Ideal Systems and Lattice Theory III

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1. Introduction.

In the two previous notes in this series we made applications to lattice theory of results on ideal systems which had their origin in ring theory. In the present note the procedure is more or less reversed. On the basis of results which were first proved in a lattice-theoretic setting we have obtained generalizations of these results to ideal systems in such a way that they also have some bearing on rings which are not Boolean.

There is little doubt that the most important idea which ever originated within a basically lattice-theoretic context was Stone's idea of topologizing a family of prime ideals. Innumerable variations on this theme have appeared over the last few decades and a bewildering profusion of both functional and sectional representation theorems have been proved (see [4]). The present note adds a couple of new representation theorems to the already existing collection. It will be shown that the notion of an ideal system provides a convenient common ground for some of the developments in this area, with particular relevance to distributive lattices and (von Neumann) regular rings.

2. An algebraic representation theorem.

When statements about morphisms of (generalized) ideal systems are applied to rings one cannot be sure that one gets a statement about ring homomorphisms as a result. However, in the case of lattices this is so. We restate the relevant result in the following form (see [3], Theorem   ).

Theorem 1. If a map between two lattices gives rise to a morphism between the corresponding generalized ideal systems, then this map is also a lattice morphism.

In symbols this is just the implication $\phi(a \cap b) = \phi(a) \cap \phi(b)$ & $\phi(A_1) \subset (\phi(A))_1 \Rightarrow \phi(a \cup b) = \phi(a) \cup \phi(b)$. For the terminology concerning ideal systems the reader is referred to [1] and [2].

By applying Theorem 1 we shall see that the following theorem represents an immediate generalization of Stone's representation theorem for complete Boolean algebras.

Theorem 2. Let (D,x) denote an ideal system with zero 0 for which the sum of x-ideals is completely distributive with respect to the intersection of x-ideals within the ideal lattice of (D,x). If $\sqrt{0}'$ denotes the nilpoint radical of 0 then we have a canonical injective morphism

$$\Phi: \ (D/\sqrt{0}, \ x_0) \rightarrow \prod_{i \in I} (D/P_x^{(i)}, \ x_i)$$

where x_0 denotes the canonical ideal system in $D/\sqrt{0}$, x_i denotes the canonical ideal system in $D/P_x^{(i)}$ and the product sign indicates product in the category of ideal systems, here taken over all prime x-ideals $P_x^{(i)}$ in (D,x).

Proof: First of all we have canonical morphisms

2.1 $\qquad (D/\sqrt{0}, \ x_0) \rightarrow (D/P_x^{(i)}, x_i)$

because $\sqrt{0} \subset P_x^{(i)}$. It is a routine matter to verify that the category of ideal systems with zero has products. This fact together with 2.1 gives rise to the morphism Φ in the theorem. Injectivity of Φ means that $a \equiv b(P_x^{(i)})$ for all i implies $a \equiv b(\sqrt{0})$. But $a \equiv b(P_x^{(i)}$ for all i implies that

2.2 $\qquad \bigcap_{i \in I} (P_x^{(i)} + \{a\}_x) = \bigcap_{i \in I} (P_x^{(i)} + \{b\}_x)$

which by the distributivity assumption in the theorem gives

2.3 $\qquad (\bigcap_{i \in I} P_x^{(i)}) + \{a\}_x = (\bigcap_{i \in I} P_x^{(i)}) + \{b\}_x$

By the Krull-Stone theorem for x-ideals ([1] p.17) we obtain $\sqrt{0} + \{a\}_x = \sqrt{0} + \{b\}_x$ which means that $a \equiv b(\sqrt{0})$.

By imposing the rather strong distributivity assumption in the above theorem we have sacrificed some generality in order to get a simple proof in return. It is clear, however, that we can infer 2.3 from 2.2 in case of the 1-system in a complete Boolean algebra B (which _itself_ then automatically verifies the required infinite distributive law). For 2.2 means in this case that to every i there exists a $p_i \in P_i^{(i)}$ such that $a \vee p_i = b \vee p_i$. By infinite distributivity in B this implies $a \vee (\wedge p_i) = b \vee (\wedge p_i)$ which gives 2.3. In a Boolean algebra we also have $\sqrt{0} = 0$, furthermore every prime 1-ideal is maximal and hence each $B/P_i^{(i)}$ is isomorphic to the two-element Boolean algebra. (Note that in an ideal system (D,x) it is a general fact that A_x is maximal in D if and only if D/A_x consists of exactly two elements.) Now, it remains only to use Theorem 1 in order to get the familar representation theorem for a complete Boolean algebra:

Corollary. Any complete Boolean algebra is isomorphic to a Boolean algebra of characteristic functions on some set.

