function \( f \) and some \( p \in J_v \cap X \), \( x = f(p, J_v) \).

By lemma 12 and claim 1, \( \pi(x) = f(\pi(p), J_{\pi(v)}) \). But \( p \in J_v \cap X \) so \( \pi(p) \in J_{\pi(v)} \). Hence \( \pi(x) \in \text{rud}(J_{\pi(v)}) \). This proves \( \subseteq \).

Conversely, suppose \( y \in \text{rud}(J_{\pi(v)}) \). Then \( y \in \text{rud}(\pi(J_v)) \), by claim 1, so for some rud function \( f \) and some \( p \in \pi(J_v) \), \( y = f(p, \pi(J_v)) \). Now, \( \pi(J_v) = \pi''(J_v \cap X) \), so for some \( q \in J_v \cap X \), \( p = \pi(q) \) and we have \( y = f(\pi(q), \pi(J_v)) = \pi(f(q, J_v)) = \pi'' \text{rud}_X(J_v) \).

Hence \( \supseteq \), and claim 3 is proved.
By claims 2 and 3, we have \( W = \pi'X = \pi'(U_{\nu \in X \cap \alpha} \text{rud}_X(J_\nu)) = U_{\nu \in X \cap \alpha} \pi' \text{rud}_X(J_\nu) = U_{\nu \in X \cap \alpha} \text{rud}(J_{\pi(\nu)}) = U_{\eta < \beta} \text{rud}(J_\eta) = J_\beta \), where \( \beta = \pi'(X \cap \alpha) \).

Note. It is easily seen that we may regard the following as part of the statement of theorem 29: If \( Y \subset X \) is transitive, then \( \pi[Y] = \text{id}_Y \). And for \( \nu \in X \cap \alpha \), \( \pi(\nu) \leq \nu \), and for all \( x \in X \), \( \pi(x) \leq J_\alpha x \).

By an argument well known to all set theorists, it is easily shown that \( J = U_{\alpha \in \text{OR}} J_\alpha \) is a model of ZFC. (In fact, setting \( < J = U_{\alpha \in \text{OR}} < J_\alpha \), \( < J \) is a \( J \)-definable well-ordering of the entire class \( J \), so \( J \) satisfies the axiom of choice in a strong way.). Using the condensation lemma, an equally well-known argument shows that \( J \models \text{GCH} \). However, in the next section we will prove (and have already indicated this fact in our preamble) that \( J = L \), so all that the above says is that we can use the Jensen hierarchy in place of the \( L \)-hierarchy in order to establish the classical results on the constructible universe.

§ 5. On The Fine Structure of the Jensen Hierarchy.

As mentioned in the introduction, a theory similar to the one following can be developed for the usual \( L \)-hierarchy, if desired. Central in our discussion will be the concept of a "uniformising function" for a relation, which is a sort of "choice function" for a given relation. Specifically, a function \( r \) is said to uniformise a relation \( R \) iff \( \text{dom}(r) = \text{dom}(R) \) and for all \( \bar{x} \), \( \exists y R(y, \bar{x}) \iff R(r(\bar{x}), \bar{x}) \).
Let $\mathcal{M} = \langle M, A \rangle, n \geq 1$. We say $\mathcal{M}$ is $\Sigma_n$-uniformisable iff every $\Sigma_n(\mathcal{M})$ relation on $M$ is uniformised by a $\Sigma_n(\mathcal{M})$ function.

A few moments reflection will reveal that $\Sigma_n$-uniformisability is a very strong condition to demand of an arbitrary structure $\mathcal{M}$, since in the more obvious cases, the definition of a uniformisary function for a given relation would appear to increase the logical complexity by one or more quantifier switches. However, it will turn out that for all $\alpha$, all $n \geq 1$, $J_\alpha$ is $\Sigma_n$-uniformisable. For $n = 1$, this will be easy to prove, but for $n > 1$, the corresponding argument will only work when $J_\alpha$ is $\Sigma_{n-1}$-admissible, so a more indirect approach will be necessary. We shall outline the approach required after we dispose of some of the more easy results. First, $\Sigma_1$-uniformisability. The $\Sigma_1(J_\alpha)$ well-ordnary of each $J_\alpha$ gives us this with little effort. In fact, we have a much stronger result, of importance in applications of $\Sigma_1$-uniformisability.

Let $F$ be a class of structures $\mathcal{M} = \langle M, A \rangle, n \geq 1$. Say $F$ is uniformly $\Sigma_n$-uniformisable if, whenever $\phi$ is a $\Sigma_n$-formula w.p.f. $\cap \{M | M \in F \}$ such that $\phi^M$ is a relation on $M$ for each $M \in F$, there is a $\Sigma_n$-formula $\psi$ (w.p.f. $\cap \{M | M \in F \}$) such that for each $M \in F$, $\psi^M$ is a function uniformising $\phi^M$.

**Theorem 30**

$\langle J_\alpha, A \rangle$ is $\Sigma_1$-uniformisable. In fact, the class of all $\langle J_\alpha, A \rangle$ is uniformly $\Sigma_1$-uniformisable.

**Proof:** Let $\phi$ be a $\Sigma_1$-formula w.p.f. $J_\alpha$ such that $[\phi(y, z)]_{\langle J_\alpha, A \rangle}$ is a $\Sigma_1$ relation on $J_\alpha$. By contraction of quantifiers, we can, in a uniform way, find a $\Sigma_0$ formu-
Define \( g \) by: 
\[
g(x) \sim \text{the } <_J \text{-least } w \text{ such that } \\
\langle J_{\alpha}, A \rangle \psi((w)_0, (w)_1, \bar{x})
\]
Then \( g \) is (uniformly) \( \Sigma_1(<J_{\alpha}, A>) \), since if 
\[
w = g(x) \sim \langle J_{\alpha}, A \rangle \psi((x)_0, (w)_1, \bar{x})
\]
\[
\land \exists t[ t = \text{pr}_{\alpha}(w) \land (\forall w \in t) \neg \psi((w')_0, (w')_1, \bar{x})]
\]
Set \( r(x) \sim (g(x))_1 \). Then \( r \) is (uniformly) \( \Sigma_1(<J_{\alpha}, A>) \), and clearly uniformises \( \psi(y, \bar{x}) \langle J_{\alpha}, A \rangle \).

Remark. We call the above construction the \textit{canonical} \( \Sigma_1 \)-uniformisation procedure. Observe that if \( R(y, \bar{x}) \) is a \( \Sigma_1(<J_{\alpha}, A>) \) predicate, then the canonical \( \Sigma_1 \)-uniformisation of \( R \) is a function whose \( \Sigma_1(<J_{\alpha}, A>) \) definition involves \textbf{only those parameters} which occur in the definition of \( R \). Let us take a little time off to examine the above construction more closely. Suppose \( R(y, \bar{x}) \) is a given \( \Sigma_1 \) relation, say 
\[
R(y, \bar{x}) \sim \exists z P(z, y, \bar{x}), \text{ where } P \text{ is } \Sigma_0.
\]
To obtain the \( \Sigma_1 \) uniformisation of \( R \), we first obtain a \( \Sigma_1 \) uniformisation of the \( \Sigma_0 \) relation \( \{ \langle w, \bar{x} \rangle | P((w)_0, (w)_1, \bar{x}) \} \), and then simply pick out the requisite component of the result as our required function. And since \( <_J \) is a \( \Sigma_1 \) well order of \( J_{\alpha} \) the result is also \( \Sigma_1 \). However, returning now to the notation of theorem 30, we see that, if we try to extend this procedure to the case \( n > 1 \), we cannot conclude that the function \( g \) is \( \Sigma_n \), the problem being the last conjunct in the explicit definition of \( g \). Let \( \Psi(w, \bar{x}) \) denote the predicate \( \sim \Psi((w)_0, (w)_1, \bar{x}) \langle J_{\alpha}, A \rangle \). For \( n = 1 \), there was no problem, since \( \Psi(w, \bar{x}) \) is \( \Sigma_0 \), so is \( (\forall w \in t)\Psi(w, \bar{x}) \). However, for \( n > 1 \), \( \Psi(w, \bar{x}) \) is \( \Sigma_{n-1} \), and we can only conclude that \( (\forall w \in t)\Psi(w, \bar{x}) \) is \( \Sigma_{n-1} \) if \( \langle J_{\alpha} \rangle \) is \( \Sigma_{n-1} \)-admissible. Otherwise
it is merely \( \Pi_n \) of course, and so the resulting uniformisation of the original \( \Sigma_n \) relation turns out to be \( \Sigma_{n+1} \). So, in order to establish the general \( \Sigma_n \)-uniformisation lemma, it is not altogether unreasonable to try and "reduce" all \( \Sigma_n \) relations on an arbitrary \( J_\alpha \) to \( \Sigma_n \) relations on some \( \Sigma_{n-1} \)-admissible \( J_\beta \), for which we have a \( \Sigma_n \)-uniformisation procedure. In practice, it will turn out that this hint is slightly off target, but in its general tone it is worth bearing in mind.

Closely connected with \( \Sigma_n \)-uniformisability is the notion of a "\( \Sigma_n \) skolem function".

Let \( \mathcal{M} = (\mathcal{M}, A) \) be transitive and rud closed. By a \( \Sigma_n \) skolem function for \( \mathcal{M} \) we mean a \( \Sigma_n(\mathcal{M}) \) function \( h \) with \( \text{dom}(h) \subseteq \omega \times \mathcal{M} \), such that for some \( p \in M \), \( h \) is \( \Sigma_n(\mathcal{M}, \{p\}) \), and whenever \( P \in \Sigma_n^\mathcal{M}(\{x,p\}) \) for some \( x \in \mathcal{M} \), then \( \forall yP(y) \rightarrow (\exists i \in \omega)P(h(i,x)) \).

(With \( h, p \) as above, we say that \( p \) is a good parameter for \( h \).)

