Uniqueness of the physical vacuum and the Wightman functions in the infinite volume limit for some non polynomial interactions

by

Sergio Albeverio and Raphael Høegh-Krohn

Institute of Mathematics
University of Oslo
Blindern, Oslo, Norway.

A B S T R A C T

We consider quantum field theoretical models in n dimensional space-time given by interaction densities which are bounded functions of an ultraviolet cut-off boson field. Using methods of enclidean Markov field theory and of classical statistical mechanics we prove, for small coupling constants, the uniqueness of the vacuum ω as limit of the ground states of the space cut-off Hamiltonians when the space cut-off is taken away. In the physical Hilbert space, ω is the unique state invariant under space-time translations. The corresponding Wightman functions and vacuum energy density are given as analytic functions of the coupling constant. The Wightman functions have cluster properties with respect to space translations.

August 1972.

1. Introduction.

In recent years the mathematical construction of quantum field theoretical models has made an impressive progress. [1)]

For the polynomial interactions [2)] in two-dimensional space-time all the Haag-Kastler axioms for a quantum field theory of local observables have been verified, as well as most of the Wightman axioms. [3)]

In particular in these polynomial models (and also for certain 2-dimensional boson models with exponential interactions [4]) the existence of a vacuum state has been proven. [4)]

This was sufficient for J. Glimm and A. Jaffe to build a theory in which the Wightman functions exist and have some of the important physical properties embodied in Wightman's axioms.

The question of the uniqueness of the vacuum has not been tackled. yet. The vacuum state is only obtained by a compactness argument as limit of a subsequence of space cut-off vacua, so that the possibility of different subsequences giving rise to different vacua is not ruled out. [5)]

In this paper we would like to remark that for certain non polynomial interactions in n space-time dimensions with ultraviolet cut-off but no space cut-off uniqueness of the vacuum can be proven for small values of the coupling constant. Moreover the corresponding Wightman functions can be constructed and studied.

The formal Hamiltonian of the boson models which we study has the form

$$H_o + \lambda \int_{\mathbb{R}^{n-1}} e^{is\,\varphi_\varepsilon(\vec{x})} d\nu(s) d\vec{x} ,$$

where φ_ε is an ultraviolet cut-off, free, time zero, field and $d\nu(s)$ is a measure with bounded support on the real line (and $d\nu(-s) = d\overline{\nu(s)}$, - meaning complex conjugate). [6)]

We first prove that the space cut-off Schwinger functions (imaginary time Wightman functions) have unique limits when the space cut-off is removed, provided the coupling constant λ is sufficiently small. These limit Schwinger functions are given explicitely by Liouville-Neumann series with known kernel as convergent power series in λ . Moreover they have cluster properties with respect to space and time translations. Using the fact that the strong limit, when the space cut-off is taken away, of the time automorphism exists as C*-automorphism of an algebra of quasi local observables, we obtain from the Schwinger functions the correspondent Wightman functions and the unique physical vacuum ω . They are all invariant under space and time translations. ω is the only state in the physical Hilbert space which has this invariance property.

It is the unique ekgenvector to the eigenvalue 0 of the non negative generator of time translations in the physical Hilbert space. The Wightman functions are proved to have the cluster property with respect to translations in space. They are analytic functions of the coupling constant in a circle containing $\lambda = 0$.

The limit $\tilde{\varepsilon}$ of the ground state energy densities of the space cut-off Hamiltonians exists, is analytic in λ for $|\lambda|$ small and concave in λ . It also exists for arbitrary negative λ and positive $d\nu$ and $\tilde{\varepsilon}$ is then negative, decreasing for $|\lambda|$ increasing and concave in $\ln(-\lambda)$.

The idea of the proofs is suggested by the analogy between euclidean field theory and classical statistical mechanics, on one hand 8) and, on the other hand, by the relation between Minkowski quantum field theory and euclidean Markov field theory as recently established by E. Nelson [8] 9).

2. The space cut-off models.

Let $\mathcal{F}$ be the Fock space for free, scalar, uncharged bosons of strictly positive mass m, moving in n dimensional space-time. Thus $\mathcal{F}$ is the direct sum $\mathcal{F} = \overset{\infty}{\underset{r=0}{\oplus}} \mathcal{F}^{(r)}$, where $\mathcal{F}^{(o)} \equiv \mathbb{C}$ = complex number and $\mathcal{F}^{(r)}$, for $r = 1,2$, is the r-fold symmetric tensor product $\mathcal{F}^{(r)} = \mathcal{H} \underset{s}{\otimes} \dots \underset{s}{\otimes} \mathcal{H}$, $\mathcal{H}$ being the Lebesgue L^2-space of (equivalence classes of) functions of a (momentum) variable p running over the euclidean $n-1$ dimensional space $\mathbb{R}^{n-1}$.

Let H_o be the free Hamiltonian in $\mathcal{F}$. It is a self-adjoint operator with domain $D(H_o) \equiv D_o$.
For $\vec{x}$ in $\mathbb{R}^{n-1}$ the free time zero fields are given by

$$\varphi(\vec{x}) = 2^{-\frac{1}{2}}(2\pi)^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} \frac{e^{i\vec{p}\vec{x}}}{\mu(\vec{p})^{\frac{1}{2}}} [a^*(-\vec{p})+a(\vec{p})]d\vec{p} \ , \tag{2.1}$$

where $\mu(\vec{p}) = \sqrt{\vec{p}^2+m^2}$. $a(\vec{p})$ and $a^*(\vec{p})$ are the usual formal annihilation-creation operators for free scalar, uncharged bosons, mormalized so that $[a(\vec{p}), a^*(\vec{p}\,')] \equiv a(\vec{p})a^*(\vec{p}\,') - a^*(\vec{p}\,')a(\vec{p}) = \delta(\vec{p}-\vec{p}\,')$.
Let $\chi(\vec{x})$ be a positive symmetric C^∞ function in $\mathbb{R}^{n-1}$ with support in the unit ball such that $\int\chi(\vec{x})d\vec{x} = 1$. Set $\chi_\varepsilon = \varepsilon^{n-1}\chi(\varepsilon^{-1}\vec{x})$, with $\varepsilon > 0$, and define the ultraviolet cut-off free time zero field by

$$\varphi_\varepsilon(\vec{x}) = \int\varphi(\vec{y})\chi_\varepsilon(\vec{x}-\vec{y})d\vec{y} \ . \tag{2.2}$$

Then $\varphi_\varepsilon(\vec{x})$ are self-adjoint operators in $\mathcal{F}$ with definition domain containing D_o and they are essentially self-adjoint on D_o . They are bounded from $\mathcal{F}^{(r)}$ into $\mathcal{F}^{(r-1)} \oplus \mathcal{F}^{(r+1)}$.

Let now $v(\alpha)$ be a real-valued function on $\mathbb{R}$, so chosen as to be the Fourier transform of a finite measure $d\nu$ of bounded support on the real line:

$$v(\alpha) = \int e^{is\alpha}\, d\nu(s) \tag{2.3}$$

with $\int d|\nu| < \infty$ and $\nu(-s) = \overline{\nu(s)}$.

The interaction density is given by $\lambda v(\varphi_\varepsilon(\vec{x}))$, which is a well defined bounded self-adjoint operator since $v(\alpha)$ is a bounded continuous function.

We note that

$$v(\varphi_\varepsilon(\vec{x})) = \int e^{is\varphi_\varepsilon(\vec{x})}\, d\nu(s) \ , \tag{2.4}$$

where the integral is taken in the strong sense. This is of the same form as the bounded interaction densities studied in [6].

The space cut-off interaction corresponding to this interaction density is given by

$$\lambda V_l \equiv \lambda \int_{|\vec{x}|\leq l} v(\varphi_\varepsilon(\vec{x}))d\vec{x} \ , \tag{2.5}$$

where the integral is again to be understood as a strong one.

This defines λV_l as a bounded self-adjoint operator on $\mathcal{F}$ for all l .

Hence $H_l \equiv H_o + \lambda V_l$ is a self-adjoint operator, bounded from below, with the same domain D_o as H_o .

Moreover we have from [6c] (Th.3) that, for arbitrary λ , the bottom of the spectrum of H_l consists of the simple eigenvalue E_l with (unique) eigenvector Ω_l . 10)

From regular perturbation theory alone one has the additional result (which we are going to extend, in a certain sense, also for $l \to \infty$) that for $|\lambda|$ sufficiently small (depending on l) E_l and Ω_l are analytic in λ . Moreover E_l is a concave function of λ i.e. satisfies $E_l(\alpha\lambda_1+(1-\alpha)\lambda_2) \geq \alpha E_l(\lambda_1)+(1-\alpha)E_l(\lambda_2)$ for all $0 \leq \alpha \leq 1$, λ_1,λ_2 .