3. Topological representation theorems.

We shall here generalize the classical topological representation of distributive lattices to a certain class of principal ideal systems which we shall call radical Bezout systems, and which appear

as a direct generalization of the notion of a von Neumann regular
ring. More precisely we shall show that a morphism from such a
'von Neumann regular ideal system' into a 'Boolean ideal system'
may be uniquely represented by a certain type of continuous map
(going in the reverse direction) between the corresponding prime
spectra - and vice versa. Again it is Theorem 1 which helps us
to recover the classical representation theorems for Boolean rings
and distributive lattices as special cases of this result.

The language of ideal systems offers a common basis for the
various theories of structure spaces which have been developed in
connection with rings, distributive lattices, lattice ordered
groups, lattice ordered vector spaces, lattice ordered rings etc.
In partucular, the prime x-ideals of an ideal system (D,x) form a
topological space $\mathrm{Spec}(D,x)$ (or simply $\mathrm{Spec}\ D$) with respect to the
spectral topology (Zariski topology). The fact that we really get
a topology in this general setting relies heavily on the continuity
axiom. (See [1], p.35 and for a fuller account [5]). A basis for
this topology is given by the sets $D(a) = \{P_x \mid a \notin P_x\}$; more
generally a subset of $\mathrm{Spec}\ D$ is open if and only if it is of the
form $D(A_x) = \{P_x \mid P_x \not\supseteq A_x\}$ for some x-ideal A_x. The equations

3.1 $\qquad D(A_x) \cap D(B_x) = D(A_x \cap B_x)$

3.2 $\qquad D(\sum_{i \in I} A_x^{(i)}) = \bigcup_{i \in I} D(A_x^{(i)})$

and

3.3 $\qquad D(a) = D(\{a\}_x)$

show that the sets $D(A_x)$ form the smallest family which contains
the $D(a)$'s and which is closed under arbitrary unions and finite
intersections. We also note that

3.4 $D(A_x) = D(B_x)$ if and only if $\sqrt{A_x} = \sqrt{B_x}$.

(This is a consequence of the general Krull-Stone theorem for ideal systems proved in [1] p. 17).

To get further we must impose rather restrictive conditions on our ideal system (D,x). First of all we must suppose it to be principal, i.e. $\{a\}_x = Da$ for every $a \in D$. Secondly we shall assume that it is a radical ideal system in the sense that $\sqrt{A_x} = A_x$ for any ideal A_x in (D,x). Thirdly we shall assume that (D,x) is a Bezout system in the sense that any finitely generated ideal in (D,x) is principal. All these conditions do not enter at the same point in the proof of Theorem 3 and we shall try to emphasize which conditions are needed at each stage of the proof.

Lemma 1.   If (D,x) is a Bezout system, then a subset of Spec(D) is open and quasi-compact if and only if it is of the form $D(a)$ for some $a \in D$.

Proof: Assume first that $D(a) \subset \bigcup_{i \in I} D(A_x^{(i)}) = D(\sum_{i \in I} A_x^{(i)})$ which means that $a \in \sqrt{\sum_{i \in I} A_x^{(i)}}$. By the finite character axiom ((2.5) in [2]) there exists a finite subset J of I such that $a \in \sqrt{\sum_{i \in J} A_x^{(i)}}$ and hence $D(a) \subset D(\sum_{i \in J} A_x^{(i)}) = \bigcup_{i \in J} D(A_x^{(i)})$. Thus $D(a)$ is quasi-compact.

Assume conversely that the open set $D(A_x)$ is quasi-compact. Since $D(A_x) \subset \bigcup_{a \in A_x} D(a)$ we can pick a finite set $B \subset A_x$ such that $D(A_x) \subset D(B_x)$ and hense also $D(A_x) = D(B_x)$. Since (D,x) is a Bezout system and B is finite there exists an element $b \in B_x$ such that $D(A_x) = D(b)$.