Note that \( \Sigma_n \) skolem functions need not be (and in general are not) total! As far as existence of \( \Sigma_n \) skolem functions is concerned, we can get away with slightly less than might first appear.

In fact:

**Lemma 31**

Let \( \mathcal{M} = (\mathcal{M}, A) \) be transitive and rud closed. Let \( h \) be a \( \Sigma_n(\mathcal{M}, \{p\}) \) function with \( \text{dom}(h) \subseteq \omega \times \mathcal{M} \). Suppose that whenever \( P \in \Sigma_n^\mathcal{M}(\{x\}) \) for some \( x \in \mathcal{M} \), then \( \forall yP(y) \rightarrow (\exists i \in \omega)P(h(i,x)) \).

Then \( \mathcal{M} \) has a \( \Sigma_n \) skolem function.

**Proof:** Set \( \tilde{h}(i,x) \equiv h(i,\langle x,p \rangle) \). It is easily seen that \( \tilde{h} \) is a \( \Sigma_n \) skolem function for \( \mathcal{M} \).
Note that in the above, if $h$ is actually $\Sigma^M_n$, then $\tilde{h} = h$. This is used in establishing the following result:

**Lemma 32**

If $\langle J_\alpha, A \rangle$ is amenable, then it has a $\Sigma_1$ skolem function. In fact, there is a $\Sigma_1$ skolem function $h_{\alpha, A}$ for $\langle J_\alpha, A \rangle$ which is uniformly $\Sigma_1 \langle J_\alpha, A \rangle$ for all amenable $\langle J_\alpha, A \rangle$.

**Proof:** Let $\langle \varphi_i | i < \omega \rangle$ be a recursive enumeration of $\text{Fml}^{\Sigma_1}$.

Let $\langle J_\alpha, A \rangle$ be amenable. By lemma 9, $\models_{\langle J_\alpha, A \rangle}$ is (uniformly) $\Sigma_1 \langle J_\alpha, A \rangle$. Let $h = h_{\alpha, A}$ be the canonical $\Sigma_1$-uniformisation of the $\Sigma_1 \langle J_\alpha, A \rangle$ relation $\{ \langle y, i, x \rangle | \models_{\langle J_\alpha, A \rangle} \varphi_i[y, x] \}$. (By lemma 30 and the ensuing remark, $h$ is thus uniformly $\Sigma_1 \langle J_\alpha, A \rangle$ for amenable $\langle J_\alpha, A \rangle$.) By the remark following lemma 31, it is clear that $h$ is a $\Sigma_1$ skolem function for $\langle J_\alpha, A \rangle$.

We refer to $h_{\alpha, A}$ as the _canonical_ $\Sigma_1$ skolem function for (amenable) $\langle J_\alpha, A \rangle$.

By a similar argument, we have:

**Lemma 33**

If $\langle J_\alpha, A \rangle$ is amenable and $\Sigma_n$-uniformisable, it has a $\Sigma_n$ skolem function.

The following lemmas indicate our reason for using the word "skolem" here.

**Lemma 34**

Let $M$ be transitive and $\text{rud}$ closed, and let $h$ be a $\Sigma_n$ skolem function for $M$. Then whenever $x \in M$, $x \in h''(\omega \times \{x\}) \not\subseteq \Sigma_n M$. 
Proof: Set $X = h''(w \times \{x\})$. Clearly, $x \in X$. Let $P \in \Sigma^M_n(X)$, $P \neq \emptyset$. We must show that $P \cap X \neq \emptyset$. Let $p$ be a good parameter for $h$, and pick $y_1, \ldots, y_m \in X$ with $P \in \Sigma^{M}_n(\{y_1, \ldots, y_m\})$. By definition of $X$, there are $j_1, \ldots, j_m \in \omega$ such that $y_1 = h(j_1, x), \ldots, y_m = h(j_m, x)$. Since $h$ is $\Sigma^M_n(\{p\})$, it follows that $P \in \Sigma^M_n(\{p, x\})$. Hence, $P \neq \emptyset \to \exists y P(y) \to (\exists i \in \omega)P(h(i, x)) \to (\exists y \in X)P(y)$.

Lemma 35

Let $\mathcal{M}$ be transitive and rud closed, and let $h$ be a $\Sigma_n$ skolem function for $\mathcal{M}$. If $X \subseteq M$ is closed under ordered pairs, then $X \subseteq h''(w \times X) \subseteq \Sigma_n \mathcal{M}$.

Proof: Set $Y = h''(w \times X)$. By lemma 34, $X \subseteq Y$. Let $P \in \Sigma^M_n(Y)$, $P \neq \emptyset$. We must show that $P \cap Y \neq \emptyset$. Let $p$ be a good parameter for $h$, and pick $y_1, \ldots, y_m \in Y$ with $P \in \Sigma^M_n(\{y_1, \ldots, y_m\})$. Pick $j_1, \ldots, j_m \in \omega$ and $x_1, \ldots, x_m \in X$ such that $y_1 = h(j_1, x_1), \ldots, y_m = h(j_m, x_m)$. Let $x = \langle x_1, \ldots, x_m \rangle$. By assumption, $x \in X$. But clearly, as $h$ is $\Sigma^M_n(\{p\})$, $P$ is then $\Sigma^M_n(\{p, x\})$, so $P \neq \emptyset \to \exists y P(y) \to (\exists i \in \omega)P(h(i, x)) \to (\exists y \in Y)P(y)$.

Corollary 36

Let $\mathcal{M}$, $h$ be as above. Let $X \subseteq M$ and suppose $h''(w \times X)$ is closed under ordered pairs. Then $X \subseteq h''(w \times X) \subseteq \Sigma_n \mathcal{M}$.

Proof: Let $Y = h''(w \times X)$. Clearly, $Y = h''(w \times Y)$, so the result follows by the lemma.

Lemma 37 (Gödel)

There is a bijection $\xi : \text{OR}^2 \leftrightarrow \text{OR}$ such that $\xi(\alpha, \beta) \geq \alpha, \beta$ for all $\alpha, \beta$, and $\xi^{-1} \Gamma \alpha$ is uniformly $\Sigma^J_\alpha$ for all $\alpha$. 
Proof: Define a well-order $<^*$ of OR$^2$ by
\[(\alpha, \beta) <^*(\gamma, \delta) \iff \max(\alpha, \beta) < \max(\gamma, \delta) \lor \max(\alpha, \beta) = \max(\gamma, \delta) \land (\alpha < \gamma \lor (\alpha = \gamma \land \beta < \delta)).\]

Let $\prec : <^* \cong \text{OR}$. By induction on $\alpha$, $\prec^{-1}\upharpoonright w_\alpha$ is $\Sigma^+_1$$\alpha$
(uniformly).

Lemma 38
There is a $\Sigma_1(J_\alpha)$ map of $w_\alpha$ onto $(w_\alpha)^2$ for all $\alpha$.

Proof: Let $Q = \{\alpha | \tilde{\delta}(0,\alpha) = \alpha\}$. Thus $\tilde{\delta}$ is closed and unbounded in OR. Clearly, $Q = \{\alpha | \tilde{\delta}(\alpha^2 - \alpha)\}$, so $w_\alpha \in Q \Rightarrow w_\alpha = \alpha$.

We prove the lemma by induction on $\alpha$. Assume it is true for all $\nu < \alpha$.

Case 1: $w_\alpha \in Q$. Then $\tilde{\delta}^{-1}\upharpoonright \alpha$ suffices.

Case 2: $\alpha = \beta + 1$. If $\beta = 0$, then $w_\alpha = w \in Q$, so we are done by Case 1. Hence we may assume $\beta \geq 1$. Then clearly, there is a $\Sigma_1(J_\alpha)$ map $j : w_\alpha \rightarrow w_\beta$. By hypothesis, there is a $\Sigma_1(J_\beta)$ map of $w_\beta$ onto $(w_\beta)^2$, so there is certainly a $\Sigma_1(J_\beta)$ map $g$ of $(w_\beta)^2$ one-one into $w_\beta$. Then $g \in \text{rud}(J_\beta) = J_\alpha$, so for $\nu, \tau \in w_\alpha$, define

$$f(\langle \nu, \tau \rangle) = g(\langle j(\nu), j(\tau) \rangle).$$

Then $f$ is $\Sigma_1(J_\alpha)$ and $f$ maps $(w_\alpha)^2$ one-one into $w_\beta$. Now $\text{ran}(f) = \text{ran}(y) \in J_\alpha$, so if we define $h$ by (for $\nu \in w_\alpha$)

$$h(\nu) = \begin{cases} f^{-1}(\nu) & \text{if } \nu \in \text{ran}(f) \\ \langle 0, 0 \rangle & \text{otherwise} \end{cases}$$

we see that $h$ is $\Sigma_1(J_\alpha)$ and $h : w_\alpha \leftrightarrow (w_\alpha)^2$.

Case 3: $\lim(\alpha) \wedge w_\alpha \notin Q$. In this case let $\langle \nu, \tau \rangle = \tilde{\delta}^{-1}(w_\alpha)$.
Thus \( \nu, \tau < \omega \alpha \). Set \( c = \{ z \mid z \in \langle \nu, \tau \rangle \} (\in J_\alpha) \). Thus \( \xi \upharpoonright c \) maps \( c \) one-one onto \( \omega \alpha \), and is \( \Sigma_1(J_\alpha) \). Pick \( \gamma < \alpha \) with \( \nu, \tau < \omega \gamma \). Then \( \xi^{-1} \upharpoonright \omega \alpha \) is a \( \Sigma_1(J_\alpha) \) map of \( \omega \alpha \) one-one into \( \omega \gamma \). And by assumption, there is a map \( g \in J_\alpha \) mapping \( (\omega \gamma)^2 \) one-one into \( \omega \gamma \). So, setting \( f(\langle \iota, \theta \rangle) = g(\langle g(\xi^{-1}(\iota)), g(\xi^{-1}(\theta)) \rangle) \), \( \iota, \theta < \omega \alpha \), we see that \( f \) is a \( \Sigma_1(J_\alpha) \) map of \( (\omega \alpha)^2 \) one-one into \( d = g''(g''c)^2 \). But \( d \in J_\alpha \), so we can define a \( \Sigma_1(J_\alpha) \) map \( h \) on \( \omega \alpha \) by

\[
h(\theta) = \begin{cases} f^{-1}(\theta), & \text{if} \ \theta \in d \\ \langle 0, 0 \rangle, & \text{otherwise.} \end{cases}
\]

Clearly, \( h = \omega \alpha^{\text{onto}} (\omega \alpha)^2 \).

