Closed faces with internal points

by

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We give necessary and sufficient conditions for face(x), the smallest face of a compact convex set K containing x, to be closed. In case face(x) is closed, we give necessary and sufficient conditions for ∂_eface(x), the set of extreme points in face(x), to be closed. M. Capon has shown that if K is an α-polytope and face(x) is closed, then face(x) is finite-dimensional. We give a new proof of this result, and we show that this also holds in β-polytopes.

Notation.

Let K be a compact convex set in a locally convex Hausdorff vector space E. We will use the following symbols:

$C(K)$ = The Banach space of all real-valued continuous functions on K.

$A(K) = \{f \in C(K): f \text{ is affine}\}$

$P(K) = \{f \in C(K): f \text{ is convex}\}$

$M(K) = C^*(K)$ = The set of all Randon-measures on K.

$M^+(K) = \{\mu \in M(K): \mu \geq 0\}$

$M_1^+(K) = \{\mu \in M(K): \mu(1) = 1 = \|\mu\|\}$

$M_1^+(K)$ is w^*-compact convex and the barycenter map $r: M_1^+(K) \to K$ is continuous, affine and surjective. For each $x \in K$, we write

$$M_x = r^{-1}(x) = \{\mu \in M_1^+(K): \mu(f) = f(x), \text{ all } f \in A(K)\}$$

Each M_x is w^*-compact convex and non-empty.

We denote by Z the subset of $M_1^+(K)$ of all elements which are maximal in Choquet's ordering $\prec$, (if $\mu, \nu \in M_1^+(K)$, then $\mu \prec \nu$ if $\mu(f) \leq \nu(f)$ for all $f \in P(K)$,) and we write $Z_x = M_x \cap Z$. (See [2],[7]) It should be noted that Z is a norm-closed face of $M_1^+(K)$, and that $Z_x \neq \emptyset$ for all $x \in K$.

If $f \in C(K)$, we define $\check{f}, \hat{f}: K \to \mathbb{R}$ by

$$\hat{f}(x) = \inf\{g(x): g \in A(K), g \geq f\}$$

$$\check{f}(x) = \sup\{g(x): g \in A(K), g \leq f\}$$

$\hat{f}(\check{f})$ is the least (greatest) u.s.c. (l.s.c.) concave (convex) function majorizing (minorizing) f.

If $x \in K$, we denote by face(x) the smallest face of K containing x. [1], [2].

For each $\alpha \geq 1$, we write

$$D_\alpha(x) = (\alpha x - (\alpha-1)K) \cap K .$$

Then we have that $D_\alpha(x)$ is compact convex for each α and that

$$\mathrm{face}(x) = \bigcup_{n=1}^{\infty} D_n(x)$$

$\partial_e K$ will denote the set of extreme points in K. The closure of a set S will be denoted $\mathrm{cl}(S)$, and the convex hull of S will be denoted $\mathrm{co}(S)$.

1. Compactness of face(x)

Definition: If F is a face of K and $x \in F$, then we shall say that x is an internal point of F if $F = \mathrm{face}(x)$.

Other names for internal points are core points and radial points.

If $x \in K$, then the smallest face of $M_1^+(K)$ containing M_x is given by

$$\mathrm{face}(M_x) = \bigcup_{\alpha \geq 1} (\alpha M_x - (\alpha-1)M_1^+(K)) \cap M_1^+(K) .$$

It is easy to see that if $\mu \in M_1^+(K)$, then $\mu \in \mathrm{face}(M_x)$ if and only if $r(\mu) \in \mathrm{face}(x)$. Hence it follows easily:

Proposition 1.1: Let $x \in K$. Then $\mathrm{face}(x)$ is closed if and only if $\mathrm{face}(M_x)$ is w*-closed.

Proposition 1.2: Let $x \in K$. Then $\mathrm{face}(Z_x)$ is w*-closed if and only if $\mathrm{face}(x)$ and $\partial_e \mathrm{face}(x)$ are closed.