Lemma 2. Let $\phi: (D_1, x_1) \to (D_2, x_2)$ be a morphism of ideal systems and put $\text{Spec}\phi(P_{x_2}) = \phi^{-1}(P_{x_2})$. Then $\text{Spec } \phi$ is a continuous map of $\text{Spec } (D_2)$ into $\text{Spec } (D_1)$ with the property that the inverse image of an open quasi-compact set is quasi-compact.

Proof: If $U = D(a)$, $a \in D_1$ then

$$(\text{Spec}\phi)^{-1}(U) = \{P_{x_2} \mid \phi^{-1}(P_{x_2}) \in U\} = \{P_{x_2} \mid a \notin \phi^{-1}(P_{x_2})\}$$

$$= \{P_{x_2} \mid \phi(a) \notin P_{x_2}\} = D(\phi(a))$$

and the assertions in the lemma follow from this by using one half of Lemma 1.

We note that Lemma 2 is valid for general ideal systems. It is for the following converse that we need extra conditions. But first some definitions. By the Boolean part of an ideal system (D,x) we shall understand the ideal system which is induced by x on the submonoid $B(D)$ consisting of all the idempotents of D. We shall denote this ideal system by $(B(D), x)$ and recall that an x-ideal in $B(D)$ is nothing but a set $A_x \cap B(D)$ where A_x is an x-ideal in D. We shall call (D, x) a Boolean ideal system if it is identical with its Boolean part. When we consider the prime spectra of some class of ideal systems as a category, a morphism will always be a continuous map with the extra property that inverse images of quasi-compact open sets are quasi-compact. Accordingly we shall use the notation $\text{Hom}_{\text{comp}}(\text{Spec}(D_2), \text{Spec}(D_1))$.

We also give the following lemma for ready reference

Lemma 3. The following conditions are equivalent for a principal ideal system (D,x)

1. (D,x) is a radical ideal system

2. Every principal ideal in (D,x) is an intersection of prime x-ideals.

3. Every principal ideal in (D,x) has a unique idempotent generator.

4. D is a regular monoid in the sense of von Neumann, i.e. a^2 is always a divisor of a.

The simple proof of this lemma is the same as in the case of rings and may be left to the reader. (Actually, condition 4 shows that the closure operation x does not really play any part in this lemma . Several other properties are also equivalent to the von Neumann regularity of D. For instance that ideal multiplication coincides with intersection and that every irreducible ideal is prime. However, some of the other characterizations of von Neumann regular rings will, in the case of ideal systems lead to a stronger regularity condition than the one occurring in the above lemma. This is for instance the case with the property that every primary ideal is maximal.)

Theorem 3. The contravariant functor Spec from the category of principal and radical Bezout systems into the category of prime spectra of such systems is full. More precisely, the canonical map

$$\text{Hom}((D_1,x_1),(B(D_2),x_2)) \to \text{Hom}_{comp}(\text{Spec}(D_2),\text{Spec}(D_1))$$

is a bijection.

Proof: In order to show that Spec is full, let

$\psi \in \text{Hom}_{\text{comp}}(\text{Spec}(D_2), \text{Spec}(D_1)$ and let $U = D(a)$ with $a \in D_1$. Then $\psi^{-1}(U)$ is open and quasi-compact and hence of the form $D(b)$ for som $b \in D_2$ according to Lemma 1. By 3.3, 3.4 and Lemma 3 (using both the properties 2. and 3. of that lemma), there exists a unique element $c \in B(D_2)$, independent of the choice of b, such that $\{b\}_x = \{c\}_x$. We then put $\phi(a) = c$ such that ϕ defines a map of D_1 into $B(D_2)$. We proceed to show that $\phi \in \text{Hom}((D_1,x_1),(B(D_2),x_2))$ and that $\psi = \text{Spec}\,\phi$. The definition of ϕ gives

$$D(\phi(a_1 a_2)) = \psi^{-1}(D(a_1 a_2)) = \psi^{-1}(D(a_1) \cap D(a_2)) =$$

$$= \psi^{-1}(D(a_1)) \cap \psi^{-1}(D(a_2)) = D(\phi(a_1)) \cap D(\phi(a_2)) =$$