The lemma is proved.

Using this lemma, we may now establish the following important result:

**Theorem 39**

There is a \( \Sigma_1(J_\alpha) \) map of \( \omega \alpha \) onto \( J_\alpha \) for all \( \alpha \).

**Proof:** Let \( f : \omega \alpha \to (\omega \alpha)^2 \) be \( \Sigma_1^J([p]) \), where \( p \in J_\alpha \) is the \( <_J \)-least element of \( J_\alpha \) for which such an \( f \) exists. Define \( f^0, f^1 \) by demanding that \( f(\nu) = \langle f^0(\nu), f^1(\nu) \rangle \) for all \( \nu \in \omega \alpha \). By induction, define \( f_n : \omega \alpha \to (\omega \alpha)^n \) thus: \( f_0 = \text{id} \upharpoonright \omega \alpha ; f_{n+1}(\nu) = \langle f^0(\nu), f_n \circ f^1(\nu) \rangle \). Hence each \( f_n \) is \( \Sigma_1^J([p]) \). Let \( h = h_\alpha \), the canonical \( \Sigma_1 \) skolem function for \( J_\alpha \). Set \( X = h''(\omega \times (\omega \alpha \times [p])) \).

**Claim 1:** \( X \) is closed under ordered pairs.
To see this, let \( y_1, y_2 \in X \), say \( y_1 = h(j_1, \langle v_1, p \rangle) \), \( y_2 = h(j_2, \langle v_2, p \rangle) \). Let \( \langle v_1, v_2 \rangle = f_2(\tau) \). Then \( \{\langle y_1, y_2 \rangle\} \) is a \( \Sigma^0_1 \{\{\tau, p\}\} \) predicate, so by definition of \( h \), \( \langle y_1, y_2 \rangle \in X \), as claimed.

So by corollary 36, \( X < \Sigma^1_\alpha \). By the condensation lemma, let \( \pi : X \cong J_\beta, \beta \leq \alpha \). Since \( \omega \alpha \subset X \), we clearly have \( \beta = \alpha \) here.

**Claim 2:** For all \( i \in \omega \), \( x \in X \), \( \pi(h(i, x)) \simeq h(i, \pi(x)) \).

To see this, observe first that as \( h \) is \( \Sigma^0_1 \alpha \), there is a rud function \( H \) such that \( y = h(i, x) \iff (\forall t \in J_\alpha)[H(t, i, x, y) = 1] \).

Now let \( i \in \omega \), \( x \in X \). Since \( X < \Sigma^1_\alpha \), \( y = h(i, x) \in X \) (if defined). Thus, by the above, since \( x, y \in X < \Sigma^1_\alpha \),

\[
(\forall t \in X)[H(t, i, x, y) = 1].
\]

By lemma 12, therefore,

\[
(\forall t \in X)[H(\pi(t), i, \pi(x), \pi(y)) = 1].
\]

Since \( \pi''X = J_\alpha \), this can be rewritten as

\[
(\forall t \in J_\alpha)[H(t, i, \pi(x), \pi(y)) = 1].
\]

Thus \( \pi(y) = h(i, \pi(x)) \), as claimed.

Now, \( f \subset (\omega \alpha)^3 \), so as \( \pi \upharpoonright \omega \alpha = \text{id} \upharpoonright \omega \alpha \), \( \pi''f = f \). And by isomorphism, \( \pi''f \) is \( \Sigma^0_1 \alpha(\{\pi(p)\}) \). So as \( \pi(p) \leq J_\alpha \), the choice of \( p \) shows that \( \pi(p) = p \).

So, by claim 2, if \( i \in \omega \), \( v \in \omega \alpha \), \( \pi(h(i, \langle v, p \rangle)) \simeq h(i, \langle v, p \rangle) \),

which is to say \( \pi \upharpoonright X = \text{id} \upharpoonright X \). Thus \( X = J_\alpha \).

Now define \( \tilde{h} : (\omega \alpha)^3 \to J_\alpha \) by setting

\[
\tilde{h}(i, v, \tau) = \begin{cases} y, & \text{if } (\forall t \in J_\tau)[H(t, i, \langle v, p \rangle, y) = 1] \\ \emptyset, & \text{otherwise}. \end{cases}
\]

Then \( \tilde{h} \) is \( \Sigma^1_\alpha(J_\alpha) \), and clearly \( \tilde{h}''(\omega \alpha)^3 = h''(x \times (\omega \alpha \times \{p\})) = X = J_\alpha \). Therefore, \( \tilde{h} \circ f_3 \) is as required by the theorem.

Observe that in lemmas 38, 39, the maps constructed generally have
parameters in their definitions. Note also that, being total, these maps are in fact $\Delta_1(\beta)$. Recalling the results of § 3, we now investigate those ordinals $\alpha$ for which $J_\alpha$ is an admissible set.

Let us call an ordinal $\alpha$ admissible iff $\alpha = \omega \beta$ and $J_\beta$ is an admissible set.

**Theorem 40**

$\omega \alpha$ is admissible iff there is no $\Sigma_1(J_\alpha)$ map of any $\gamma = \omega \alpha$ cofinally into $\omega \alpha$. (Note that such a map, having domain $\gamma \in J_\alpha$, would in fact be $\Delta_1(J_\alpha)$.)

**Proof:** ($\Rightarrow$). Let $\gamma < \omega \alpha$ and suppose $f : \gamma \to \omega \alpha$ is $\Sigma_1(J_\alpha)$. Then $(\forall \xi \in \gamma)(\exists \zeta \in \omega \alpha)(f(\xi) = \zeta)$. If $J_\alpha$ is admissible, then by $\Sigma_1$-Replacement, $(\exists \eta \in \omega \alpha)(\forall \xi \in \gamma)(\exists \zeta \in \eta)(f(\xi) = \zeta)$, so $f$ is not cofinal in $\omega \alpha$.

($\Leftarrow$) Assume $\omega \alpha$ is not admissible. If $\alpha = \beta + 1$, then the $\Sigma_1(J_\alpha)$ map $\langle (\omega \beta + n, n) \mid n \in \omega \rangle$ map $\omega$ cofinally into $\omega \alpha$, so we are done. Assume then that $\lim(\alpha)$. Since $J_\alpha$ is not admissible, there must be a $\Sigma_1(J_\alpha)$ relation $R$ and a $u \in J_\alpha$ such that $(\forall x \in u)(\exists y \in R(x, y))$ but for all $z \in J_\alpha$, $\neg (\forall x \in u)(\exists y \in z)R(x, y)$. Take $\gamma < \alpha$ with $u \in J_\gamma$. By Theorem 39, let $f$ be a $\Sigma_1(J_\gamma)$ map of $\omega \gamma$ onto $\beta \gamma$. Thus $f \in J_\gamma$, and $u \in f"\omega \gamma$.

Define $g : \omega \gamma \to \omega \alpha$ by

$$g(\nu) = \begin{cases} \text{the least } \tau \text{ such that } (\exists y \in S_\tau)R(f(\nu), y), & \text{if } f(\nu) \in u \\ 0 & \text{if } f(\nu) \notin u. \end{cases}$$

Then $g$ is a $\Sigma_1(J_\alpha)$ map of $\omega \gamma$ cofinally into $\omega \alpha$. 

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Recalling our discussion at the end of § 3, let us call an ordinal $\alpha$ strongly admissible (or non-projectible admissible) iff $\alpha = \omega \beta$ and $J_{\beta}$ is strongly admissible. Imitating the proof of Theorem 40, we have:

**Theorem 41**

$\omega \alpha$ is strongly admissible iff there is no $\Sigma_1(J_{\alpha})$ map of a bounded subset of $\omega \alpha$ cofinally into $\omega \alpha$.

The above two results illustrate our earlier remark concerning the difference between a function being partial and being total, and the corresponding difference between a predicate being $\Sigma_n$ and being $\Delta_n$. The next two results, which strengthen the last two, and are also due to Kripke and Platek, also highlight this distinction.

**Theorem 42**

The following are equivalent:

(i) $\omega \alpha$ is admissible.

(ii) $\langle J_{\alpha} A \rangle$ is amenable for all $A \in \Delta_1(J_{\alpha})$.

(iii) There is no $\Sigma_1(J_{\alpha})$ function mapping a $\gamma < \omega \alpha$ onto $J_{\alpha}$.

(Of course, any such function would in fact be $\Delta_1(J_{\alpha})$.)

**Proof:** (i) $\rightarrow$ (ii). By lemma 15 ($\Delta_1$-Comprehension)

(ii) $\rightarrow$ (iii). Assume (ii) $\land$ (iii). Let $\gamma < \omega \alpha$, and let $f : \gamma \xrightarrow{\text{onto}} J_{\alpha}$ be $\Sigma_1(J_{\alpha})$. Then $f$ is $\Delta_1(J_{\alpha})$, so $d = \{\nu | \nu \in f(\nu)\}$ is $\Delta_1(J_{\alpha})$. Thus by (ii), $d = d \land \gamma \in J_{\alpha}$. So, $d = f(\nu)$ for some $\nu < \gamma$, so $\nu \in f(\nu) \rightarrow \nu \in d \land \forall \nu \notin f(\nu)$, a contradiction.