Proof: Suppose face(Z_x) is w*-closed. Let $\{x_\alpha\}$ be a net in face(x) converging to $y \in K$, and let $\mu_\alpha \in Z_{x_\alpha} \subseteq$ face(Z_x). Then there exists a subnet $\{\mu_\beta\}$ of $\{\mu_\alpha\}$ and a $\mu \in$ face(Z_x) such that $\mu_\beta \to \mu(w^*)$. Then $x_\beta \to r(\mu) \in$ face(x) and $y = r(\mu)$. Thus face(x) is closed. By Theorem 1.9 below, there exists n such that face(x) $= D_n(x)$. Now let $\{x_\alpha\}$ be a net in ∂_eface(x) converging to y. Then

$$x = \frac{1}{n}x_\alpha + (1-\frac{1}{n})z_\alpha$$

for some $z_\alpha \in K$, and we can assume z_α converges to $z \in K$. Let $\nu_\alpha \in Z_{z_\alpha}$. Then, going to a subnet if necessary, we can assume $\nu_\alpha \to \nu \in M_z$ in the w*-topology. Now

$$\frac{1}{n}\epsilon_{x_\alpha} + (1-\frac{1}{n})\nu_\alpha \in Z_x$$

converges to

$$\frac{1}{n}\epsilon_y + (1-\frac{1}{n})\nu \in M_x \cap \text{face}(Z_x) \subseteq Z .$$

Hence $y \in \partial_e K$, so $y \in \partial_e$face(x).

Suppose face(x) and ∂_eface(x) are closed. If $\mu \in Z_x$, then supp(μ) $\subseteq$ face(x) and it is easy to see that supp(μ) $\subseteq$ ∂_eface(x) $\subseteq \partial_e K$. In fact, if $X \subseteq$ face(x) $\setminus \partial_e$face(x), then since $\mu(\{x \in K: f(x) = \hat{f}(x)\}) = 1$ for all $f \in C(K)$, $\mu(X) = 0$ and by regularity $\mu(\partial_e$face(x)$) = 1$. Thus supp(μ) $\subseteq \partial_e$face(x) $\subseteq \partial_e K$ for all $\mu \in$ w*-cl(face(Z_x)). Let $\{\mu_\alpha\}$ be a net in face(Z_x) and suppose $\mu_\alpha \to \mu(w^*)$. Then $r(\mu_\alpha) \to r(\mu)$, so $r(\mu) \in$ face(x). Hence $\mu \in$ face(Z_x), and the proof is complete.

We will now prove that if face(x) is closed, then face(x) $= D_n(x)$ for some n. But first we need some lemmas and definitions.

Definition: If $x \in K$, we define

$$J_x = \{f \in C(K): \mu(f) = 0, \text{ all } \mu \in M_x\}$$

and

$$K_x = \{f \in C(K): \mu(f) = 0, \text{ all } \mu \in Z_x\}$$

Lemma 1.3: $J_x = \{f \in C(K): f(x) = 0$ and f is affine on face$(x)\}$.

Proof: From Corollary I.3.6. in [2] we get

$$J_x = \{f \in C(K): \hat{f}(x) = f(x) = \check{f}(x) = 0\}$$

It is easy to see that $\hat{f}(x) = \check{f}(x) = 0$ is equivalent to $f(x) = 0$ and f is affine on face(x), and the lemma is proved.

Lemma 1.4: We have

$$J_x = \mathrm{cl}(J_x + C^+(K)) \cap \mathrm{cl}(J_x - C^+(K))$$

and

$$K_x = \mathrm{cl}(K_x + C^+(K)) \cap \mathrm{cl}(K_x - C^+(K)) .$$

Proof: Suppose $f \in \mathrm{cl}(J_x + C^+(K)) \cap \mathrm{cl}(J_x - C^+(K))$ and let $\epsilon > 0$. Then we can find $f_1, f_2 \in J_x$, $g_1, g_2 \in C^+(K)$ and $h_1, h_2 \in C(K)$ with $\|h_i\| < \epsilon$ such that

$$f = f_1 + g_1 + h_1 = f_2 - g_2 + h_2$$

Then

$$f_1 + h_1 \leq f \leq f_2 + h_2$$

and if $\mu \in M_x$, then

$$-\epsilon \leq \mu(f) \leq \epsilon .$$

Since ε is arbitrary $\mu(f) = 0$, so $f \in J_x$. Thus the first equality is proved. The second equality is proved in the same manner.