3. The associated euclidean Markov field.

For any real Hilbert space $\mathcal{H}$ let $\Phi_{\mathcal{H}}(h)$, $h \in \mathcal{H}$ be the Gaussian generalized stochastic process indexed by $\mathcal{H}$ 11), with mean zero and covariance $E(\Phi_{\mathcal{H}}(g)\Phi_{\mathcal{H}}(h)) = (g,h)_{\mathcal{H}}$. So that $\Phi_{\mathcal{H}}(h)$ maps $h \in \mathcal{H}$ into a measurable function (Gaussian random variable) on a probability space $(\Omega_{\mathcal{H}}, d\mu_{\mathcal{H}})$. Let $L_2(d\mu_{\mathcal{H}})$ be the L^2-space over $\Omega_{\mathcal{H}}$ with respect to the measure $d\mu_{\mathcal{H}}$. $L_2(d\mu_{\mathcal{H}})$ is isomorphic [13,14] with the Fock space $\overset{\infty}{\underset{n=0}{\oplus}} \mathcal{H}^{(n)}$ over $\mathcal{H}$, where $\mathcal{H}^{(n)}$ is the n-fold symmetric tensorproduct of $\mathcal{H}$. Using this isomorphism we see that any strongly continuous unitary group on $\mathcal{H}$ induces through a group of measure preserving transformations on $\Omega_{\mathcal{H}}$ a strongly continuous unitary group on $L_2(d\mu_{\mathcal{H}})$.

Let Δ be the Laplacian as a self-adjoint operator in $L_2(\mathbb{R}^n)$. Let $\mathcal{H}_n^{\alpha}$ be the real Sobolev space, which is the completion of $C_0^{\infty}(\mathbb{R}^n)$ with respect to the inner product in $\mathcal{H}_n^{\alpha}$ given by

$$(f,g)_{\alpha} = (f,(-\Delta+m^2)^{\alpha}g), \tag{3.1}$$

where $(,)$ is the inner product in $L_2(\mathbb{R}^n)$, and m is chosen to be the mass of the free field discussed in section 2.

For $\alpha < 0$, $\mathcal{H}_n^{\alpha}$ will be a space of distrubutions.

The generalized Gaussian stochastic process $\Phi_{\mathcal{H}_n^{-1}}(h)$ is called the free euclidean Markov field. Using ideas introduced by Nelson [9] in the constructive study of models, we associate to the free time zero field over $\mathbb{R}^{n-1}$, $\varphi(g) = \int\varphi(\vec{x})g(\vec{x})d\vec{x}$ of section 2, the euclidean Markov field $\Phi_{\mathcal{H}_n^{-1}}(h)$.

For any open set U with smooth boundary in $\mathbb{R}^n$ let $\mathcal{O}(U)$ be the family of random variables generated by $\Phi(h)$, with $h \in \mathcal{H}_n^{-1}$ and support of h in U. Let $E\{\Phi(h) \mid \mathcal{O}(U)\}$ be the conditional expectation of $\Phi(h)$ given $\mathcal{O}(U)$. Nelson proved that $\Phi(h)$

has the following "Markovian property":

$$E\{\Phi(h) \mid \mathcal{O}(\complement U)\} = E\{\Phi(h) \mid \mathcal{O}(\partial U)\} \tag{3.2}$$

where $\complement U$ is the complement of U and ∂U is the boundary. The property (3.2) is taken as the characterizing property of a Markov field.

The Fock space of the free boson field as given in section 2 is just the Fock space over $\mathcal{H}_{n-1}^{-\frac{1}{2}}$, moreover the free time zero field itself $\varphi(g)$ is a generalized Gaussian stochastic process with mean zero and covariance function

$$E(\varphi(f)\varphi(g)) = (f,g)_{-\frac{1}{2}} \,. \tag{3.3}$$

Hence the free time zero field $\varphi(g)$ may be identified with the generalized Gaussian stochastic process $\Phi_{\mathcal{H}_{n-1}^{-\frac{1}{2}}}(g)$.

We define now a mapping $W_t : \mathcal{H}_{n-1}^{-\frac{1}{2}} \to \mathcal{H}_n^{-1}$ by $(W_t f)(x) = \delta(x_0 - t) f(\vec{x})$. One verifies easily that W_t is anisometry of $\mathcal{H}_{n-1}^{-\frac{1}{2}}$ onto the closed subspace of $\mathcal{H}_n^{-1}$ generated by elements of $\mathcal{H}_n^{-1}$ with support on the hyperplane $x_0 = t$.

The Fock space of the free boson field $\mathcal{F}$ is the Fock space over $\mathcal{H}_{n-1}^{-\frac{1}{2}}$, hence identified with $L_2(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$. Since W_0 is an isometry, we have that the generalized Gaussian stochastic processes $\Phi_{\mathcal{H}_{n-1}^{\frac{1}{2}}}(g)$ and $\Phi_{\mathcal{H}_n^{-1}}(W_0 g)$ have the same mean and covariance functions, hence may be identified. This then identifes $L_2(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$ with a closed subspace of $L_2(d\mu_{\mathcal{H}_n^{-1}})$.

Let $F \in L_2(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$ be of the form $F = f(\Phi_{\mathcal{H}_{n-1}^{-\frac{1}{2}}}(g_1), \ldots$ $\ldots, \Phi_{\mathcal{H}_{n-1}^{-\frac{1}{2}}}(g_k))$, where f is a bounded continuous function of k real variables. Then we define $F_t \in L_2(d\mu_{\mathcal{H}_n^{-1}})$ by

$F_t = f(\Phi_{\mathcal{H}_n^{-1}}(W_t g_1),\dots,\Phi_{\mathcal{H}_n^{-1}}(W_t g_k))$. Using that W_t is an isometry one gets that $F \to F_t$ extends to an isometry of $L_2(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$ into $L_2(d\mu_{\mathcal{H}_n^{-1}})$. Moreover in $\mathcal{H}_n^{-1}$ the translation group acts unitarily and strongly continuously. Using the identification of $L_2(d\mu_{\mathcal{H}_n^{-1}})$ with the Fock space over $\mathcal{H}_n^{-1}$ we get a unitary and strongly continuous representation $U(x)$ of the translation group in $\mathbb{R}^n$ on $L_2(d\mu_{\mathcal{H}_n^{-1}})$. Since $F_t = U(t,\vec{0})\,F_o\,U(-t,\vec{0})$, we see that F_t depends continuously on t in the L_2-norm for any F in $L_2(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$.

One verifies that

$$E(\Phi_{\mathcal{H}_n^{-1}}(h_1)\ \dots\ \Phi_{\mathcal{H}_n^{-1}}(h_n)) = \begin{cases} \displaystyle\sum_{\substack{\text{all partitions}\\ n_1<n_2,\dots,n_{r-1}<n_r}} (h_{n_1},h_{n_2})_{-1}\cdots(h_{n_{r-1}},h_{n_r})_{-1} \\ 0 \quad \text{for } r \text{ odd}, \end{cases} \tag{3.4}$$

from which it follows that the distributions of r-variables defined by $E(\Phi_{\mathcal{H}_n^{-1}}(h_1)\ \dots\ \Phi_{\mathcal{H}_n^{-1}}(h_r))$ are the imaginary time free field Wightman functions. Hence, for $t_1 \le t_2\ \dots\ \le t_r$,

$$E(\Phi_{\mathcal{H}_n^{-1}}(W_{t_1}g_1)\ \dots\ \Phi_{\mathcal{H}_n^{-1}}(W_{t_r}g_r)) = (\Omega_o,\varphi(g_1)e^{-(t_2-t_1)H_o}\varphi(g_2)e^{-(t_3-t_2)H_o}\cdots \\ \cdots\varphi(g_r)\Omega_o)\ , \tag{3.5}$$

where $\Omega_o \in \mathcal{F}$ is the vacuum for the free scalar boson field and H_o is the free energy. Using now the identification of $\mathcal{F}$ with $L_2(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$ and taking sums and limits of expressions of the form (3.5) we get the following lemma.

<u>Lemma 3.1</u>

Let $F^{(1)},\dots,F^{(r)}$ be in $L_\infty(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$.

Then, for $t_1 \le\dots\le t_r$,

$$E(F_{t_1}^{(1)} \dots F_{t_r}^{(r)}) = (\Omega_o, F^{(1)} e^{-(t_2-t_1)H_o} F^{(2)} e^{-(t_3-t_2)H_o} \dots F^{(r)} \Omega_o). \blacksquare$$

We will now consider self-adjoint operators of the form $H = H_o + V$, where H_o is the free energy and V is a bounded operator on $\mathcal{F}$ which commutes with all the free time zero fields $\varphi(g)$. Since the $L_2(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$ is a spectral representation of $\mathcal{F}$ with respect to the maximal abelian algebra generated by $\varphi(g)$, we see that, in $L_2(d\mu_{\mathcal{H}^{-\frac{1}{2}}})$, V is a multiplication operator by a function, which we will also denote V.

Lemma 3.2

Let V be as above, and let F and G be in $L_2(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$, then

$$E(F_o\, e^{-\int_0^t V_\tau d\tau} G_t) = (\Omega_o, F\, e^{-t(H_o+V)} G\, \Omega_o),$$

where the integral over V_τ is taken in the strong $L_2(d\mu_{\mathcal{H}_n^{-1}})$ sense.

Proof.

The Trotter product formula gives us

$$e^{-t(H_o+V)} = \operatorname{st\,lim}_{n\to\infty} (r^{-t/_n H_o} e^{-t/_n V})^n .$$ Now, by lemma 3.1,

$$(\Omega_o, F\, e^{-t/_n H_o} e^{-t/_n V} \dots e^{-t/_n H_o} e^{-t/_n V} G\, \Omega_o) = E(F_o\, e^{-t/_n \sum_{k=1}^{n} V_{kt/n}} G_t). \tag{3.6}$$

Since V is in $L_\infty(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$ we know that V_t is in $L_\infty(d\mu_{\mathcal{H}_n^{-1}})$ and is continuous in t in the strong L_2-sense. Hence $t/_n \sum_{k=1}^{n} V_{kt/_n}$ converges strongly in $L_2(d\mu_{\mathcal{H}_n^{-1}})$ to $\int_0^t V_\tau d\tau$ for $n \to \infty$.