(iii) $\rightarrow$ (i). Assume (iii) $\land$ (i). If $\alpha = \beta + 1$, we can easily construct a $\Sigma_1(J_{\alpha})$ map of $\omega \beta$ onto $\omega \alpha$, so
Theorem 39 yields the required contradiction. Assume \( \lim(\alpha) \).

By Theorem 40, there must be a \( \tau < \omega \alpha \) and a \( \Sigma_1(J_\alpha) \) map \( f \) of \( \tau \) cofinally into \( \omega \alpha \). Let \( f \) be \( \Sigma_1(J_\alpha) \). Pick \( \gamma < \alpha \) with \( \tau, p \in J_\gamma \). Let \( h = h_\alpha \) be the canonical \( \Sigma_1 \) skolem function for \( J_\alpha \).

Set \( X = h''(\omega \times J_\gamma) \). As \( J_\gamma \) is closed under ordered pairs, lemma 35 tells us that \( X < \Sigma_1 J_\alpha \). Let \( \pi : X \equiv J_\beta \). Thus \( \pi \upharpoonright J_\gamma = \text{id} \upharpoonright J_\gamma \). By an argument as in Theorem 39, \( \pi \upharpoonright X = \text{id} \upharpoonright X \), so \( X = J_\beta \). Now, \( f \) is \( \Sigma_1(J_\alpha) \) and \( p \in X < \Sigma_1 J_\alpha \), so \( X \) is closed under \( f \). But \( \tau \subset X \) and so \( f'' \tau \subset X \), which means, since \( f'' \tau \) is cofinal in \( \omega \alpha \) and \( X = J_\beta \) is transitive, that \( \omega \alpha \subset J_\beta \).

Thus \( \beta = \alpha \), and \( X = J_\alpha \). Define a \( \Sigma_1(J_\alpha) \) map \( \tilde{h} : \omega \times \tau \times J_\gamma \to J_\alpha \) as follows. Let \( H \) be a \( \Sigma_0 \) relation such that \( y = h(i, x) \to (\exists t \in J_\alpha)H(t, i, x, y) \).

Set \( \tilde{h}(i, \nu, x) = \begin{cases} y, & \text{if } (\exists t \in S_f(\nu))H(t, i, x, y) \\ \emptyset, & \text{otherwise.} \end{cases} \)

Then \( \tilde{h} \) is total on \( \omega \times \tau \times J_\gamma \), and \( \tilde{h}''(\omega \times \tau \times \{x\}) = h''(\omega \times \{x\}) \) for any \( x \), as \( f'' \tau \) is cofinal in \( \omega \alpha \). Hence \( \tilde{h}''(\omega \times \tau \times J_\gamma) = X = J_\alpha \).

By Theorem 39 there is \( g \in J_\alpha \), \( g : \omega \gamma \to \omega \times \tau \times J_\gamma \). Then \( \tilde{h} \circ g \) is a \( \Sigma_1(J_\alpha) \) map of \( \omega \gamma \) onto \( J_\alpha \), contrary to (iii).

**Theorem 43**

The following are equivalent:

(i) \( \omega \alpha \) is strongly admissible.

(ii) \( \langle J_\alpha, A \rangle \) is amenable for all \( A \in \Sigma_1(J_\alpha) \).

(iii) There is no \( \Sigma_1(J_\alpha) \) function mapping a bounded subset of \( \omega \alpha \) onto \( J_\alpha \).

**Proof:** (i) \to (ii) \to (iii). Similar to the above.
(iii) → (i). Assume (iii) ∩ (i) and proceed much as before. So, we assume \( \lim(\alpha) \), \( f \) is (by Theorem 41) a \( \Sigma_1^1(\alpha) \) map of some \( a \subset \tau < \omega_\alpha \) cofinally into \( \omega_\alpha \), \( f \in \Sigma_1^1(\{p\}) \), and \( \tau < \omega_\gamma \), \( p \in J_\gamma \), \( \gamma < \alpha \). As before, if \( h = h_\alpha \) and \( X = h''(\omega \times J_\gamma) \), then \( X = J_\alpha \). Now, since we do not need to bother about functions being total, we can easily contradict (iii). By Theorem 39, let \( g \in J_\alpha \), \( g : \omega_\gamma \rightarrow \omega \times J_\gamma \). Set \( \overline{f}(\nu) \approx h(g(\nu)) \). Then \( \overline{f} \) is a \( \Sigma_1^1(\alpha) \) map of a subset of \( \omega_\gamma \) onto \( J_\alpha \).

Note that an immediate corollary of Theorem 42 is:

**Theorem 44**

If \( \kappa \) is a cardinal, then \( \kappa \) is an admissible ordinal.

Using admissibility theory, we can give a quick proof that \( L = \bigcup_{\alpha \in \text{OR}} J_\alpha \).

**Theorem 45**

If \( \omega \alpha \) is admissible, then \( J_\alpha = L_{\omega \alpha} \).

**Proof:** If \( \alpha = 1 \), then \( J_1 = L_\omega \) is the hereditarily finite sets. Assume \( \alpha > 1 \). Thus \( \omega \in J_\alpha \). Since \( J_\alpha \) is admissible, the recursion theorem tells us that \( \text{rud}(x) = U_{\eta < \omega} \Sigma^1_\eta(x) \) is \( \Sigma_1^1(\alpha) \). But if \( u \) is transitive, then \( \Sigma_{\omega}(u) = \mathcal{P}(u) \cap \text{rud}(u) \). Hence the map \( L_\gamma \mapsto \Sigma_{\omega}(L_\gamma) = L_{\gamma + 1} \) is \( \Sigma_1^1(\alpha) \) (\( \gamma < \omega \alpha \)). So, by the recursion theorem again, we see that \( \langle L_\gamma | \nu < \omega \alpha \rangle \) is \( \Sigma_1^1(\alpha) \). Hence \( L_{\omega \alpha} = U_{\nu < \omega \alpha} L_{\nu} \subset J_\alpha \).

For the converse inclusion, it suffices to show that \( L_{\omega \alpha} \) is admissible. (For then, by the recursion theorem, \( \langle S_\nu | \nu < \omega \alpha \rangle \) is \( \Sigma_1(\omega_\alpha) \), so \( J_\alpha = U_{\nu < \omega \alpha} S_\nu \subset L_{\omega \alpha} \).)
Let $R$ be $\Sigma_0(L_{\omega_1})$, $x \in L_{\omega_1}$, and assume $(\forall z \in x)(\exists y R(y, z))$. Since $\langle L_\nu \mid \nu < \omega_1 \rangle$ is $\Sigma_1(J_\alpha)$, we may define a $\Sigma_1(J_\alpha)$ predicate $R'$ by $R'(\nu, z) \iff z \in x \land (\exists y \in L_\nu) R(y, z)$. Since $J_\alpha$ is admissible, there is $\tau < \omega_1$ with $(\forall z \in x)(\exists y < \tau) R'(\nu, z)$. Hence $(\forall z \in x)(\exists y \in L_\tau) R(y, z)$. So as $L_\tau \in L_{\omega_1}$, $L_{\omega_1}$ satisfies the $\Sigma_0$-Replacement axiom.

Let $\alpha, n \geq 0$. The $\Sigma_n$-projectum of $\alpha$, $\rho_n^\alpha$, is the largest $\rho \leq \alpha$ such that $\langle J_\rho, A \rangle$ is amenable for all $A \in \Sigma_n(J_\alpha) \cap \wp(J_\rho)$. Roughly speaking, our reason for introducing the $\Sigma_n$ projectum is this. We have seen that, for example, we can reasonably handle $\Sigma_n(J_\alpha)$ predicates when $J_\alpha$ is $\Sigma_n$-admissible. This is because $\Sigma_n$-admissibility is a sort of "hardness" condition on $J_\alpha$ for $\Sigma_n$ predicates. For, if we take an arbitrary $J_\alpha$, it may be "soft" for $\Sigma_n(J_\alpha)$ predicates; we may, for instance, find $\Sigma_n(J_\alpha)$ subsets of members of $J_\alpha$ which are not themselves members of $J_\alpha$, or even $\Sigma_n(J_\alpha)$ functions which project a subset of a member of $J_\alpha$ onto all of $J_\alpha$. But if $J_\alpha$ is $\Sigma_n$-admissible, none of these situations can arise. Thus, we try to isolate that part of $J_\alpha$ which is "hard" for $\Sigma_n(J_\alpha)$ predicates, a sort of "$\Sigma_n$-admissible core" of $J_\alpha$. One natural way of formalising these ideas is provided by the $\Sigma_n$ projectum. Clearly, $J_\rho_n^\alpha$ is a reasonable interpretation of the notion of a "$\Sigma_n$-hard core" of $J_\alpha$. We shall eventually give two characterisations of the $\Sigma_n$-projectum which make it appear even more reasonable - if not inevitable. One of these is that $\rho_n^\alpha$ is the smallest $\rho \leq \alpha$ for which there is a $\Sigma_n(J_\alpha)$ map of a subset of $\omega_\rho$ onto $J_\alpha$. Then, since we clearly have, for $\omega_1$ admissible, that $\omega_1$ is strongly admissible.
iff \( \rho^1_\alpha = \alpha \), we obtain some justification for our alternative name of "non-projectible admissible" for strong admissibility.

It is convenient, at this point, for us to define an obvious generalisation of the notion of the \( \Sigma_n \)-projectum of an ordinal.

Let \( \langle J_\alpha, A \rangle \) be amenable. The \( \Sigma_n \)-projectum of \( \langle J_\alpha, A \rangle \), \( \rho^n_\alpha \), is the largest \( \rho \leq \alpha \) such that \( \langle J_\rho, B \rangle \) is amenable for all \( B \in \Sigma_n(\langle J_\alpha, A \rangle) \).

Note that by Theorem 43, \( \omega^n_\alpha \) is always strongly admissible.