We write $J_x^\perp = \{\mu \in M(K): \mu(f) \leq 1, \text{ all } f \in J_x\}$. It is easy to see that the following inclusions hold:

$$Z_x \subseteq K_x^\perp \cap M_1^+(K) \subseteq J_x^\perp \cap M_1^+(K) = M_x$$

Lemma 1.5: We have

$$w^* - cl(\text{lin}M_x) = J_x^\perp = w^* - cl((J_x^\perp \cap M^+(K)) - (J_x^\perp \cap M^+(K)))$$

and

$$w^* - cl(\text{lin}Z_x) = K_x^\perp = w^* - cl((K_x^\perp \cap M^+(K)) - (K_x^\perp \cap M^+(K)))$$

Proof: Taking polars, we get:

$$J_x^\perp = (cl(J_x + C^+(K)) \cap cl(J_x - C^+(K)))^\circ$$

$$= ((J_x + C^+(K))^{\circ\circ} \cap (J_x - C^+(K))^{\circ\circ})^\circ$$

$$= w^* - cl((J_x + C^+(K))^\circ + (J_x - C^+(K))^\circ)$$

$$= w^* - cl((J_x^\perp \cap M^+(K)) - (J_x^\perp \cap M^+(K)))$$

$$= w^* - cl(\text{lin}M_x)$$

In the same manner, we get

$$K_x^\perp = w^* - cl((K_x^\perp \cap M^+(K)) - (K_x^\perp \cap M^+(K)))$$

and

$$K_x^\perp \supseteq w^* - cl(\text{lin}Z_x)$$

If $\mu \notin w^* - cl(\text{lin}Z_x)$, then by Hahn-Banach there exists a $g \in C(K)$ such that $\mu(g) \neq 0$ and $\nu(g) = 0$ for all $\nu \in Z_x$. Thus $g \in K_x$ and $\mu \notin K_x^\perp$, so $K_x^\perp = w^* - cl(\text{lin}Z_x)$, and the proof is complete.

<u>Lemma 1.6</u>: If $\mu = a\mu_1 - b\mu_2 \in J_x^{\perp}$ where $a,b \geq 0$ with $a+b = 1$, $\mu_1,\mu_2 \in M_1^+(K)$ and $r(\mu_1), r(\mu_2) \in \text{face}(x)$, then $\mu \in \text{lin}M_x$. Furthermore, if $\mu_1,\mu_2 \in Z$, then $\mu \in \text{lin}Z_x$.

<u>Proof</u>: Let $f \in A(K)$. Then $f - f(x) \in J_x^{\perp}$, so

$$0 = \mu(f-f(x))$$
$$= a\mu_1(f) - b\mu_2(f) - af(x) + bf(x) .$$

Hence

$$f(ar(\mu_1)+bx) = a\mu_1(f) + bf(x)$$
$$= b\mu_2(f) + af(x) = f(br(\mu_2)+ax)$$

so

$$ar(\mu_1) + bx = br(\mu_2) + ax$$

Since $r(\mu_1), r(\mu_2) \in \text{face}(x)$, we can find $y \in K$ and $\alpha,\beta \in \langle 0,1]$ such that

$$x = \alpha r(\mu_2) + (1-\alpha)y$$
$$x = \beta r(\mu_1) + (1-\beta)y$$

Then we have

$$(a-b)x = ar(\mu_1) - br(\mu_2)$$
$$= (a\beta^{-1}-b\alpha^{-1})x + (a-a\beta^{-1}-b+b\alpha^{-1})y$$

so $x = y$ or

$$a - b - a\beta^{-1} + b\alpha^{-1} = 0$$

If $x = y$, then $\mu \in \text{lin}M_x$, so suppose $x \neq y$ and let $\nu \in Z_y$. Then

$$a\beta^{-1}(\beta\mu_1+(1-\mu)\nu) - b\alpha^{-1}(\alpha\mu_2+(1-\alpha)\nu)$$
$$= \mu + (a\beta^{-1}-a-b\alpha^{-1}+b)\nu = \mu \in \text{lin}M_x ,$$

and the proof is complete.