The strong L_2-convergence allows us to conclude that any subsequence has a subsequence n_j such that the convergence is almost everywhere. The almost everywhere convergence together with the uniform boundedness gives that

$$E(F_o\, e^{-t/n_j \sum_{k=1}^{n_j} V_{kt/n_j}}\, G_t) \xrightarrow[j\to\infty]{} E(F_o\, e^{-\int_0^t V_\tau d\tau}\, G_t) .$$

This implies that the right hand side of (3.6) converges to $E(F_o\, e^{-\int_0^t V_\tau d\tau}\, G_t)$, which proves the lemma. ∎

The interaction of section 2 ,

$$\lambda V_1 = \lambda \int_{|\vec{x}|\le 1} v(\varphi_\varepsilon(\vec{x}))d\vec{x} , \tag{3.7}$$

is of the form considered in lemma 3.2. Moreover the function V_τ in $L^\infty(d\mu_{\mathcal{H}_n^{-1}})$ of lemma 3.2 may be given explicitely in this case:

$$V_\tau = \lambda \int_{|\vec{x}|\le 1} v(\Phi_{n}^{-1}(f_{\tau,\vec{x}}))d\vec{x} , \tag{3.8}$$

where $f_{\tau,\vec{x}}(y) = \delta(\tau-y_o)\chi_\varepsilon(\vec{x}-\vec{y})$. This follows from the identification of $\varphi(g)$ with $\Phi_{\mathcal{H}_{n-1}^{-\frac{1}{2}}}(g)$ and the definition of the mapping $F \to F_t$ from $L_2(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$ into $L_2(d\mu_{\mathcal{H}_n^{-1}})$. Since

$$v(\Phi_{\mathcal{H}_n^{-1}}(f_{\tau,\vec{x}})) = U(-\tau,-\vec{x})v(f_{o,\vec{o}})U(\tau,\vec{x}) ,$$

we see that the integrand in (3.8) is continuous in $\vec{x}$ as well as in τ in the strong L_2-sense. Hence in this case lemma 3.2 takes the form

<u>Lemma 3.3</u>

Let $v(\alpha)$ be as in section 2. Then

$$(\Omega_o, F\, e^{-t(H_o+\lambda V_1)} G\Omega_o) = E(F_o\, e^{-\lambda \int_0^t \int_{|x|\le 1} v(\Phi_{\mathcal{H}_n^{-1}}(f_{\tau,\vec{x}}))d\vec{x}d\tau} G_t) \;,$$

where F and G are in $L_2(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$, and $f_{\tau,\vec{x}}(y) = \delta(\tau-y_o)\chi_\epsilon(\vec{x}-\vec{y})$. ▨

From (3.4) it follows that $\Phi_{\mathcal{H}_n^{-1}}(h)$ for $h \in \mathcal{H}_n^{-1}$ is in all L_p for $1 \le p < \infty$. For V in $L_\infty(d\mu_{\mathcal{H}_{n-1}^{-\frac{1}{2}}})$ we may therefore consider $E(\Phi(h_1)\ldots \Phi(h_n)\, e^{-\int_a^b V_\tau d\tau})$, where we have written $\Phi(h)$ for $\Phi_{\mathcal{H}_n^{-1}}(h)$. Take $h_1,\ldots,h_n$ in $C_o^\infty(\mathbb{R}^n)$ and set $g_i^t(\vec{x}) = h_i(t,\vec{x})$.

Then $h_i(x_o,\vec{x}) = \int \delta(x_o-t)g_i^t(\vec{x})dt$ and the integrand $\delta(x_o-t)g_i^t(\vec{x})$ is strongly continuous in $\mathcal{H}_n^{-1}$. Therefore if the support of h_i is bounded by the hyperplanes $x_o = a$ and $x_o = b$, then

$$E(\Phi(h_1)\ldots \Phi(h_n)\, e^{-\int_a^b V_\epsilon d\tau}) \tag{3.9}$$
$$= \frac{1}{n!}\int \cdots \int_{a\le t_1\le\ldots\le t_n\le b} E\, \Phi(W_{t_1} g_1^{t_1})\ldots\Phi(W_{t_n} g_n^{t_n})e^{-\int_a^b V_\tau d\tau})dt_1\ldots dt_n,$$

which by formula (3.5) and lemma 3.2) is equal to

$$\frac{1}{n!}\int\cdots\int_{t_1\le\ldots\le t_n} (\Omega_o, e^{-(t_1-a)H}\varphi(g_1^{t_1})e^{-(t_2-t_1)H}\varphi(g_2^{t_2})\ldots$$
$$\ldots\varphi(g_n^{t_n})e^{-(b-t_n)H}\Omega_o)dt_1\ldots dt_n \;, \tag{3.10}$$

with $H = H_o + V$.

Let E be the infimum of the spectrum of H , and set $\bar{H} = H-E$. Since V is bounded we have

$$\|\varphi(g_i^{t_i})e^{-(t_{i+1}-t_i)\bar{H}}\| \le c\|(\bar{H}+1)^{\frac{1}{2}}e^{-(t_{i+1}-t_i)\bar{H}}\| \;, \tag{3.11}$$

with C independent of t_i and i . On the other hand for any positive self-adjoint operator A we have

$$\|(A+1)^{\frac{1}{2}} e^{-tA}\| \leq \sup_{x>0} (x+1)^{\frac{1}{2}} e^{-tx} = (2t)^{\frac{1}{2}} e^{t-\frac{1}{2}} . \tag{3.12}$$

Using (3.11), (3.12) and the fact that $\varphi(g_i^t)$ is zero for t outside a bounded interval, we get that

$$(\Omega_o, e^{-(t_1-a)\bar{H}} \varphi(g_1^{t_1}) e^{-(t_2-t_1)\bar{H}} \dots \varphi(g_n^{t_n}) e^{-(b-t_n)\bar{H}} \Omega_o) \tag{3.13}$$

is bounded in absolute value uniformly in a and b by an integrable function over $t_1 \leq \dots \leq t_n$.

Let us assume that H has a simple eigenvalue at E and let Ω be the corresponding eigenvector. Then $e^{-(t_1-a)\bar{H}} \Omega_o$ as well as $e^{-(b-t_n)\bar{H}} \Omega_o$ converge to $(\Omega,\Omega_o)\Omega$ as $a \to -\infty$ and $b \to +\infty$. By (3.11) $\varphi(g_i^{t_i}) e^{-(t_{i+1}-t_i)\bar{H}}$ is a bounded operator for $t_1 < t_2 < \dots < t_n$. Hence (3.13) converges to

$$|(\Omega,\Omega_o)|^2 (\Omega, \varphi(g_1^{t_1}) e^{-(t_2-t_1)\bar{H}} \dots e^{-(t_n-t_{n-1})\bar{H}} \varphi(g_n^{t_n})\Omega) \tag{3.14}$$

as $a \to -\infty$ and $b \to +\infty$ for $t_1 < t_2 < \dots < t_n$.

From Lebesgue's dominated convergence theorem we then get that

$$(\Omega_o, e^{-(b-a)H}\Omega_o)^{-1} \cdot \int_{t_1 \leq \dots \leq t_n} \dots \int (\Omega_o, e^{-(t_1-a)H} \varphi(g_1^{t_1}) \dots \varphi(g_n^{t_n}) e^{-(b-t_n)H} \Omega_o) dt_1 \dots dt_n$$

converges to

$$\int_{t_1 \leq \dots \leq t_n} \dots \int (\Omega, \varphi(g_1^{t_1}) e^{-(t_2-t_1)\bar{H}} \dots e^{-(t_n-t_{n-1})\bar{H}} \varphi(g_n^{t_n})\Omega) dt_1 \dots dt_n$$

as $a \to -\infty$ and $b \to +\infty$. This proves the following lemma.

<u>Lemma 3.4</u>

Let $h_1, \dots, h_n \in C_o^\infty(\mathbb{R}^n)$, then

$$\lim_{t\to+\infty} \left(E\left(e^{-\int_{-t}^{t} V_\tau d\tau}\right)\right)^{-1} \cdot E\left(\Phi(h_1) \ldots \Phi(h_n) e^{-\int_{-t}^{t} V_\tau d\tau}\right)$$

$$= \frac{1}{n!} \int \ldots \int (\Omega, \varphi(g_1^{t_1}) e^{-(t_2-t_1)\bar{H}} \ldots e^{-(t_n-t_{n-1})\bar{H}} \varphi(g_n^{t_n})\Omega) dt_1 \ldots dt_n .$$

∎

Remark 1: For $V = \lambda V_1$, the interaction of section 2, this lemma holds since we know that $H_1 = H_o + \lambda V_1$ has a simple lowest eigenvalue.

Remark 2: Lemma (3.4) shows that the limit is the time imaginary Wightman function for the space cut-off interaction integrated with $h_1(x_1) \ldots h_n(x_n)$.