We shall make strong use of the \( \Sigma_n \)-projectum in proving that every \( J_\alpha \) is \( \Sigma_n \)-uniformisable, all \( n \geq 1 \). Since most of the following lemmas are directed towards this goal, it is worth indicating briefly our strategy.

We already know that \( \langle J_\delta, A \rangle \) is \( \Sigma_1 \)-uniformisable for all \( \langle J_\delta, A \rangle \). What we shall do is attempt to "reduce" \( \Sigma_n(J_\alpha) \) predicates to \( \Sigma_1(\langle J_\rho_\alpha, A \rangle) \) predicates for some \( A \subset J_\rho_\alpha \) which is itself \( \Sigma_n(J_\alpha) \). To carry out this reduction, we need to have at our disposal a \( \Sigma_n(J_\alpha) \) map of a subset (at least) of \( J_\rho_\alpha \) onto \( J_\alpha \). Thus, what we shall do is to simultaneously prove, by induction on \( n, \alpha \), the following two propositions:

(P 1) \( J_\alpha \) is \( \Sigma_{n+1} \)-uniformisable

(P 2) There is a \( \Sigma_n(J_\alpha) \) map of a subset of \( \omega^n_\alpha \) onto \( J_\alpha \).

The proof of (P 1) goes roughly as follows. Let \( R \) be a \( \Sigma_{n+1}(J_\alpha) \) predicate on \( J_\alpha \). Let \( f : \subset \omega^n_\alpha \overset{onto}{\longrightarrow} J_\alpha \) be \( \Sigma_n(J_\alpha) \). Now, \( f^{-1} \) is a \( \Sigma_n(J_\alpha) \) relation, so by assuming \( \Sigma_n \)-uniformisability, \( f^{-1} \) can be "shrunk" to a \( \Sigma_n(J_\alpha) \) map of \( J_\alpha \) into \( \omega^n_\alpha \). This reduces \( R \) to a \( \Sigma_{n+1}(J_\alpha) \) predicate \( R' \) on \( J_\rho_\alpha \).
Now find a $\Sigma_n(J_\alpha)$ predicate $A \subset J_\rho^\alpha$ such that $R'$ is in fact $\Sigma_1(\langle J_\rho^\alpha, A \rangle)$. Uniformise $R'$ by a $\Sigma_1(\langle J_\rho^\alpha, A \rangle)$ function, and then reverse the procedure to recover a $\Sigma_{n+1}(J_\alpha)$ uniformising function for $R$. There is one doubtful point in the above outline. Can we in fact find a set $A$ as required. That we can has to be proved as we proceed, so we shall in fact simultaneously prove three propositions, ($P 1$), ($P 2$), and a proposition ($P 3$) to be formulated precisely later.

**Lemma 46**

Let $n \geq 1$, and assume $J_\alpha$ is $\Sigma_n$-uniformisable. Let $\gamma \leq \alpha$ be the least ordinal such that $\Theta(\psi \gamma) \cap \Sigma_n(J_\alpha) \not\subset J_\alpha$. Then there is a $\Sigma_n(J_\alpha)$ map of a subset of $\psi \gamma$ onto $J_\alpha$.

**Proof:** By lemma 33, $J_\alpha$ has a $\Sigma_n$ skolem function, $h$. Let $h$ be $\Sigma_n^\alpha([p])$. We may assume $p$ is the $J_\gamma$-least element of $J_\alpha$ for which such an $h$ exists.

Let $a \subset \psi \gamma$, $a \in \Sigma_n(J_\alpha)$, $a \not\subset J_\alpha$. Let $q$ be the $J_\gamma$-least element of $J_\alpha$ such that $a \in \Sigma_n^\alpha([q])$. Define $\tilde{h}$ by $\tilde{h}(i, x) \approx h(i, \langle x, p, q \rangle)$. It is easily seen that $\tilde{h}$ is a $\Sigma_n$ skolem function for $J_\alpha$ and that $\langle p, q \rangle$ is a good parameter for $\tilde{h}$.

Set $X = \tilde{h}''(\psi \times J_\gamma)$. Now, there is a $\Sigma_1(J_\gamma)$ map $g : \psi \gamma \overset{onto}{\rightarrow} J_\gamma$, so $\tilde{h} \circ g$ is a $\Sigma_n(J_\alpha)$ map of a subset of $\psi \gamma$ onto $X$. Hence it suffices to show that $X = J_\alpha$.

Clearly, $X \prec \Sigma_n J_\alpha$. Let $\pi : X \cong J_\beta$, $\beta \leq \alpha$. Then $\pi \upharpoonright J_\gamma = \text{id} \upharpoonright J_\gamma$, so in particular, $\pi''a = a$. Also $\pi''a$ is $\Sigma_n^\beta([\pi(q)])$. But look, this implies that $a = \pi''a \in J_{\beta+1}$. Hence we must have $\beta = \alpha$ (and here we have used our hypothesis that $\Theta(\psi \gamma) \cap \Sigma_n(J_\alpha) \not\subset J_\alpha$). Thus, in particular, $a = \pi''a$ is
\[ \Sigma^J_n([\pi(q)]) \], so by the choice of \( q \), we see that \( \pi(q) = q \). Again, it is easy to see that \( h' = \pi \circ h \circ \pi^{-1} \) is a \( \Sigma^J_n([\pi(p)]) \) \( \Sigma_n \) skolem function for \( J_\alpha \), so by choice of \( p \), \( \pi(p) = p \). But then \( h, h' \) are both defined by the same \( \Sigma_n \) formula (with parameter \( p \)) in \( J_\alpha \), so \( h = h' \). It follows immediately that \( \pi \circ \tilde{h} \circ \pi^{-1} = \tilde{h} \), of course. So for \( i \in \omega, x \in J_\gamma, \pi \circ \tilde{h}(i,x) \rightleftharpoons \tilde{h} \circ \pi(i,x) \rightleftharpoons \tilde{h}(i,x) \). Thus \( \pi \upharpoonright X = \text{id} \upharpoonright X \), and \( X = J_\alpha \).

Lemma 46 plays a direct part in the proof of (P1)-(P3). The next lemma, however, is only used during the proof of the lemma which follows it, and may, on first sight, appear somewhat uninspiring.

**Lemma 47**

Let \( \langle J_\alpha, A \rangle \) be amenable, \( p = p_1^\alpha \). If \( B \in J_\rho \) is \( \Sigma_1(\langle J_\alpha, A \rangle) \), then \( \Sigma_1(\langle J_\rho, B \rangle) \subseteq \Sigma_2(\langle J_\alpha, A \rangle) \).

**Proof:** Case 1. There is a \( \Sigma_1(\langle J_\alpha, A \rangle) \) map of some \( \gamma < \omega \), cofinally into \( w_\alpha \).

Let \( g \) be such a map, and let \( \overline{B} \) be \( \Sigma_0(\langle J_\alpha, A \rangle) \) such that \( B(x) \rightarrow \exists z \overline{B}(z,x) \) for each \( x \in J_\rho \). Define \( B' \) by \( B'(\langle \nu, x \rangle) \rightarrow (\exists z \in S_\rho(\nu)) \overline{B}(z,x) \), for \( \nu \in \gamma, x \in J_\rho \). Thus \( B' \) is \( \Delta_1(\langle J_\alpha, A \rangle) \). And since \( B(x) \rightarrow (\exists \nu \in \gamma) B'(\langle \nu, x \rangle) \), \( \Sigma_1(\langle J_\rho, B \rangle) \subseteq \Sigma_1(\langle J_\rho, B' \rangle) \). Thus, we need only prove that \( \Sigma_1(\langle J_\rho, B' \rangle) \subseteq \Sigma_2(\langle J_\alpha, A \rangle) \). It clearly suffices to prove that \( \Sigma_0(\langle J_\rho, B' \rangle) \subseteq \Sigma_2(\langle J_\alpha, A \rangle) \).

Let \( R \) be \( \Sigma_0(\langle J_\rho, B' \rangle) \). Thus \( R \) is rud in \( B' \) and some parameter \( p \in J_\rho \). By choice of \( p \), \( \langle J_\rho, B' \rangle \) is amenable, so by lemmas 3 and 4, there is a \( \Sigma_0(J_\rho) \) predicate \( P \) and functions \( f_1, \ldots, f_{m+k} \), rud in parameter \( p \), such that \( R(\overline{x}) \rightarrow P(\overline{x}, f_1(\overline{x}), \ldots, f_m(\overline{x}), B' \cap f_{m+1}(\overline{x}), \ldots, B' \cap f_{m+k}(\overline{x})) \). Hence
\( R(\vec{x}) \leftarrow \exists y_1, \ldots, \exists y_k[y_1 = B' \cap f_{m+1}(\vec{x}) \wedge \ldots \wedge y_k = B' \cap f_{m+k}(\vec{x}) \wedge P(\vec{x}, f_1(\vec{x}), \ldots, f_m(\vec{x}), y_1, \ldots, y_k)]. \) Now \( P \) is certainly \( \Sigma_0(J_\alpha) \), and \( f_1, \ldots, f_{m+k} \) are rud in parameter \( p \), so it suffices to show that the function \( b(u) = B' \cap u \) is \( \Sigma_2(J_\alpha, A) \). It is in fact \( \Pi_1(J_\alpha, A) \), because: \( y = b(u) \leftarrow \forall x [x \in y \leftarrow x \in u \wedge B'(x)] \), and \( B' \) is \( \Delta_1(J_\alpha, A) \).

**Case 2.** Otherwise.

As before, we must show that \( \Sigma_0(J_\rho, B) \subseteq \Sigma_2(J_\alpha, A) \). Again as before, this reduces, by the amenability of \( J_\rho, B \), to proving that the function \( b(u) = B \cap u \) is \( \Sigma_2(J_\alpha, A) \) on \( J_\rho \). Now, we clearly have

\[
y = b(u) \leftarrow (\forall x \in y)(x \in u \land B(x)) \land (\forall x \in u)(B(x) \rightarrow x \in y).
\]

Now, the second conjunct here is \( \Pi_1(J_\alpha, A) \). We show that the first conjunct is \( \Sigma_1(J_\alpha, A) \), which is sufficient. It reduces to showing that \( (\forall x \in y)B(x) \) is \( \Sigma_1(J_\alpha, A) \). But look, we know that Case 1 fails to hold, so this is proved just as in lemma 14.