$$= D(\phi(a_1)\phi(a_2))$$

By the unicity of the idempotent generator (Lemma 3) we obtain $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$. Furthermore $a \in \phi^{-1}(P_{x_2}) \iff \phi(a) \in P_{x_2}$ $\iff P_{x_2} \notin D(\phi(a)) \iff P_{x_2} \notin \psi^{-1}(D(a)) \iff \psi(P_{x_2}) \notin D(a) \iff a \in \psi(P_{x_2})$. Hence $\phi^{-1}(P_{x_2}) = \psi(P_{x_2})$. This shows that $\psi = \text{Spec}\,\phi$ and also that $\phi^{-1}(P_{x_2})$ is an x_1-ideal since $\psi(P_{x_2})$ by definition is an x_1-ideal. Since (D_2,x_2) is a radical ideal system any x_2-ideal in D_2 is an intersection of prime x_2-ideals and it follows that the inverse image by ϕ of any x_2-ideal is an x_1-ideal proving that ϕ is a morphism of ideal systems.

Assume finally that $\phi_1, \phi_2 \in \text{Hom}((D_1,x_1), (B(D_2),x_2))$ and $\phi_1 \neq \phi_2$, i.e. $\phi_1(a) \neq \phi_2(a)$ for a suitable $a \in D_1$. By Lemma 3,3 we can infer that $\{\phi_1(a)\}_{x_2} \neq \{\phi_2(a)\}_{x_2}$. Using Lemma 3,2, there

exists a prime ideal P_{x_2} which contains exactly one of the two elements $\phi_1(a)$ and $\phi_2(a)$. This implies that $Spec \ \phi_1 \neq Spec \ \phi_2$ and this concludes the proof of the theorem.

The usual ideal system in a distributive lattice L is a principal and radical Bezout system such that $B(L) = L$. Invoking Theorem 1 this gives the

Corollary. Two distributive lattices (or Boolean rings) are isomorphic if and only if their prime spectra are homeomorphic.

Proof. Assume that $\psi \in Hom_{Comp}(Spec \ L_2, Spec \ L_1)$ is a homeomorphism. By Theorem 3 there is a unique morphism of ideal systems $\phi \in Hom((L_1, \ell_1), (L_2, \ell_2))$ which by Theorem 1 is a lattic homeomorphism. The injectivity of ϕ follows from $a_1 \neq a_2 \Rightarrow D(a_1) \neq D(a_2) \Rightarrow \psi^{-1}(D(a_1)) \neq \psi^{-1}(D(a_2)) \Rightarrow D(\phi(a_1)) \neq D(\phi(a_2)) \Rightarrow \phi(a_1) \neq \phi(a_2)$ (The first and last implication uses Lemma 3,2.) The surjectivity is equally clear: If $b \in L_2$ then $\psi(D(b))$ is open and quasi-compact. Hence $\psi(D(b)) = D(a)$ for some $a \in L_1$ and $\phi(a) = b$.

The above corollary cannot be derived from Theorem 3 by a purely functorial or 'general nonsense' argument. In fact, there are situations where we have the bijection of Theorem 3 but where a homeomorphism comes from an algebraic morphism which is not an injection. This is for instance the case if R_1 is a non-Boolean and regular ring and $R_2 = B(R_1)$ is its Boolean part. Then the canonical homeomorphism between $Spec \ R_1$ and $Spec \ R_2$ comes from a non-injective morphism of ideal systems $\phi : (R_1, d_1) \to (R_2, d_2)$.

References.

[1]   K.E. Aubert, Theory of x-ideals, Acta Mathematica 107(1962),
1-52.

[2]   K.E. Aubert, Additive ideal systems, Journal of Algebra
18(1971), 511-528.

[3]   K.E. Aubert, Ideal systems and lattice theory II (to appear).

[4]   K.H. Hofmann, Representation of algebras by continuous sections.
Bull. Amer. Math. Soc. 78(1972) 291-373.

[5]   A. Holme, A general theory of structure spaces,
Fundamenta Mathematica, 58(1966) 335-347.