4. Connection with some quantities of classical statistical mechanics.

Let us denote the random variable $\Phi_{\mathcal{H}_n^{-1}}(f_{x_o,\vec{x}})$ by $\Phi_\varepsilon(x_o,\vec{x}) \equiv$ $\equiv \Phi_\varepsilon(x)$, where $f_{x_o,\vec{x}}(y) = \delta(x_o - y_o)\chi_\varepsilon(\vec{x}-\vec{y})$, and define for any bounded measurable $\Lambda \subset \mathbb{R}^n$ and for any $h_1,\dots,h_k$ in $C_o(\mathbb{R}^n)$

$$Z_\Lambda \equiv E(e^{-\lambda\int_\Lambda v(\Phi_\varepsilon(x))dx}) ,$$

$$F_\Lambda^k(h_1,\dots,h_k) \equiv E(\Phi(h_1) \dots \Phi(h_k)e^{-\lambda\int_\Lambda v(\Phi_\varepsilon(x))dx}),$$

and

$$G_\Lambda^k(h_1,\dots,h_k) = Z_\Lambda^{-1}F_\Lambda^k(h_1,\dots,h_k).$$

From lemma 3.4 we see that if we take $\Lambda = \Lambda_{t,l} \equiv$ $\equiv \{x; |x_o| \leq {}^t/_2, |\vec{x}| \leq l\}$, then the $G_{\Lambda_{t,l}}^k(h_1,\dots,h_k)$ converge for $t \to \infty$ to the imaginary time Wightman functions for the space cut-off interaction. In order to remove the space cut-off we will therefore naturally be interested in taking the limit as $l \to \infty$ as well as $t \to \infty$ in $V_{\Lambda_{t,l}}^k$. We intend by using methods from classical statistical mechanics to prove that the limit of V_Λ^k exists for Λ expanding to $\mathbb{R}^n$. This will then give us the time imaginary Wightman functions for the model without cut-off.

So let Λ be bounded. Since $v(\Phi_\varepsilon(x))$ is a bounded random variable and strongly L_2-continuous in x , Z_Λ and F_Λ are entire functions of λ . Let us set

$$F_\Lambda(h) = E(e^{i\Phi(h)}\, e^{-\lambda\int_\Lambda v(\Phi_\varepsilon(x))dx}) \quad \text{and} \quad G_\Lambda(h) = Z_\Lambda^{-1}F_\Lambda(h) .$$

Since $v(\Phi_\varepsilon(x))$ is a bounded random variable we see from the definition of $F_\Lambda(h)$ that $F_\Lambda(\sum_{i=1}^k t_i h_i)$ is k times differentiable with respect to $t_1,\dots,t_k$ and that $\frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_k} F_\Lambda(\sum_{i=1}^k t_i h_i) =$ $= (i)^k F_\Lambda^k(h_1,\dots,h_k)$ for $t_1 = t_2\dots = t_k = 0$. Hence $F_\Lambda(h)$ determines $F_\Lambda^k(h_1,\dots,h_k)$.

Since $v(\Phi_\varepsilon(x))$ is a bounded random variable, $F_\Lambda(h)$ is also an entire function of λ . By expanding in powers of λ we get

$$F_\Lambda(h) = E(e^{i\Phi(h)}) + \sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\int_{\Lambda^n}\!\!\cdots\!\int E(e^{i\Phi(h)}v(\Phi_\varepsilon(x_1))\dots v(\Phi_\varepsilon(x_n)))\prod_{j=1}^n dx_j\,.$$

Using now that $v(\alpha) = \int e^{is\alpha}d\nu(s)$ we get

$$\int_{\Lambda^n}\!\!\cdots\!\int E(e^{i\Phi(h)}v(\Phi_\varepsilon(x_1))\dots v(\Phi_\varepsilon(x_n)))\prod_{j=1}^n dx_j$$

$$= \int_{\Lambda^n}\!\!\cdots\!\int E(e^{i(\Phi(h)+\sum_{j=1}^n s_j\Phi_\varepsilon(x_j))})\prod_{j=1}^n d\nu(s_j)dx_j$$

$$= \int_{\Lambda^n}\!\!\cdots\!\int E(e^{i\Phi(h+\sum_{j=1}^n s_j f_{x_j})})\prod_{j=1}^n d\nu(s_j)dx_j\,,$$

where $f_x(y) = \delta(x_0-y_0)\chi_\varepsilon(\vec{x}-\vec{y})$ by the definition of $\Phi_\varepsilon(x)$. On the other hand, for any $g \in \mathcal{H}_n^{-1}$

$$E(e^{i\Phi(g)}) = e^{-\frac{1}{2}(g,g)_{-1}} \qquad \text{and}$$

setting

$$g = h + \sum_{j=1}^n s_j f_{x_j} \qquad \text{we get}$$

$$E(e^{i\Phi(h+\sum_{j=1}^n s_j f_{x_j})}) = E(e^{i\Phi(h)})e^{-\frac{1}{2}\sum_{i,j=1}^n s_i s_j G_\varepsilon(x_i-x_j)-\sum_{j=1}^n s_j h^\varepsilon(x_j)}\,,$$

where $G_\varepsilon(x-y) = (f_x,f_y)_{-1}$ and $h^\varepsilon(x) = (h,f_x)_{-1}$.

Hence the integral over Λ^n above is

$$E(e^{i\Phi(h)})\int_{\Lambda^n}\!\!\cdots\!\int e^{-\frac{1}{2}\sum_{i,j=1}^n s_i s_j G_\varepsilon(x_i-x_j)}\prod_{j=1}^n[(e^{-s_j h^\varepsilon(x_j)}-1)+1]\prod_{j=1}^n d\nu(s_j)dx_j\,.$$

Computing now the product and using that $\sum_{i,j}^{n} s_i s_j G_\varepsilon(x_i - x_j)$ is symmetric under permutations of $x_1 s_1, \dots, x_n s_n$, we get this equal to

$$E(e^{i\Phi(h)}) \sum_{r=o}^{n} \binom{n}{r} \cdot \int \dots \int_{\Lambda^n} e^{-\frac{1}{2}\sum_{i,j} s_i s_j \, G_\varepsilon(x_i - x_j)} \prod_{j=1}^{r} (e^{-s_j h^\varepsilon(x_j)} - 1) \prod_{j=1}^{n} d\nu(s_j) dx_j \, .$$

From this it follows that

$$F_\Lambda(h) = E(e^{i\Phi(h)}) [Z_\Lambda + \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{n=o}^{\infty} \frac{(-\lambda)^{n+r}}{n!} \int \dots \int_{\Lambda^{n+r}} e^{-\frac{1}{2}\sum_{i,j=1}^{n+r} s_i s_j G_\varepsilon(x_i - x_j)} \cdot \tag{4.1}$$

$$\prod_{j=1}^{r} (e^{-s_j h^\varepsilon(x_j)} - 1) \prod_{j=1}^{n+r} d\nu(s_j) dx_j \, ,$$

where we already have used that the expansion for Z_Λ is given by

$$Z_\Lambda = \sum_{n=o}^{\infty} \frac{(-\lambda)^n}{n!} \int \dots \int_{\Lambda^n} e^{-\frac{1}{2}\sum_{i,j=1}^{n} s_i s_j G_\varepsilon(x_i - x_j)} \prod_{j=1}^{n} d\nu(s_j) dx_j \, . \tag{4.2}$$

We remark that $G_\varepsilon(x)$ is a bounded real positive definite function, which tends to zero as $\frac{1}{|x|} e^{-m|x|}$ for $|x| \to \infty$. Since $G_\varepsilon(x)$ is positive definite, we have that $|G_\varepsilon(x)| \leq G_\varepsilon(0)$. We notice that, for negative λ, Z_Λ is in fact the grand canonical partitionfunction for a gas in n-dimensional space with variably charged particles and activity $z = -\lambda$. The interaction energy between a particle at x_i with charge s_i and a particle at x_j with charge s_j is $s_i s_j G_\varepsilon(x_i - x_j)$, and the self energy of a particle with charge s is given by $\frac{1}{2} s^2 G_\varepsilon(0)$. So the charge s is an internal degree of freedom for these particles, and s may be discrete or continuous, depending on $d\nu$. We are going to exploit this connection with the grand canonical ensemble of a gas of variably charged particles, by introducing the corresponding correlation functions and we shall see that $G_\Lambda(h)$ can be expressed explicitely by these correlation functions. [12] The correlation functions $\rho_\Lambda^k(x_1 s_1,\ldots,x_k s_k)$ are defined for $x_i \in \mathbb{R}^n$ and s_i in the support of $d\nu$ by

$$\rho_\Lambda^k(x_1 s_1,\ldots,x_k s_k) = Z_\Lambda^{-1} \sum_{n=0}^{\infty} \frac{(-\lambda)^{n+k}}{n!} \int_{\Lambda^{n+k}} \cdots \int e^{-\sum_{i,j=1}^{n+k} s_i s_j G_\epsilon(x_i - x_j)} \prod_{j=k+1}^{n} d\nu(s_j)\, dx_j \ , \tag{4.3}$$

for all $x_i \in \Lambda$ and zero elsewhere, for those values of λ for which $Z_\Lambda \neq 0$. Since $\sum_{i,j} s_i s_j G_\epsilon (x_i - x_j) \geq 0$ we see that the series converge for all complex λ . From (4.1) it follows that $G_\Lambda(h)$ is given in terms of ρ_Λ^k by

$$G_\Lambda(h) = e^{-\frac{1}{2}(h,h)_{-1}} \left[1 + \sum_{r=1}^{\infty} \frac{1}{r!} \int_{\Lambda^r} \cdots \int \prod_{j=1}^{r} \left(e^{-s_j h^\epsilon (x_j)} - 1\right) \rho_\Lambda^r(x_1 s_1,\ldots,x_r s_r) \prod_{j=1}^{r} d\nu(s_j)\, dx_j \right. . \tag{4.4}$$

As in classical statistical mechanics [13] we shall now introduce the Banach spaces B_ξ of sequences $\underset{\sim}{\psi} = \{\psi_k((xs)_k)\}_{k\geq 1} = \{\psi_k(x_1 s_1,\ldots,x_k s_k)\}_{k\geq 1}$ of bounded $dxd\nu$-measurable functions. The norm in B_ξ is given by

$$\|\underset{\sim}{\psi}\|_\xi = \sup_n \xi^{-n} \operatorname*{ess\,sup}_{\substack{x_1 \ldots x_n \\ s_1 \ldots s_n}} |\psi_n(x_1 s_1,\ldots,x_n s_n)| \ ,$$

where ξ is a positive number .