The next lemma is the key step involved in proving, by induction, the as yet unformulated (P 3).

**Lemma 48**

Let \( \langle J_\alpha, A \rangle \) be amenable, \( \rho = \rho_{\alpha, A}^1 \). Suppose there is a \( \Sigma_1(J_\alpha, A) \) map of a subset of \( \omega^\rho \) onto \( J_\alpha \). Then there is a \( B \subseteq J_\rho, B \in \Sigma_1(J_\alpha, A) \), such that \( \Sigma_n(J_\rho, B) = \varnothing(J_\rho) \cap \Sigma_{n+1}(J_\alpha, A) \) for all \( n \geq 1 \).

**Proof:** Let \( u \subseteq \omega^\rho \), and let \( f : u \overset{onto}{\rightarrow} J_\alpha \) be \( \Sigma_1(J_\alpha, A) \).

Pick \( p \in J_\alpha \) such that \( f \) is \( \Sigma_1(J_\alpha, A)(\{p\}) \). Let \( \langle \varphi_1 | i < \omega \rangle \) be a recursive enumeration of \( \text{Fml} \Sigma_1^1 \).
\[ B = \{ \langle i, x \rangle \mid i \in w \land x \in J_\rho \land |x| \leq \sigma_1 \langle J_\alpha, A \rangle \circ \omega_1[x, p] \}. \]

Now, \( \langle J_\alpha, A \rangle \) is amenable, and hence rud closed, so by lemma 9, \( B \in \Sigma_1(\langle J_\alpha, A \rangle) \). And of course \( B \subseteq J_\rho \).

Commencing with lemma 47, an easy induction shows that for all \( n \geq 1 \), \( \Sigma_n(\langle J_\rho, B \rangle) \subseteq \Sigma_{n+1}(\langle J_\alpha, A \rangle) \).

For the converse, let \( R(\vec{x}) \) be a \( \Sigma_{n+1}(\langle J_\alpha, A \rangle) \) relation on \( J_\rho \), \( n \geq 1 \). Assume, for the sake of argument, that \( n \) is even. Let \( P \) be a \( \Sigma_1(\langle J_\alpha, A \rangle) \) relation such that, for \( \vec{x} \in J_\rho \),
\[ R(\vec{x}) \iff \exists y_1 y_2 \ldots y_n P(\vec{y}, \vec{x}). \]
Define \( \tilde{P} \) by \( \tilde{P}(\vec{z}, \vec{x}) \iff [\exists \vec{y} \in J_\rho \land P(f(\vec{y}), \vec{x})] \). By choice of \( f \), any \( x \in J_\alpha \) is \( \Sigma_1 \langle J_\alpha, A \rangle(\{p, \nu\}) \)
for some \( \nu < \omega_\nu \), so by definition of \( B \), \( \tilde{P} \) is rud in \( B \) and some parameter \( \nu < \omega_\nu \). In particular, \( \tilde{P} \) is \( \Delta_1(\langle J_\rho, B \rangle) \).

Again, \( D = \text{dom}(f) \) is rud in \( B \) and some parameter \( \tau < \omega_\nu \), so \( D \) is also \( \Delta_1(\langle J_\rho, B \rangle) \).

But for \( \vec{x} \in J_\rho \),
\[ R(\vec{x}) \iff (\forall z_1 \in D)(\forall z_2 \in D) \ldots (\forall z_n \in D)\tilde{P}(\vec{z}, \vec{x}), \]
which is thus \( \Sigma_n(\langle J_\rho, B \rangle) \).

We are now ready to formulate (P 3) and prove our promised uniformisation theorem.

Let \( \alpha, n \geq 0 \). A \( \Sigma_n \)-master code for \( J_\alpha \) is a set \( A \subseteq J_{\rho n} \), \( A \in \Sigma_n(J_\alpha) \), such that whenever \( m \geq 1 \), \( \Sigma_m(\langle J_{\rho n}, A \rangle) = \emptyset(J_{\rho n}) \cap \Sigma_{n+m}(J_\alpha) \).

**Theorem 49**

Let \( \alpha, n \geq 0 \). Then:

(P 1) \( J_\alpha \) is \( \Sigma_{n+1} \)-uniformisable.

(P 2) There is a \( \Sigma_n(J_\alpha) \) map of a subset of \( \omega^{n}_{\alpha} \) onto \( J_\alpha \).

(P 3) \( J_\alpha \) has a \( \Sigma_n \) master code.
Proof: We prove the theorem (for all $n$) by induction on $\alpha$.

For $\alpha = 0$, it is trivial. So assume $\alpha > 0$ and that
$(P\ 1)$-$(P\ 2)$ hold (for all $n$) for all $\beta < \alpha$. We prove
$(P\ 1)$-$(P\ 3)$ at $\alpha$ by induction on $n$.

Case 1: $n = 0$. $(P\ 1)$ is already proved (Theorem 30)

$(P\ 2)$ $\rho^o_\alpha = \alpha$, so $(P\ 2)$ is already proved (Theorem 39)

$(P\ 3)$ Since $\rho^o_\alpha = \alpha$, $\lambda = \emptyset$ is a $\Sigma_\alpha$ master
code for $J_\alpha$.

Case 2: $n = m + 1$, $m \geq 0$. Let $\rho = \rho^m_\alpha$ for convenience.

We first prove that $\rho$ is the least ordinal such that some $\Sigma_n(J_\alpha)$
function maps a subset of $\omega \rho$ onto $J_\alpha$.

To this end, let $\delta$ be the least such ordinal. Suppose first
that $\delta < \rho$. Then $B = \{ \xi \in \omega \delta \mid \xi \notin f(\xi) \}$ is a $\Sigma_n(J_\alpha)$ subset of
$J_\rho$, so by definition of $\rho$, $\langle J_\rho, B \rangle$ is amenable. Thus, as $\delta < \rho$,
$B = B \cap \omega \delta \in J_\rho \subset J_\alpha$. So $B = f(\xi)$ for some $\xi \in \omega \delta$, whence
$\xi \in f(\xi) \rightarrow \xi \in B \rightarrow \xi \notin f(\xi)$, which is absurd. Hence $\rho \leq \delta$.

Suppose $\rho < \delta$. By definition of $\rho$, this means that for some
$\Sigma_n(J_\alpha)$ set $B \subset J_\delta$, $\langle J_\delta, B \rangle$ is not amenable. Since $\langle J_1, B \rangle$ must be
amenable, $\delta > 1$. If $\delta = \gamma + 1$, then since there is a $\Sigma_1(J_\alpha)$
map of $\omega \gamma$ onto $\omega \delta$, there is a $\Sigma_n(J_\alpha)$ map of a subset of $\omega \gamma$
onto $J_\alpha$, contrary to the choice of $\delta$. Hence $\lim(\delta)$. It follows,
since $\langle J_\delta, B \rangle$ is not amenable, that there is $\tau < \delta$ with
$B \cap J_\tau \notin J_\delta$. By induction hypothesis, $J_\alpha$ is $\Sigma_n$-uniformisable.

So as $\tau < \delta$, lemma 46 implies that $\bar{\rho}(\omega \tau) \cap \Sigma_n(J_\alpha) \subset J_\alpha$. But
there is $h \in J_\alpha$, $h : \omega \tau \rightarrow J_\tau$, so this implies $\bar{\rho}(J_\tau) \cap \Sigma_n(J_\alpha) \subset J_\alpha$.

In particular, $B \cap J_\tau \in J_\alpha$. Hence for some $\beta < \alpha$, $B \cap J_\tau$ is
$J_\beta$-definable. Let $\beta$ be the least such, and let $r$ be least
such that $B \cap J_\tau$ is $\Sigma_r(J_\beta)$. By definition, $\langle J_\rho^r_\beta, B \cap J_\tau \rangle$ is
amenable, so if \( \tau < \rho^r_\beta \), then \( B \cap J_\tau = (B \cap J_\tau) \cap J_\tau \in J_{\rho^r_\beta} \), contrary to the choice of \( \beta \). Hence \( \tau \geq \rho^r_\beta \). By induction hypothesis, there is a \( \Sigma_r(J_\beta) \) map \( g \) from a subset of \( \omega_{\rho^r_\beta} \) onto \( J_\beta \). And since \( B \cap J_\tau \in J_{\beta+1} \) and \( B \cap J_\tau \notin J_\delta, \beta+1 > \delta \), or \( \beta \geq \delta \). Hence there is a \( \Sigma_r(J_\beta) \) map \( g' \) from a subset of \( \omega_{\rho^r_\beta} \) onto \( \omega_\delta \). Then \( f \cdot g' \) is a \( \Sigma_n(J_\alpha) \) map of a subset of \( \omega_{\rho^r_\beta} \) onto \( J_\alpha \). But we have established that \( \rho^r_\beta \leq \tau < \delta \), so this contradicts the choice of \( \delta \). Hence \( \delta = \rho \).

\((P \, 2)\) follows immediately from the above result of course.