By the <u>σ-convex hull</u> of a set $S \subseteq K$ we mean the set of all $y = \sum_{i=1}^{\infty} \lambda_i y_i \in K$ where $\{y_i\}_{i=1}^{\infty}$ is a sequence from S and

$\{\lambda_i\}_{i=1}^{\infty}$ is a sequence from $[0,1]$ such that $\sum_{i=1}^{\infty} \lambda_i = 1$. We shall say that S is σ-convex if S coincides with its σ-convex hull.

<u>Lemma 1.7</u>: If $\mu \in \text{norm-cl}(\text{lin}M_x)$, then there exist $\mu_1, \mu_2 \in M_1^+(K)$ and $a, b \geq 0$ such that $\mu = a\mu_1 - b\mu_2$ and $r(\mu_1)$ and $r(\mu_2)$ are in the σ-convex hull of $\text{face}(x)$.

<u>Proof</u>: Let $\{\nu_n\}_{n=1}^{\infty}$ be a sequence from $\text{lin}M_x$ such that $\|\mu - \nu_n\| \to 0$ as $n \to \infty$.
Choose $a, b \geq 0$ and $\mu_1, \mu_2 \in M_1^+(K)$ such that $\mu^+ = a\mu_1$ and $\mu^- = b\mu_2$. By Corollary II.7.5. in [8], there exist $\nu \in M_x$, $\sigma_1, \sigma_2 \in M(K)$ such that $\mu = \sigma_1 + \sigma_2$, $\sigma_1 \ll \nu$ and $\sigma_2 \perp \theta$, all $\theta \in M_x$. From this it follows that

$$\|\mu - \nu_n\| = \|\sigma_1 - \nu_n\| + \|\sigma_2\| \to 0 \quad \text{as} \quad n \to \infty$$

so $\sigma_2 = 0$. Hence $\mu \ll \nu$, so $\mu^+, \mu^- \ll \nu$.
Let $f, g \in L^1(\nu)$ be such that $\mu_1 = f \cdot \nu$ and $\mu_2 = g \cdot \nu$. If we define for $n = 1, 2, \ldots$

$$B_n = \{x \in K : f(x) \in [n-1, n\rangle\},$$

$$\lambda_n = \int_{B_n} f \, d\nu$$

and

$$\sigma_n = \lambda_n^{-1} f \cdot \chi_{B_n} \cdot \nu \qquad (\text{define } \sigma_n = \nu \text{ if } \lambda_n = 0)$$

then

$$\sum_{n=1}^{\infty} \lambda_n = \int_K f \, d\nu = \mu_1(1) = 1,$$

$$\mu_1 = \sum_{n=1}^{\infty} \lambda_n \sigma_n$$

and $r(\sigma_n) \in \text{face}(x)$ (see Proposition 1.2. in [13]).

Hence $r(\mu_1) = \sum_{n=1}^{\infty} \lambda_1 r(\sigma_1)$ is in the σ-convex hull of face(x) . In the same manner, we see that $r(\mu_2)$ is in the σ-convex hull of face(x) , and the proof is complete.

As a corollary of Lemma 1.7. and Lemma 1.6. we get:

Corollary 1.8: $\mathrm{lin}Z_x$ is norm-closed if face(x) is closed.

It is well known that we may imbed K as a subset of A(K)* [2]. By the norm topology on K we mean the relative topology on K as a subset of the Banach space A(K)* . Now we can prove:

Theorem 1.9: Let $x \in K$. The following statements are equivalent:

(i) $\mathrm{face}(x) = D_n(x)$ for some n .

(ii) face(x) is closed.

(iii) face(x) is norm-closed.

(iv) face(x) is σ-convex.

(v) $\mathrm{lin}M_x$ is w*-closed.

(vi) $\mathrm{lin}M_x$ is norm-closed.

Proof: (i) ==> (ii) ==> (iii) ==> (iv) and (v) ==> (vi) are trivial.

(ii) ==> (v). Let $\mu \in \text{w*-cl}(\mathrm{lin}M_x) = J_x^{\perp}$. We may assume $\|\mu\| = 1$. Choose $a, b \geq 0$; $\mu_1, \mu_2 \in M_1^+(K)$ such that $\mu^+ = a\mu_1$ and $\mu^- = b\mu_2$. Then $a+b = \mu^+(1) + \mu^-(1) = \|\mu\| = 1$. Since face(x) is closed, we have $\mathrm{support}(\mu_1), \mathrm{support}(\mu_2) \subseteq \mathrm{face}(x)$, and hence $r(\mu_1), r(\mu_2) \in \mathrm{face}(x)$. (v) now follows from Lemma 1.6.