In B_ξ we define the projection operator P_Λ of norm one given by

$$(P_\Lambda \underset{\sim}{\psi})(xs)_n = \chi_\Lambda(x)_n \, \psi(xs)_n \ , \tag{4.5}$$

where $\chi_\Lambda(x)_n = \chi_\Lambda(x_1) \ldots \chi_\Lambda(x_n)$, with $\chi_\Lambda(x)$ being the characteristic function for the set Λ . Also in analogy with statistical mechanics we introduce an operator K on B_ξ given by

$$(K\underset{\sim}{\psi})(xs)_m = e^{-\sum_{i=2}^{m} s_1 s_i G_\epsilon(x_i - x_1)} \Big\{ \psi_{m-1}(x_2 s_2, \ldots, x_m s_m)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int \prod_{j=1}^{n} \Big[(e^{-s_1 t_j G_\epsilon(y_j - x_1)} - 1)\, e^{-\frac{1}{2} t_j^2 G_\epsilon(0)} \Big] \psi_{m+n-1}(x_2 s_2, \ldots \tag{4.6}$$

$$\ldots, x_m s_m, y_1 t_1 \ldots y_n t_n) \prod_{j=1}^{n} d\nu(t_j) dy_j \Big\} .$$

For $m = 1$ the first term in the curly bracket is set equal to zero.

Let $\underset{\sim}{\alpha}$ be the sequence where $\alpha_1(x_1 s_1) = 1$ and $\alpha_n(x_1 s_1, \ldots, x_n s_n) = 0$ for $n > 1$. We then verify that the sequence $\underset{\sim}{\rho}_\Lambda$ given by the correlation functions $\rho_\Lambda^n(x_1 \ldots x_n)$ satisfies the equation

$$\rho_\Lambda = -\lambda P_\Lambda \underset{\sim}{\alpha} - \lambda P_\Lambda K \rho_\Lambda . \tag{4.7}$$

Since the correlation functions $\rho_\Lambda^n(x_1 s_1, \ldots, x_n s_n)$ are symmetric, we find from (4.7) that $\underset{\sim}{\rho}_\Lambda$ also will satisfy the equation

$$\underset{\sim}{\rho}_\Lambda = -\lambda P_\Lambda \underset{\sim}{\alpha} - \lambda P_\Lambda \Pi K \underset{\sim}{\rho}_\Lambda , \tag{4.8}$$

where Π is an operator of the form

$$(\Pi \underset{\sim}{\psi})_n (x_1 s_1, \ldots, x_n s_n) = \psi_n(x_{\sigma(1)} s_{\sigma(1)}, \ldots, x_{\sigma(n)} s_{\sigma(1)}) , \tag{4.9}$$

σ being, for each n, a permutation of $1, \ldots, n$ which may depend measurably on $x_1, \ldots, x_n$ and $s_1, \ldots, s_n$.

We note that such a Π will have norm equal to one.

Since $G_\epsilon(x)$ is positive definite [14] we have that

$$\sum_{i \neq j}^{m} s_i s_j G_\epsilon(x_i - x_j) \geq -2\, G_\epsilon(0) \sum_{i=1}^{m} s_i^2 .$$

Let $B = G_\epsilon(0) \sup\{s^2 ; s \in \text{supp of } d\nu\}$; then

$$\sum_{i \neq j}^{m} s_i s_j G_\epsilon(x_i - x_j) \geq -2\, m\, B . \tag{4.10}$$

It follows from (4.10) that for any $x_1, \ldots, x_m$ and $s_1, \ldots, s_m$

there exists an index i such that

$$\sum_{\substack{j=1\\ j\neq i}}^{m} s_i s_j G_\varepsilon(x_i-x_j) \geq 2B \,. \tag{4.11}$$

For any m and any $x_1,\dots,x_m$ and $s_1,\dots,s_m$ we now choose a permutation σ of $1,\dots,m$ such that $\sigma(1) = i$ where i is the index i of (4.11). σ is then a permutation depending on the x's and the s's , and let Π be the corresponding operator on B_ξ defined by (4.9).

We now estimate the operatornorm on B_ξ of the operator ΠK of (4.8).

From (4.6) and (4.11) we have

$$|\Pi K \underset{\sim}{\psi}(xs)_m| \leq e^{2B}[\sup_{x,s}|\psi_{m-1}(x_1 s_1,\dots,x_{m-1}s_{m-1})| \,,$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n!} C^n \sup_{x,s}|\psi_{m+n-1}(x_1 s_1,\dots,x_{m+n-1},s_{m+n-1})|$$

with $C = \sup_s\{\int|e^{-stG_\varepsilon(x)}-1|e^{-\frac{1}{2}t^2 G_\varepsilon(0)} d|\nu|(t)dx; s\in \operatorname{supp} d\nu\}$.

It follows from the exponential decrease of $G_\varepsilon(x)$ that C is finite. 15)

Using now that $\sup_{x,s}|\psi_k(x_1 s_1,\dots,x_k s_k)| \leq \xi^k \|\underset{\sim}{\psi}\|_\xi$,
we get

$$|(\Pi K \underset{\sim}{\psi})_m(xs)_m| \leq e^{2B}\xi^{m-1}e^{\xi C} \,.$$

Hence

$$\|\Pi K \underset{\sim}{\psi}\|_\xi \leq \xi^{-1} e^{2B+\xi C}\|\underset{\sim}{\psi}\|_\xi \,. \tag{4.12}$$

So that $\|\Pi K\| \leq C\, e^{2B+1}$ if we choose $\xi = C^{-1}$, which is seen to be the best choice of ξ . This proves that (4.8) has a unique solution for $|\lambda| < C^{-1}e^{-2B-1}$, which then is $\underset{\sim}{\rho}_\Lambda$. From this we also get that the correlation functions $\rho^k_\Lambda(x_1 s_1,\dots,x_k s_k)$ are analytic in λ uniformly in Λ for $|\lambda| < C^{-1}e^{-2B-1}$. Moreover we may define $\rho^k(x_1s_1,\ldots,x_ks_k)$ by

$$\underset{\sim}{\rho} = -\lambda\underset{\sim}{\alpha} - \lambda\Pi K\underset{\sim}{\rho} \tag{4.13}$$

for $|\lambda| < C^{-1}e^{-2B-1}$.

Lemma 4.1

For $|\lambda| < C^{-1}e^{-2B-1}$ the infinite volume correlation functions $\rho^k(x_1s_1,\ldots,x_ks_k)$ defined as the unique solution of (4.13) exist and are analytic in λ .[16] Moreover they satisfy

$$|\rho^k(x_1s_1,\ldots,x_ks_k)| \le C^{-n}\frac{|\lambda|}{1-|\lambda|Ce^{2B+1}},$$

are continuous in $x_1,\ldots,x_k$ and $s_1\ldots s_k$, and translation invariant in the x's. The finite volume correlation functions $\rho^k_\Lambda(x_1s_1,\ldots,x_ks_k)$ converge to $\rho^k(x_1s_1,\ldots,x_ks_k)$ as $\Lambda \to \mathbb{R}^n$ such that $d(x,C\Lambda) \to \infty$ for any $x \in \mathbb{R}^n$ and $d(x,C\Lambda)$ is the distance from x to the complement of Λ . The convergence is such that

$$|\rho^k_\Lambda(x_1s_1,\ldots,x_ks_k) - \rho^k(x_1s_1,\ldots,x_ks_k)| \le C^{-n}\eta(d),$$

where η is a function that goes to zero at infinity and is independent of Λ , k and $x_1\ldots x_k$, $s_1\ldots s_k$, and $d = \min\{d(x_i,C\Lambda)\}$.

Lemma 4.2

For $|\lambda| < C^{-1}e^{-2B-1}$ we have the clusterproperty for the correlation functions:

$$\rho^{k+l}(x_1s_1,\ldots,x_ks_k,y_1+a,t_1,\ldots,y_l+a,t_l) \to \rho^k(x_1s_1,\ldots,x_ks_k)\,\rho^l(y_1t_1,\ldots,y_lt_l),$$

pointwise as a tends to infinity in $\mathbb{R}^n$.

These two lemmas are proved as in classical statistical mechanics [15] by using that $s_is_jG_\epsilon(x_i-x_j)$ corresponds to a stable and regular interaction in the language of classical statistical mechanics. The proofs require only a slight modification of the proofs given in Ref.15 Ch.4, and will therefore not be given here.