We turn now to \((P \, 3)\). By induction hypothesis, let \( A \) be a \( \Sigma_m \) master code for \( J_\alpha \). Set \( \eta = \rho^m_\alpha \) for convenience. By the above, let \( f \) be a \( \Sigma_n(J_\alpha) \) map of a subset of \( \omega_\rho \) onto \( J_\alpha \). By choice of \( A \), \( f' = f \left( f^{-1} \right)' \) is a \( \Sigma_1(<J_\eta, A>) \) map of a subset of \( \omega_\rho \) onto \( J_\eta \). By choice of \( A \), it is clear that \( \rho = \rho^1_\alpha = \rho^1_{\eta, A} \). Finally, of course, \( <J_\eta, A> \) is amenable. So, we may apply lemma 48 to \( <J_\eta, A> \) to obtain a \( \Sigma_1(<J_\eta, A>) \) set \( B \subset J_\rho \) such that \( \Sigma_x(<J_\rho, B>) = \mathcal{O}_1(J_\rho) \cap \Sigma_{x+1}(<J_\eta, A>) \) for all \( r \geq 1 \). By choice of \( A \), \( B \in \Sigma_n(J_\alpha) \) and \( \Sigma_x(<J_\rho, B>) = \mathcal{O}_1(J_\rho) \cap \Sigma_{n+r}(J_\alpha) \) for all \( r \geq 1 \). Hence \( B \) is a \( \Sigma_n \) master code for \( J_\alpha \).

Finally we prove \((P \, 1)\). Let \( B \) be, as above, a \( \Sigma_n \) master code for \( J_\alpha \). Let \( R(y, \vec{x}) \) be a \( \Sigma_{n+1}(J_\alpha) \) relation on \( J_\alpha \). Define, with \( f \) as above, a relation \( \tilde{R} \) on \( J_\rho \) by \( \tilde{R}(y, \vec{x}) \Leftrightarrow [y, \vec{x} \in J_\rho \land R(f(y), f(\vec{x}))] \). Then \( \tilde{R} \) is \( \Sigma_{n+1}(J_\alpha) \), and hence \( \Sigma_1(<J_\rho, B>) \).

Let \( \tilde{f} \) be a \( \Sigma_1(<J_\rho, B>) \) function uniformising \( \tilde{R} \). Since \( f \) is \( \Sigma_n(J_\alpha) \), so is \( f^{-1} \). But \( J_\alpha \) is \( \Sigma_n \)-uniformisable, by induction hypothesis, so we can let \( \bar{f} \) be a \( \Sigma_n(J_\alpha) \) function uniformising.
Set \( r = f \circ \tilde{r} \circ f^{-1} \). It is clear that \( r \) is a \( \Sigma_{n+1}(J_\alpha) \) function which uniformises \( R \). The proof is complete.

The above results give us two (intuitive) equivalent formulations of the \( \Sigma_n \)-projectum:

**Theorem 50**

Let \( \alpha, n \geq 0 \). Let \( \delta \) be the least ordinal such that some \( \Sigma_n(J_\alpha) \) function maps a subset of \( \omega^\delta \) onto \( J_\alpha \). Let \( \gamma \) be the least ordinal such that \( \varphi(\omega^\gamma) \cap \Sigma_n(J_\alpha) \notin J_\alpha \). Then \( \delta = \gamma = \rho^\alpha_n \).

**Proof:** That \( \delta = \rho^\alpha_n \) was actually proved during the proof of Theorem 49. Since we now know that \( J_\alpha \) is \( \Sigma_n \)-uniformisable, lemma 46 tells us that \( \delta \leq \gamma \). Assume \( \delta < \gamma \). Now by definition, let \( u \subset \omega^\delta \), and let \( f : u \xrightarrow{\text{onto}} J_\alpha \) be \( \Sigma_n(J_\alpha) \). Let \( Z = \{ \xi \mid \xi \notin f(\xi) \} \). Then \( Z \subset \omega^\delta \) and \( Z \in \Gamma_n(J_\alpha) \), so by definition of \( \gamma \), \( Z \in J_\alpha \). Thus \( Z = f(\xi) \) for some \( \xi \), so \( \xi \in f(\xi) \rightarrow \xi \notin f(\xi) \), which is absurd. Hence \( \delta = \gamma \).

There is, of course, a concept which, for \( \Delta_n \) predicates, plays the role that the \( \Sigma_n \) projectum plays for \( \Sigma_n \) predicates. And, as might be expected, there is a corresponding "total function" or \( \Delta_n \) equivalent of Theorem 50 for this concept.

Let \( \alpha, n \geq 0 \). The \( \Delta_n \)-projectum of \( \alpha \) (sometimes called the weak \( \Sigma_n \)-projectum), \( \eta^n_\alpha \), is the largest \( \eta \leq \alpha \) such that \( \langle J_\eta, A \rangle \) is amenable for all \( \Delta_n(J_\alpha) \) sets \( A \subset J_\eta \).

Thus the \( \Delta_n \)-projectum of \( \alpha \) represent the "hard core" of \( J_\alpha \) with regards to \( \Delta_n \) predicates on \( J_\alpha \).

Clearly, \( \eta^n_\alpha \geq \rho^\alpha_n \). We do not, however, necessarily have equality
here. For example, let $\alpha$ be the first admissible ordinal $> \omega$. Then it is easily seen that $\eta_\alpha^1 = \alpha$, whereas $\rho_\alpha^1 = \omega$.

Corresponding to lemma 46, we have:

**Lemma 51**

Let $n \geq 1$, and let $\gamma$ be the least ordinal such that $(\rho(\omega \gamma) \cap \Delta_n(J_\alpha) \neq J_\alpha)$. Then there is a $\Sigma_n(J_\alpha)$ (and hence $\Delta_n(J_\alpha)$) map of $\omega \gamma$ onto $J_\alpha$.

**Proof:** Let $n = m+1$, $n \geq 0$. Since $\Sigma_m(J_\alpha) \subseteq \Delta_n(J_\alpha) \subseteq \Sigma_n(J_\alpha)$, Theorem 50 implies that $\rho_\alpha^m \leq \gamma \leq \rho_\alpha^n$. Theorem 50 also implies that there is a $\Sigma_m(J_\alpha)$ map of a subset of $\omega \rho_\alpha^m$ onto $J_\alpha$.

So, we can clearly define a $\Sigma_n(J_\alpha)$ map of $\omega \rho_\alpha^m$ itself onto $J_\alpha$. This reduces our problem to showing that there is a $\Sigma_n(J_\alpha)$ map of $\omega \gamma$ onto $\omega \rho_\alpha^m$. As a first step, we have the:

**Claim:** There is a $\Sigma_n(J_\alpha)$ map $g$ from $\omega \gamma$ cofinally into $\omega \rho_\alpha^m$.

Let $A$ be a $\Sigma_m$ master code for $J_\alpha$. By hypothesis, let $b \subseteq \omega \gamma$, $b \in \Delta_n(J_\alpha)$, $b \notin J_\alpha$. By choice of $A$, $b$ is $\Delta_1(\langle J^m_\rho, A \rangle)$. Suppose $b$ is in fact defined by:

$$\forall \nu \in b \rightarrow \exists y B_0(y, \nu), \quad \forall \nu \notin b \rightarrow \exists y B_1(y, \nu),$$

where $B_0, B_1$ are $\Sigma_0(\langle J^m_\rho, A \rangle)$. Then

$$(\forall \nu \in \omega \gamma)(\exists y)[B_0(y, \nu) \lor B_1(y, \nu)].$$

But $\langle J^m_\rho, A \rangle$ is amenable, and hence $\text{rud}$ closed, so as $b \notin J^m_\rho$, there can be no $\tau < \omega \rho_\alpha^m$ such that

$$(\forall \nu \in \omega \gamma)(\exists y \in S_\tau)[B_0(y, \nu) \lor B_1(y, \nu)].$$
Define $g : \omega \gamma \rightarrow \omega^m_{\alpha}$ by

$$g(\gamma) = \text{the least } \tau \text{ such that } (\exists y \in S_{\gamma})[B_o(y, \eta) \vee B_1(y, \eta)].$$

Clearly, $g$ is $\Sigma_n(J_\alpha)$ and cofinal in $\omega^m_{\alpha}$, proving the claim.

We now prove the lemma. Since $\rho^1_\alpha \leq \gamma$, there must be a $\Sigma_n(J_\alpha)$ map $f$ from a subset of $\omega\gamma$ onto $\omega^m_{\alpha} (\leq \omega)$, and of course such an $f$ will then be $\Sigma_1(\langle \rho^m_{\alpha}, A \rangle)$. Define $\overline{f} : (\omega\gamma)^2 \rightarrow \omega^m_{\alpha}$ as follows. Let $f$ be given by $f(\gamma) = \tau \rightarrow \exists y F(y, \tau, \eta)$, where $F$ is $\Sigma_0(\langle \rho^m_{\alpha}, A \rangle)$. Set

$$\overline{f}(\gamma, \tau) = \begin{cases} 0, & \text{if } (\exists y \in S_{\gamma}(\tau))F(y, \eta, \eta) \\ \theta, & \text{otherwise} \end{cases}$$

Then $\overline{f}$ is $\Sigma_1(\langle \rho^m_{\alpha}, A \rangle)$, and hence $\Sigma_n(J_\alpha)$. And $f$ clearly maps $(\omega\gamma)^2$ onto $\omega^m_{\alpha}$, as $g$ is cofinal in $\omega^m_{\alpha}$.

Since we have (by lemma 38) a $\Sigma_1(J_\alpha)$ map of $\omega\gamma$ onto $(\omega\gamma)^2$, the lemma follows.

Corresponding to Theorem 50, we have:

**Theorem 52**

Let $\alpha, n > 0$. Let $\delta$ be the least ordinal such that some $\Sigma_n(J_\alpha)$ (and hence $\Delta_n(J_\alpha)$) function maps $\omega \delta$ onto $J_\alpha$. Let $\gamma$ be the least ordinal such that $\overline{f}(\omega \gamma) \cap \Delta_n(J_\alpha) \notin J_\alpha$. Then $\delta = \gamma = \tau^n_{\alpha}.$

**Proof:** Suppose $\gamma < \tau^n_{\alpha}$. Let $B \subseteq \omega \gamma$, $B \in \Delta_n(J_\alpha)$, $B \notin J_\alpha$. Then $\omega \gamma \cap B = B \notin J_{\tau^n_{\alpha}}$, contrary to $\langle J_{\tau^n_{\alpha}}, B \rangle$ being amenable.