(iv) ==> (vi). Let $\mu \in \text{norm-cl}(\mathrm{lin}M_x)$. By Lemma 1.7. and Lemma 1.6. $\mu \in \mathrm{lin}M_x$.

(vi) ==> (i). $\mathrm{lin}M_x$ is a Banach spacs with a generating positive cone $\bigcup_{s \geq o} sM_x$, so by Lemma 1 in [4] there is a constant C such that if $\mu \in \text{lin}M_x$, then $\mu = \mu_1 - \mu_2$ for some $\mu_1, \mu_2 \in \bigcup_{s \geq o} sM_x$ with

$$\|\mu_1\| + \|\mu_2\| \leq C \|\mu\|$$

Hence by Krein-Šmulian $\text{lin}M_x$ is w*-closed.

Let $u \in \text{face}(x)$ and let $v = \frac{1}{2}(u+v)$. Then $\mu = \varepsilon_v - 2^{-1}\varepsilon_u \in J_x^\perp = \text{lin}M_x$. Hence, we can find $s,t \geq 0$, $\mu_1, \mu_2 \in M_x$ such that $s+t \leq 2^{-1} \cdot 3 \cdot C$ and

$$\mu = t\mu_1 - s\mu_2$$

Then

$$\varepsilon_v + s\mu_2 = t\mu_1 + 2^{-1}\varepsilon_u$$

so

$$\mu_2(\{u\}) \geq (2s)^{-1} \geq (3C)^{-1}$$

Hence

$$\mu_2 = \lambda\varepsilon_u + (1-\lambda)\nu$$

for some $\nu \in M_1^+(K)$ and some $\lambda \geq (3C)^{-1}$.

Thus $x \in D_{3C}(x)$, and the proof is complete.

<u>Remark</u>: 1) If K is the unit ball in a Hilbert space, then $\text{face}(x)$ is closed for each $x \in K$. In fact, if $\text{face}(x) \neq K$, then $x \in \partial_e K$.

2) If $K = \text{face}(x)$ and $y \in K$, then $\text{face}(y)$ do not need to be compact. In fact $\text{face}(o) = \text{co}(M_1^+(K) \cup -M_1^+(K))$ and $M_1^+(K)$ is a face of $\text{face}(o)$, but not all faces of $M_1^+(K)$ are closed.

<u>Lemma 1.10</u>: The following statements are related as follows:

(i) $\Leftarrow$ (ii) $\Leftrightarrow$ (iii)

(i) $Z_x = K_x^{\perp} \cap M_1^+(K)$

(ii) $\text{lin}Z_x$ is w*-closed

(iii) $\text{lin}Z_x$ is norm-closed and Z_x is w*-closed.

<u>Proof</u>: (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii) follows from the Krein-Šmulian theorem using that $\text{lin}Z_x$ as a Banach space with positive cone $\bigcup_{s \geq 0} sZ_x$ is boundedly positively generated (see Lemma 1 in [4] and Theorem II.5.5. in [2]).

(ii) $\Rightarrow$ (i) Let $\mu \in K_x^{\perp} \cap M_1^+(K) \subseteq M_x$. Then $\mu \in \text{lin}Z_x$, so $\mu = \alpha\mu_1 - \beta\mu_2$ where $\alpha, \beta \geq 0$ and $\mu_1, \mu_2 \in Z_x$. Now if $f \in C(K)$, then $\mu(f) = \alpha\mu_1(f) - \beta\mu_2(f) = \alpha\mu_1(\hat{f}) - \beta\mu_2(\hat{f}) = \mu(\hat{f})$, so μ is maximal, i.e. $\mu \in Z_x$, and the proof is complete.

<u>Theorem 1.11</u>: Let $x \in K$ and assume $\text{face}(x)$ is closed. Then the following statements are equivalent:

(i) $\partial_e \text{face}(x)$ is closed

(ii) Z_x is w*-closed

(iii) $\text{lin}Z_x$ is w*-closed

(iv) $\text{lin}Z_x$ is norm-closed and Z_x is w*-closed.