Lemma 4.3

For $|\lambda| < C^{-1}e^{-2B-1}$ the limit

$$\tilde{\varepsilon} = -\lim_{\Lambda \to \mathbb{R}^n} \frac{1}{|\Lambda|} \ln Z_\Lambda$$

exists when $\Lambda \to \mathbb{R}^n$ in the sense that $d(x, C\Lambda)$ tends to infinity for all $x \in \mathbb{R}^n$. Moreover $\tilde{\varepsilon}(\lambda)$ is analytic [17] in λ for $|\lambda| < C^{-1}e^{-2B-1}$ and

$$\tilde{\varepsilon}(\lambda) = -\int_0^\lambda \frac{1}{\tilde{\lambda}} \rho^1(xs;\tilde{\lambda})d\tilde{\lambda}\, d\nu(s),$$

where $\rho^1(xs,\lambda)$ is the correlation function with one argument. For $\lambda < 0$ and $d\nu$ a positive measure we have that ρ^1 is positive which gives us that $\tilde{\varepsilon}$ is negative. Moreover in this case $\tilde{\varepsilon}$ exists also for all $\lambda < 0$, decreases when $|\lambda|$ increases and is also concave in $\ln(-\lambda)$.

Proof: From the expansion of Z_Λ ((4.2)) and the expansion of ρ^1_Λ ((4.3)) we find that

$$\frac{d}{d\lambda} \ln Z_\Lambda = \frac{1}{\lambda}\int_\Lambda \rho^1_\Lambda(xs;\lambda)d\nu(s)dx .$$

From lemma 4.1 we have that $\frac{1}{\lambda}\rho^1_\Lambda(xs,\lambda)$ is uniformly bounded and analytic in λ for $|\lambda| < C^{-1}e^{-2B-1}-\delta$, for any $\delta > 0$. Moreover $|\rho^1_\Lambda(x,s;\lambda) - \rho^1(x,s;\lambda)| \leq C^{-1}\eta(d)$, and hence it follows that

$$\frac{1}{|\Lambda|}\int_\Lambda d\nu(s)dx \int_0^\lambda \frac{1}{\tilde{\lambda}} \rho^1_\Lambda(x,s;\tilde{\lambda})d\tilde{\lambda}$$

converges uniformly for $|\lambda| < C^{-1}e^{-2B-1}-\delta$ to

$$\iint_0^\lambda \frac{1}{\tilde{\lambda}} \rho^1(x,s;\tilde{\lambda})d\nu(s)d\tilde{\lambda} ,$$

since $\rho^1(x,s;\tilde{\lambda})$ is independent of x. This proves that $\frac{1}{|\Lambda|}\ln Z_\Lambda$ converges as $\Lambda \to \mathbb{R}^n$ and that the limit $-\check{\varepsilon}$ is given by the formula of the lemma. That ρ^1 is positive for $d\nu \geq 0$ and $\lambda \leq 0$ follows from the fact that $\rho^1_\Lambda \geq 0$, which one sees from (4.3). The existence of $\tilde{\varepsilon}$ for all $\lambda < 0$ in this case follows from the identification, possible in this case, of Z_Λ with a grand canonical partition function for a system with stable and tempered interactions, (see [15], p.157).
The decrease of $\tilde{\varepsilon}$ as $|\lambda|$ increases follows from the increase of Z_Λ.

Remark: All the series expansions for the $\rho^r(x_1 s_1,\ldots,x_r s_r)$ and therefore also for $\tilde{\varepsilon}$ in powers of λ can be explicitely obtained from (4.13) and are given by

$$\underset{\sim}{\rho} = -\lambda(1+\lambda\Pi K)^{-1}\underset{\sim}{\alpha} = -\lambda\sum_{n=0}^{\infty}(-\lambda)^n(\Pi K)^n\underset{\sim}{\alpha} \;. \tag{4.14}$$

5. Removal of the space cut-off for the imaginary time Wightman functions and the vacuum energy density.

Let $\Lambda_{t,l} = [-{}^t/2, {}^t/2] \times \{\vec{x} \mid |\vec{x}| \leq l\} \subset \mathbb{R}^n$ and set $Z_{t,l} \equiv Z_{\Lambda_{t,l}}$ and $F_{t,l} = F_{\Lambda_{t,l}}$ and $V_{t,l} = V_{\Lambda_{t,l}}$. It then follows from lemma 3.3 that

$$Z_{t,l} = (\Omega_o, e^{-tH_l}\Omega_o) \;, \tag{5.1}$$

with $H_l = H_o + \lambda \int_{|x|\leq l} v(\varphi_\varepsilon(\vec{x}))d\vec{x}$. From (3.8) and (3.9) we have, for $h_1,\ldots,h_k$ with support in $\Lambda_{t,l}$, that

$$F^k_{t,l}(h_1,\dots,h_k) = \frac{1}{k!}\int\cdots\int_{t_1\le\cdots\le t_k}(\Omega_o, e^{-(t_1+t/2)H_l}\varphi(\vec{x}_1)e^{-(t_2-t_1)H_l}\varphi(\vec{x}_2)\dots\varphi(\vec{x}_k)e^{-(t/2-t_k)H_l}\Omega_o) \tag{5.2}$$
$$h_1(t_1,\vec{x}_1)\dots h_k(t_k,\vec{x}_k)\prod_{j=1}^k dt_j d\vec{x}_j ,$$

and

$$G^k_{t,l} = Z^{-1}_{t,l}\, F^k_{t,l} . \tag{5.3}$$

By lemma (3.4) the limit as $t \to +\infty$ of $G^k_{t,l}$ exists and is given by

$$G^k_l(h_1,\dots,h_k) = \frac{1}{k!}\int\cdots\int_{t_1\le\cdots\le t_k}(\Omega_l,\varphi(\vec{x}_1)e^{-(t_2-t_1)\bar{H}_l}\dots e^{-(t_k-t_{k-1})\bar{H}_l}\varphi(\vec{x}_k)\Omega_l) \tag{5.4}$$
$$h_1(t_1,\vec{x}_1)\dots h_k(t_k,\vec{x}_k)\prod_{j=1}^k dt_j d\vec{x}_j ,$$

where Ω_l is the unique normalized eigenvector with eigenvalue E_l and E_l is the infimum of the spectrum of H_l, and $\bar{H}_l = H_l - E_l$. The integration over $d\vec{x}_j$ in (5.2) and (5.4) is to be understood in the sense of distributions. After integrating with respect to $\prod_{j=1}^k d\vec{x}_j$ in (5.2) and (5.4), the result is a function of $t_1,\dots,t_k$ that is translation invariant, continuous in $t_1<\dots<t_k$ and integrable over $t_1\le\dots\le t_k$. This follows from the proof of lemma (3.4). We see from (5.4) that $G^k_l(h_1,\dots,h_k)$ are the imaginary time Wightman functions (also called Schwinger functions) for the space cut-off interaction.

<u>Theorem 5.1</u> Let $|\lambda| < C^{-1}e^{-2B-1}$ and $h_1,\dots,h_k$ be in C^∞_o. Then the $G^k_l(h_1,\dots,h_k)$ converge as $l \to \infty$ to $G^k(h_1,\dots,h_k)$, where $G^k(h_1,\dots,h_k)$ are translation invariant in t and $\vec{x}$ and given by

$$G^k(h_1,\dots,h_k) = G_o^k(h_1,\dots,h_k) + (i)^k k! \sum_{r=1}^{k} \sum_{\substack{p+q=k \\ q\geq r, p\geq o}} \frac{(i)^p}{p!} \tag{5.5}$$

$$\sum_{\sigma\in S_k} G_o^k(h_{\sigma(1)},\dots,h_{\sigma(p)}) \sum_{\substack{l_1+\dots+l_r=q \\ l_i\geq 1}} \frac{1}{l_1!\dots l_r!} \int\dots\int \prod_{i=1}^{r} \left[s_i^{l_i} \prod_{j=1}^{l_i} h^{\epsilon}_{\sigma(p+l_1+\dots+l_i-j+1)}(x_i)\right]$$

$$\cdot\ \rho^r(x_1s_1,\dots,x_rs_r) \prod_{j=1}^{r} d\nu(s_j)dx_j\ .$$

S_k is the set of permutations of $1,\dots,k$ and the $G_o^k(h_1,\dots,h_k)$ are the imaginary time free Wightman functions: $G_o^k(h_1,\dots,h_k) = E(\Phi(h_1)\dots\Phi(h_k)) = \frac{1}{2^p p!} \sum_{\sigma\in S_k} (h_{\sigma(1)},h_{\sigma(2)})_{-1}\dots(h_{\sigma(2p-1)},h_{\sigma(2p)})_{-1}$ for $k = 2p$ and zero for k odd. $\rho^r(x_1s_1,\dots,x_rs_r)$ is the infinite volume correlation function of lemma 4.1, and $h_i^{\epsilon}(x) = \int G_{\epsilon}(\vec{x}-\vec{y})h_i(x_o,\vec{y})d\vec{y}$ and $G_\epsilon(x)$ is the $G_\epsilon(x)$ of section 4, which is given by

$$G_\epsilon(x) = \int \frac{e^{ipx}\ \tilde{\chi}_\epsilon^2(\vec{p})}{p^2+m^2}\, dp\ ,$$

with $\tilde{\chi}_\epsilon(\vec{p}) = \int e^{i\vec{p}\vec{x}}\chi_\epsilon(\vec{x})d\vec{x}$.