Suppose now that $\tau^n_{\alpha} < \gamma$. Then there is $A \subseteq J_\gamma$, $A \in \Delta_n(J_\alpha)$, such that $\langle J_\alpha, A \rangle$ is not amenable. In particular, $\gamma > 1$.

Suppose $\gamma = \xi + 1$. There is then a $\Sigma_1(J_\gamma)$ map of $\omega \xi$ onto $\omega \gamma$, so by lemma 51 there is a $\Sigma_n(J_\alpha)$ map, $f$, of $\omega \xi$ onto $J_\alpha$. 
Then \( Z = \{ t \in w \mid t \notin f(t) \} \) is a \( \Delta_n(J_\alpha) \) subset of \( w \). Clearly, \( Z \notin J_\alpha \), so this contradicts the choice of \( \gamma \). Hence \( \lim(\gamma) \).

Thus as \( \langle J_\gamma, A \rangle \) is not amenable, there must be \( \tau < \gamma \) with \( A \cap J_\tau \notin J_\gamma \). But by choice of \( \gamma \), \( \tau < \gamma \rightarrow A \cap J_\tau \in J_\alpha \), so for some \( \theta < \alpha \), \( A \cap J_\tau \) is \( J_\theta \)-definable. Let \( \theta \) be the least such. Then \( A \cap J_\tau \in \mathcal{P}(J_\tau) \cap \Delta_m(J_\theta) \) for some \( m \in w \), and \( A \cap J_\tau \notin J_\theta \). Thus by lemma 51 there is a \( \Sigma_m(J_\theta) \) map \( f \) of \( w \tau \) onto \( J_\theta \). (Actually the hypotheses of lemma 51 require that we have a \( \Delta_m(J_\theta) \) subset of \( w \tau \) not in \( J_\theta \), whereas we have only exhibited a subset of \( J_\tau \) with these properties. However, since there is available a \( \Sigma_1(J_\tau) \) map of \( w \tau \) onto \( J_\tau \), this point causes no problem.) Since \( \theta < \alpha \), \( f \in J_\alpha \). But \( A \cap J_\tau \notin J_\gamma \), and \( A \cap J_\tau \in J_{\theta+1} \), so \( \theta \geq \gamma \), and there is thus a map \( f' \in J_\alpha \) of \( w \tau \) onto \( w \gamma \). By lemma 51, again, this gives us a \( \Sigma_n(J_\alpha) \) map \( K \) of \( w \tau \) onto \( J_\alpha \). Then, clearly, \( K = \{ t \mid t \notin f(t) \} \) is a \( \Delta_n(J_\alpha) \) subset of \( w \tau \) not lying in \( J_\alpha \), contrary to \( \tau < \gamma \).

Hence \( \gamma = \tau^n \). Now, by lemma 51, we have \( \delta \leq \gamma \). Suppose \( \delta < \gamma \).

Let \( f : w \delta \) onto \( J_\gamma \) be \( \Sigma_n(J_\alpha) \). Let \( Z = \{ \nu \mid \nu \notin f(\nu) \} \). Then \( Z \in \mathcal{P}(w \delta) \cap \Delta_n(J_\alpha) - J_\alpha \). But this contradicts the choice of \( \gamma \). Hence \( \delta = \gamma \).

Remark: Lemmas 46 and 51 can be regarded as much sharper versions of the following, much earlier theorem of Putman:

Suppose \( \mathcal{P}(\gamma) \cap \Gamma_{\alpha+1} \notin \Gamma_\alpha \). Then \( \Gamma_{\alpha+1} \) contains a well-ordering of \( \gamma \) of order type \( \alpha \). (For \( \gamma \geq w \).)

Putman actually proved this result for the case \( \rho = w \), but his proof works in the general case.

The methods described above have, of course, many uses. We give just one, very general, example, showing that (in certain circumstances) it is possible to carry out Löwenheim-Skolem arguments...
for non-regular ordinals $\alpha$ which can generally only be done when $\alpha$ is actually a regular cardinal.

More precisely, the following theorem, is well-known:

**Theorem 53**

Let $\kappa$ be a regular cardinal. Let $\gamma < \kappa \leq \omega \theta$, and suppose that $Y \subseteq J_\beta$, $|Y| < \kappa$. Then there is $X \subseteq J_\beta$ such that $Y \cup \gamma \subseteq X$ and $\kappa \cap X \in \kappa$.

To prove this, one simply forms an $\omega$-chain $X_0 < X_1 < \cdots < X_n < \cdots J_\beta$ of elementary submodels of $J_\beta$, taking $X_0$ as the skolem hull of $Y \cup \gamma$ in $J_\beta$, and $X_{n+1}$ as the skolem hull of $X_n \cup \text{sup}(\kappa \cap X_n)$ in $J_\beta$, and then $X = \bigcup_{n<\omega} X_n$ is the required submodel of $J_\beta$.

By construction, $\kappa \cap X$ is transitive, and hence an ordinal, and since $\kappa$ is regular, $|X| < \kappa$, so $\kappa \cap X \in \kappa$.

It should be observed that $\kappa$ being regular is a necessary condition for the above procedure to work (in general).

However, providing we can, in some way, ensure that for each $n$, $\text{sup}(\kappa \cap X_n) < \kappa$, then we can, of course, get by with just $\text{cf}(\kappa) > \omega$.

The theorem below shows that, in certain cases we can do just this, providing we relax our demands somewhat.

Let $n \geq 1$, $\alpha \leq \omega \beta$. We say that $\alpha$ is $\Sigma_n$-regular at $\beta$ iff there is no $\Sigma_n(J_\beta)$ map of a bounded subset of $\alpha$ cofinally into $\alpha$.

For example, by Theorem 43, $\omega\gamma$ is strongly admissible iff $\omega\alpha$ is $\Sigma_1$-regular at $\alpha$.

**Theorem 54**

Let $n \geq 1$, $\omega \beta \geq \alpha \geq 1$. Suppose $\alpha$ is $\Sigma_n$-regular at $\beta$. Let $Y \subseteq J_\beta$, $\omega \leq |Y| < \text{cf}(\alpha)$, and let $\gamma < \alpha$. Then there is an
$X \prec \Sigma_n J^\beta$ such that $Y \cup \gamma \subseteq X$ and $\alpha \cap X \in \alpha$.

**Proof:** Since $\text{cf}(\alpha) > \omega$, it clearly suffices to prove that, under the stated hypotheses, there is $X \prec \Sigma_n J^\beta$ such that $Y \cup \gamma \subseteq X$ and $\sup(\alpha \cap X) < \alpha$.

Let $h$ be a $\Sigma_n$ skolem function for $J^\beta$ (by Theorem 49 and lemma 33). Since $\omega \leq |Y| < \text{cf}(\alpha)$, we may, without loss of generality, assume that $Y$ is closed under ordered pairs. Furthermore, let $\xi$ be the function defined in lemma 37. Since $\alpha$ is $\Sigma_n$-regular at $\beta$, $\alpha$ is certainly strongly admissible. Hence, by lemma 37, $\{\xi \in \alpha | \xi^2 \subseteq \xi \}$ is unbounded in $\alpha$. It follows that we may also, without loss of generality, assume that $\gamma^2 \subseteq \gamma$. Recall that $\xi^2 \subseteq \gamma$. Let $X = h^\gamma(w \times (Y \times \gamma))$. Then we claim that $X$ is closed under ordered pairs. To see this, let $x_1, x_2 \in X$, say

$x_1 = h(i_1, \langle y_1, \nu_1 \rangle), \quad x_2 = h(i_2, \langle y_2, \nu_2 \rangle)$. Let $y = \langle y_1, y_2 \rangle \in Y$ and $\nu = \xi(\langle \nu_1, \nu_2 \rangle) \in \gamma$. Clearly $\{\langle x_1, x_2 \rangle\}$ is $\Sigma^J_\beta(p, \langle y, \nu \rangle))$, where $p$ is a good parameter for $h$. Thus for some $i \in \omega$, $\langle x_1, x_2 \rangle = h(i, \langle y, \nu \rangle) \in X$, as required. So, by corollary 36, $X \prec \Sigma_n J^\beta$. And of course, we clearly have $Y \cup \gamma \subseteq X$. We show that $\sup(\alpha \cap X) < \alpha$.

For $y \in Y$, $i \in \omega$, define $h_{i,y} : \gamma \rightarrow \alpha$ by $h_{i,y}(\nu) = h(i, \langle y, \nu \rangle)$. Thus $h_{i,y}$ is $\Sigma_n(J^\beta)$, and so as $\alpha$ is $\Sigma_n$-regular at $\beta$, $\sup(h_{i,y}) \gamma \simeq \gamma(i, y) < \alpha$. Since $|Y| < \text{cf}(\alpha)$, it follows that $\sup_{y \in Y} \gamma(i, y) \simeq \gamma(i) < \alpha$. Since $\text{cf}(\alpha) > \omega$, we conclude finally that $\sup_{i \in \omega} \gamma(i) < \alpha$. But clearly, $\sup_{i \in \omega} \gamma(i) = \sup(\alpha \cap X)$, so we are done.
The above lemma may be used to prove that if $V = L$ and $\kappa$ is a regular uncountable, non-weakly compact cardinal, then there is a Souslin $\kappa$-tree. (Jensen.)
(1) Since we wrote this paper, a slightly revised version of these notes has been published as a research paper. See R.B. Jensen, "The Fine Structure of the Constructible Hierarchy", Annals of Mathematical Logic, Vol 4 [1972], p 227. The present paper represents a lengthy discourse on an expansion of the earlier parts of Jensen's paper, and it is hoped that the somewhat more leisurely pace adopt here (as opposed to Jensen's paper) will be of benefit to those not predominantly interested in the set theoretical consequences of the Fine Structure Theory. For those who are so inclined, the notation we use is almost identical to that of Jensen, so this paper should provide a good introduction to Jensen's.