Proof: (iv) <==> (iii) ==> (ii) follows from Lemma 1.10.
(ii) ==> (i) The proof of this implication is contained in the proof of Proposition 1.2.
(i) ==> (iii) As in the proof of Proposition 1.2 we show that if $\mu \in Z_x$, then $\text{supp}(\mu) \subseteq \partial_e \text{face}(x)$. Hence $\text{supp}(\mu) \subseteq \partial_e \text{face}(x)$ for all $\mu \in w^*\text{-cl}(\text{lin}Z_x)$. Now we proceed as in the proof of (ii) ==> (iii) of Theorem 1.9. to show that $\text{lin}Z_x$ is w*-closed, and the proof is complete.

Remark: 1) If K is a Choquet simplex and if $x \in K$ such that face(x) is not closed, then $\text{lin}Z_x$ is w*-closed, but $\text{lin}M_x$ is not w*-closed.

2) If K is the Bourbaki Example (see [2] pp 63-64), then there is a $x \in K$ such that face(x) is closed and $\partial_e \text{face}(x)$ is not closed. Then $\text{lin}M_x$ is w*-closed, but $\text{lin}Z_x$ is not w*-colsed.

Remark: The equivalence of (i) and (ii) in Theorem 1.9. is independently provedby M. Capon in [6].

2. Internal points.

Definition: K is said to be an α-polytope if Z_x is finite-dimensional for each $x \in K$.

K is said to be a generalized α-polytope if Z_x has an internal point for each $x \in K$.

K is said to be a β-polytope if there exists a compact Choquet-simplex S and $h_1,\dots,h_n \in A(S)$ such that $K = \{x \in S : h_i(x) = 0,\ i = 1,\dots,n\}$.

α-polytopes and β-polytopes have been studied in [11], [5], [6] and generalized α-polytopes has been studied in [12].

Remark: It is well known that if Z_x if finite-dimensional, then Z_x contains an internal point (see i.e. [12]). Thus every α-polytope is a generalized α-polytope.

Theorem 2.1: If K is a generalized α-polytope with an internal point x , then K is finite-dimensional.

Proof: Assume $K = \mathrm{face}(x)$. Then by Theorem 1.9. $K = D_n(x)$ for some n . Let $Z_x = \mathrm{face}(\mu)$. Then Z_x is w*-closed (see [12] or [8]) and again by Theorem 1.9. $Z_x = D_m(\mu)$ for some m . Let $y \in \partial_e K$. Then for some $z \in K$,

$$x = n^{-1}y + (1-n^{-1})z$$

If $\nu \in Z_z$, then

$$n^{-1}\epsilon_y + (1-n^{-1})\nu \in Z_x$$

so for some $\sigma \in Z_x$, we have

$$\mu = m^{-1}n^{-1}\epsilon_y + m^{-1}(1-n^{-1})\nu + (1-m^{-1})\sigma .$$

Hence $\mu(\{y\}) \geq (m\cdot n)^{-1}$, so $\partial_e K$ cannot contain more than mn extreme points. Hence K is finite-dimensional and the proof is complete.

<u>Theorem 2.2</u>: If K is a β-polytope with an internal point x, then K is finite-dimensional.

<u>Proof</u>: We have $K = \text{face}(x)$.

Let S be a compact Choquet-simplex and let $h_1, \ldots, h_n \in A(S)$ be such that

$$K = \{y \in S : h_i(y) = 0,\ i = 1, \ldots, n\}$$

Without loss of generality, we may assume that S is the smallest closed face of S containing K.

Define $K_m = \{y \in S : h_i(y) = 0,\ i = m, \ldots, n\}$; $1 \leq m \leq n$. The $K_1 = K$. Let $y \in K_2$. If $y \notin K_1$, then $h_1(y) \neq 0$, and we may assume $h_1(y) > 0$. By Lemma 3.4. in [11], there exists $z \in K_2$ with $h_1(z) < 0$. Then for some $\alpha \in \langle 0,1\rangle$

$$h_1(\alpha y + (1-\alpha)z) = \alpha h_1(y) + (1-\alpha)h_1(z) = 0$$

so $\alpha y + (1-\alpha)z \in K_1 = K$. Hence the smallest face of K_2 containing x is K_2, and by induction, the smallest face of S containing x is S, so S has an internal point. By Theorem 2.1. S is finite-dimensional, so K is finite-dimensional and the proof is complete.