Proof: It follows from (4.4) and the fact that $G_\Lambda(\sum_{i=1}^{k} t_ih_i)$ is analytic in $t_1,\dots,t_k$ that the formula (5.5) with $G_\Lambda^k(h_1,\dots,h_k)$ instead of $G^k(h_1,\dots,h_k)$ and with $\rho_\Lambda^r(x_1s_1,\dots,x_rs_r)$ instead of $\rho^r(x_1s_1,\dots,x_rs_r)$ holds. Choosing now $\Lambda = \Lambda_{t,l}$ we have by (5.4) that $G_{t,l}^k(h_1,\dots,h_k)$ converges to the limit $G_l^k(h_1,\dots,h_k)$ as $t\to\infty$. On the other hand by lemma 4.1 $\rho^r_{\Lambda_{t,l}}(x_1s_1\dots x_rs_r)$ is uniformly bounded in $x_1,\dots x_r, t, l$ and tends to a limit $\rho^r(x_1s_1,\dots,x_rs_r)$ uniformly on compacts as t and l tend to infinity. Since $h_i^\epsilon(x)$ $i = 1,\dots,k$ are all bounded integrable functions we get by dominated convergence from (5.5), with $G^k_{t,l} = G^k_{\Lambda_{t,l}}$ and $\rho^k_{\Lambda_{t,l}}$ instead of G^k and ρ^r, that $G^k_{t,l}(h_1,\dots,h_k)$ converges to the limit $G^k(h_1,\dots,h_k)$ given by (5.5) as t and l tend to infinity.

Consider now the inequality

$$|G^k_l(h_1,\dots,h_k) - G^k(h_1,\dots,h_k)|$$

$$\leq |G^k_l(h_1,\dots,h_k) - G^k_{t,l}(h_1,\dots,h_k)| + |G^k_{t,l}(h_1,\dots,h_k) - G^k(h_1,\dots,h_k)|.$$

Choose $\epsilon > 0$; then there exists a N_ϵ such that for any $t \geq N_\epsilon$ and any $l \geq N_\epsilon$ the last term is smaller than $\epsilon/2$. Choose an $l \geq T_\epsilon$. Then for this value of l we may choose a $t \geq N_\epsilon$ and large enough so that the first term is smaller than $\epsilon/2$. Then for $l \geq T_\epsilon$ we get $|G^k_l(h_1,\dots,h_k) - G^k(h_1,\dots,h_k)| \leq \epsilon$.
This proves the theorem.

<u>Theorem 5.2</u> Let $|\lambda| < C^{-1}e^{-2B-1}$, and let $h_1,\dots,h_k$, $g_1,\dots,g_l$ be in $C_0^\infty(\mathbb{R}^n)$.
Let $g_i^a(x) = g_i(x-a)$ for $a \in \mathbb{R}^n$. Then we have the following cluster properties:

$$G^{k+l}(h_1,\dots,h_k,g_1^a,\dots,g_l^a) \longrightarrow G^k(h_1,\dots,h_k)G^l(g_1,\dots,g_l)$$

as $|a| \to \infty$.

Proof: It follows from (5.5) that for any $h \in C_0^\infty(\mathbb{R}^n)$

$$G(h) = 1 + \sum_{k=1}^{\infty} \frac{(i)^k}{k!} G^k(h,\dots,h) \quad \text{is defined}$$

and the series is absolutely convergent. Remembering that (5.5) was obtained by means of (4.4), we get

$$G(h) =$$

$$e^{-\frac{1}{2}(h,h)_{-1}}\Big[1+\sum_{r=1}^{\infty}\frac{1}{r!}\int\cdots\int\prod_{j=1}^{r}(e^{-s_j h^{\epsilon}(x_j)}-1)\rho^{r}(x_1 s_1,\dots,x_r s_r)\prod_{j=1}^{r}d\nu(s_j)dx_j\Big]. \tag{5.6}$$

Therefore

$$G(h^a+g^{-a}) = e^{-\frac{1}{2}(h,h)_{-1}}\cdot e^{-\frac{1}{2}(g,g)_{-1}}e^{-(h^a,g^{-a})_{-1}} \tag{5.7}$$

$$\Big[1+\sum_{r=1}^{\infty}\frac{1}{r!}\int\cdots\int\prod_{j=1}^{r}(e^{-s_j h^{\epsilon}(x_j-a)-s_j g^{\epsilon}(x_j+a)}-1)\rho^{r}(x_1 s_1,\dots,x_r s_r)\prod_{j=1}^{r}d\nu(s_j)dx_j.$$

We observe that $(h^a,g^{-a})_{-1}\to 0$ as $|a|\to\infty$. By writing each of the integrals over x_i as the sum of the integrals over $x_i\cdot a\le 0$ and $x_i\cdot a\ge 0$, we get that the r'th term of the series above is equal to

$$\frac{1}{r!}\sum_{s=o}^{r}\binom{r}{s}\int_{x_j\cdot a\ge o}\cdots\int\cdot\int_{y_j\cdot a\le o}\cdots\int\prod_{j=1}^{s}(e^{-s_j h^{\epsilon}(x_j-a)-s_j g^{\epsilon}(x_j+a)}-1)$$

$$\cdot\prod_{j=1}^{r-s}(e^{-t_j h^{\epsilon}(y_j-a)-t_j g^{\epsilon}(y_j+a)}-1)\rho^{r}(x_1 s_1\dots x_s s_s,y_1 t_1\dots y_{r-s}t_{r-s})\cdot \tag{5.8}$$

$$\cdot\prod_{j=1}^{s}d\nu(s_j)dx_j\cdot\prod_{j=1}^{r-s}d\nu(t_j)dy_j\ .$$

From the definition of $h^{\epsilon}(x)=\int G_{\epsilon}(x-y)h(y)dy$ we get that $|h^{\epsilon}(x)|\le C\,e^{-m|x|}$, from which we obtain that $|h^{\epsilon}(y-a)|\le C\,e^{-\frac{m}{2}|a|}\cdot e^{-\frac{m}{2}|y|}$ for $y\cdot a\le 0$ and simirlarly $|g^{\epsilon}(x+a)|\le C\,e^{-\frac{m}{2}|a|}e^{-\frac{m}{2}|x|}$ for $x\cdot a\ge 0$. By the substitution $x_j\to x_j+a$ and $y_j\to y_j-a$ we get

$$\frac{1}{r!}\sum_{s=o}^{r}\binom{r}{s}\int_{x_j\cdot a\ge -a^2}\cdots\int\cdot\int_{y_j\cdot a\le a^2}\cdots\int\prod_{j=1}^{s}(e^{-s_j h^{\epsilon}(x_j)-s_j g^{\epsilon}(x_j+2a)}-1)$$

$$\cdot\prod_{j=1}^{r-s}(e^{-t_j h^{\epsilon}(y_j-2a)-t_j g^{\epsilon}(x_j)}-1)\rho_a^{r}(x_1 s_1\dots x_s s_s,y_1 t_1\dots \tag{5.9}$$

$$\dots y_{r-s}t_{r-s})\Pi d\nu(s_j)dx_j\ \Pi d\nu(t_j)dy_j\ ,$$

where $\rho_a^r(x_1,\ldots,x_s,y_1,\ldots,y_{r-s}) = \rho^r(x_1+a,\ldots,x_s+a,y_1-a,\ldots,y_{r-s}-a)$.
Let $F_a(x,s)$ be any measurable function uniformly bounded in a; then

$$\int\limits_{x\cdot a\geq -a^2}(e^{-sh^\epsilon(x)-sg^\epsilon(x+2a)}-1)F_a(x,s)dxd\nu(s)-\int\limits_{x\cdot a\geq -a^2}(e^{-sh^\epsilon(x)}-1)F_a(x,s)dxd\nu(s) \tag{5.10}$$

converges to zero when $|a|\to\infty$, because the absolute value of (5.10) is bounded by

$$\int\limits_{x\cdot a\geq -a^2} e^{-sh^\epsilon(x)}|e^{-sg^\epsilon(x+2a)}-1|\;|F_a(x,s)|dxd\nu(s)$$

$$\leq A\int\limits_{x\cdot a\geq -a^2}|e^{-sg^\epsilon(x+2a)}-1|\;dxd\nu(s)$$

$$\leq B\int\limits_{x\cdot a\geq -a^2}g^\epsilon(x+2a)dx = B\int\limits_{x\cdot a\geq o}g^\epsilon(x+a)dx$$

$$\leq B\cdot C\, e^{-\frac{m}{2}|a|}\int e^{-\frac{m}{2}|x|}dx\,.$$

Therefore for any $\epsilon > 0$ there exists an R_ϵ such that, for $|a| > R_\epsilon$, (5.9) will differ from (5.11) by an amount smaller than $\epsilon/2$:

$$\frac{1}{r!}\sum_{s=0}^{r}\int\limits_{x_j\cdot a\geq -a^2}\!\!\ldots\!\int\cdot\int\limits_{y_j\cdot a\leq a^2}\!\!\ldots\!\int\prod_{j=1}^{s}(e^{-s_jh^\epsilon(x_j)}-1)\prod_{j=1}^{r-s}(e^{-t_jg^\epsilon(y_j)}-1)$$
$$\rho_a^r(x_1s_1,\ldots,x_ss_s,y_1t_1,\ldots,y_{r-s}t_{r-s})\prod_{j=1}^{s}d\nu(s_j)dx_j\prod_{j=1}^{r-s}d\nu(t_j)dy_j\,. \tag{5.11}$$