<u>Remark</u>: In [6] M. Capon proved Theorem 2.1. for α-polytopes and Theorem 2.2. under some additional hypotheses.

<u>Definition</u>: A proper face $F (\neq \emptyset, C)$ of a convex set C is said to be a <u>split face</u> of C if there exists a face $F' \neq \emptyset$ of

K such that $F \cap F' = \emptyset$, $C = co(F \cup F')$ and every point in $C \setminus (F \cup F')$ can be uniquely represented as a convex sum of a point in F and a point in F' . $\emptyset$ and C are said to be improper split faces. F' is called the complementary face of F and is unique determined.

Split faces were first studied in [3] and [10]. (See also [2].)

Proposition 2.3: If F is a split face of K and G is a split face of F , then G is a split face of K .

Proof: Let F' be the complementary face of F in K , and let G' be the complementary face of G in F . Then by Proposition 22 in [9], $co(F' \cup G')$ is a face of K . It is easy to see that $K = co(G \cup co(F' \cup G'))$, and that G is a split face of K , and the proof is complete.

Proposition 2.4: Assume K has an internal point y , so that $K = D_m(y)$ for some m . Then there exist proper disjoint split faces $F_1,\dots,F_n$ of K with $0 \leq n \leq m$ such that if F is any split face of K , then $F = co(\bigcup_{F_i \subseteq F} F_i)$.

Proof: If there is no proper split face of K , there is nothing to prove. So assume K contains a proper split face F.

Suppose $F_1,\dots,F_n$ are disjoint proper split faces of K and define $G = \bigcap_{i=1}^{n} F_i'$. Then G is a split face , $K = co(F_1 \cup \dots \cup F_n \cup G)$ and $F_1,\dots,F_n,G$ are disjoint split faces. Thus we may assume $K = co(\bigcup_{i=1}^{n} F_i)$.

Let

$$y = \sum_{i=1}^{n} \lambda_i x_i \ , \ \sum_{i=1}^{n} \lambda_i = 1 \ , \ \lambda_i \geq 0 \ , \ x_i \in F_i \ .$$

Since $x_i \in D_m(y)$, there exists $\alpha_i \geq m^{-1}$ and $z_i \in K$ such that

$$y = \alpha_i x_i + (1-\alpha_i) z_i$$

Now

$$z_i = \sum_{k=1}^{n} \gamma_k u_k \ , \ \sum_{k=1}^{n} \gamma_k = 1 \ , \quad \gamma_k \geq 0 \ , \quad u_k \in F_k \ .$$

Hence

$$y = (\alpha_i x_i + \gamma_i u_i) + \sum_{k \neq i} (1-\alpha_i) \gamma_k u_k$$

Since F_i is a split face, we have

$$\lambda_i = \alpha_i + \gamma_i \geq m^{-1}$$

$$1 = \sum_{i=1}^{n} \lambda_i \geq n \cdot m^{-1}$$

so $n \leq m$.

If F contains a proper split face G , then F', G and G' (= the complementary face of G in F) are proper disjoint split faced of K . An easy argument now shows that there exists proper disjoint split faces $F_1, \ldots, F_n$ of F such that $n \leq m$ and $K = co(\bigcup_{i=1}^{n} F_i)$, and no F_i contain a proper split face. Thus since F is a split face of K , then $F_i \cap F = \emptyset$ or $F_i \cap F = F_i$. Now $co(\bigcup_{F_i \subseteq F} F_i)$ is a split face $\subseteq F$, and $co(\bigcup_{F_i \cap F = \emptyset} F_i)$ is a split face disjoint from F . Hence $F = co(\bigcup_{F_i \subseteq F} F_i)$, and the proof is complete.

Now it is easy to prove:

<u>Corollary 2.5</u>: Assume K has an internal point y such that $K = D_m(y)$ for some m . If $F \subseteq K$ is a split face, then there exist $x \in F$ and n such that $F = \text{face}(x) = D_n(x)$. In particular F is closed.

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