By dominated convergence and lemma 4.2 we have that (5.11) converges to (5.12) as $|a|\to\infty$:

$$\frac{1}{r!}\sum_{s=0}^{r}\binom{r}{s}\int\cdots\int\prod_{j=1}^{s}(e^{-s_j h^{\epsilon}(x_j)}-1)\rho^{s}(x_1 s_1,\ldots,x_s s_s)\prod_{j=1}^{s}d\nu(s_j)dx_j$$
$$\cdot\int\cdots\int\prod_{j=1}^{r-s}(e^{-t_j g^{\epsilon}(y_j)}-1)\rho^{r-s}(y_1 t_1,\ldots,y_{r-s}t_{r-s})\prod_{j=1}^{r-s}d\nu(t_j)dy_j \;. \tag{5.12}$$

From (5.7) and the translation invariance we now get that

$$G(h+g^{a}) \to G(h)G(g) \quad \text{as} \quad |a| \to \infty . \tag{5.13}$$

Since $G(\sum_i t_i h_i+\sum_j s_j g_j^a)$ is analytic in t and s and converges to $G(\sum_i t_i h_i)\cdot G(\sum_j s_j g_j^a)$, we have only to use that convergence of analytic functions implies the convergence of the coefficients of their powerseries to prove that $G^{k+l}(h_1,\ldots,h_k,g_1^a,\ldots,g_l^a)$ converges to $G^k(h_1,\ldots,h_k)G^l(g_1,\ldots,g_l)$. ∎

<u>Theorem 5.3</u> Set $G^k(h_1,\ldots,h_k) = \int\cdots\int G^k(x_1,\ldots,x_k)h_1(x_1),\ldots$ $\ldots,h_k(x_k)dx_1,\ldots,dx_k$, then $G^k(x_1,\ldots,x_k)$ is locally integrable and continuous for $x_i \neq x_j$, for all $i \neq j$. The singularities at $x_i = x_j$ are of the same form as the singularities of $G_o^k(x_1,\ldots,x_j)$. Moreover the $G^k(x_1,\ldots,x_j)$ are translation invariant and, for $\chi_\epsilon(\vec{x})$ rotational invariant, they are also invariant under rotations in $\mathbb{R}^{n-1}$. The $G^k(x_1,\ldots,x_k)$ depend analytically on λ for $|\lambda| < C^{-1}e^{-2B-1}$. 18)

<u>Proof</u>: This follows from (5.5) and the analyticity of the $\rho^r(x,s_1,\ldots,x_r s_r)$ as proved in lemma 4.1. ∎

<u>Theorem 5.4</u> For all $|\lambda| < C^{-1}e^{-2B-1}$ we have that the vacuum energy density

$$\tilde{\varepsilon} = \lim_{l\to\infty}|B_l|^{-1}E_l$$

exists, where $|B_l|$ is the volume of the n-1 - dimensional ball of radius 1. Moreover this limit is equal to the $\tilde{\varepsilon}$ of lemma 4.3 and is therefore analytic in λ for all $|\lambda| < C^{-1}e^{-2B-1}$ and its power series is given in terms of the powerseries for ρ^1 by

$$\tilde{\varepsilon}(\lambda) = -\int_0^\lambda \frac{1}{\tilde{\lambda}} \rho^1(x,s;\tilde{\lambda})d\tilde{\lambda}d\nu(s) .$$

The power series for all ρ^r hence also for $\tilde{\varepsilon}$ are explicitely given by (4.14). [19)]

$\tilde{\varepsilon}(\lambda)$ is a concave function of λ.
For $d\nu$ a positive measure and all $\lambda < 0$ (not necessarily $> -C^{-1}e^{-2B-1}$) the limit $\tilde{\varepsilon}$ exists, is negative, decreasing for $|\lambda|$ increasing, concave in λ and $\ln(-\lambda)$.

Proof: By lemma 4.3 $|\Lambda_{t,l}|^{-1}\ln Z_{t,l}$ converges to $\tilde{\varepsilon}$ as t and l tend to infinity. We observe that $|\Lambda_{t,l}| = t\cdot|B_l|$. By (5.1) $Z_{t,l} = (\Omega_o, e^{-tH_l}\Omega_o)$. E_l is a simple lowest eigenvalue of H_l so that $\frac{1}{t}\ln(\Omega_o, e^{-tH_l}\Omega_o) \to -E_l$ as $t \to \infty$. Therefore

$$|\Lambda_{t,l}|^{-1}\ln Z_{t,l} \to -|B_l|^{-1}\cdot E_l$$

as $t \to \infty$. Since $|\Lambda_{t,l}|^{-1}\ln Z_{t,l}$ converges to $\tilde{\varepsilon}$ for t and l tending to infinity it now follows that $|B_l|^{-1}E_l$ converges as $l \to \infty$ to $\tilde{\varepsilon}$ of lemma 4.3. That $\tilde{\varepsilon}$ is concave in λ follows from $\tilde{\varepsilon}$ being the limit of $-|B_l|^{-1}E_l$ and E_l being the lowest eigenvalue of $H_l = H_o + \lambda V_l$ is concave in λ. The rest of the theorem is contained in lemma 4.3. ∎

6. The vacuum, the interacting fields and the Wightman functions.

Let $\alpha_t^l(A) = e^{-itH_l} A e^{itH_l}$ for any bounded operator A on $\mathcal{F}$ and let α_t^o be the corresponding one parameter group of C*-automorphisms defined with H_o instead of H_l. Let $\mathcal{A}_o$ be the algebra of free quasi local observables defined as the C*-algebra generated by all $\alpha_t^o(e^{i\varphi(f)})$ for all t and all $f \in \mathcal{H}_{n-1}^{-\frac{1}{2}}$ with compact support, where $\varphi(f) = \int \varphi(\vec{x})f(\vec{x})d\vec{x}$, with $\varphi(\vec{x})$ given by (2.1).

<u>Theorem 6.1</u> α_t^l as well as α_t^o are one parameter groups of C*-automorphisms of $\mathcal{A}_o$. Moreover α_t^l converges strongly on $\mathcal{A}_o$ to a one parameter group of automorphisms α_t of $\mathcal{A}_o$ as $l \to \infty$. We also have that $\alpha_{-t}^o\alpha_t^l$ and $\alpha_t^l\alpha_{-t}^o$ converge strongly to $\alpha_{-t}^o\alpha_t$ resp. $\alpha_t\alpha_{-t}^o$, uniformly on an open interval containing $t = 0$.

The proof of this theorem is contained in Ref.[6d]. We remark that even though theorem 6.1 gives the existence of an infinite volume time automorphism α_t of the quasi local algebra $\mathcal{A}$ we do not know if α_t is strongly continuous in t. [20)]

Let now ω_l be the state on $\mathcal{A}_o$ given by the vector $\Omega_l \in \mathcal{F}$. Then ω_l is invariant under α_t^l.

<u>Theorem 6.2</u> Let $\tilde{\omega}$ be any weak limit point of $\{\omega_l\}$ in the statespace of $\mathcal{A}_o$. Then $\tilde{\omega}$ is an invariant state for α_t. Let $\mathcal{H}_{\tilde{\omega}}$ be the representation space for $\mathcal{A}_o$, generated by $\tilde{\omega}$; then the unitary group $U_{\tilde{\omega}}(t)$ generated by α_t in $\mathcal{H}_{\tilde{\omega}}$ is strongly continuous, and its infinitesimal generator $H_{\tilde{\omega}}$ satisfies $H_{\tilde{\omega}} \geq 0$.

Proof: Let $\tilde{\omega}$ be the weak limit of ω_{l_n} where $l_n \to \infty$ as $n \to \infty$, and B be in $\mathcal{A}_o$.

Then

$$\begin{aligned}\tilde{\omega}(\alpha_t B) - \tilde{\omega}(B) &= \tilde{\omega}(\alpha_t B) - \omega_{l_n}(\alpha_t B) \\ &+ \omega_{l_n}(\alpha_t B) - \omega_{l_n}(\alpha_t^{l_n} B) \\ &+ \omega_{l_n}(B) - \tilde{\omega}(B) \quad ,\end{aligned}$$

since ω_{l_n} is invariant under $\alpha_t^{l_n}$. The first and the third term converge to zero by the weak convergence and the second term converges to zero by the strong convergence of α_t^l.

Let A and B be in $\mathcal{A}_o$ and let $\Omega \in \mathcal{H}_{\tilde{\omega}}$ be a unit vector that corresponds to the state $\tilde{\omega}$. Then $(A\Omega, U_{\tilde{\omega}}(t)B\Omega) = \tilde{\omega}(A^*\alpha_t B)$, where A^* is the adjoint of A. By the same procedure as above we prove that

$$\omega_{l_n}(A^*\alpha_t^{l_n} B) \longrightarrow \tilde{\omega}(A^*\alpha_t B) \quad . \tag{6.1}$$

Since $\omega_l(A^*\alpha_t^l B) = (A\Omega_l, e^{itH_l} B\Omega_l)$ we have that $\omega_l(A^*\alpha_t^l B)$ is uniformly bounded in t and l and continuous in t. Furthermore it is analytic and uniformly bounded in l and t in the upper half plane $\mathrm{Im}\, t > 0$. From (6.1) it follows that $\tilde{\omega}(A^*\alpha_t B)$ are measurable and uniformly bounded functions with respect to t as pointwise limits of uniformly bounded continuous functions.

Let Φ and Ψ be in $\mathcal{H}_{\tilde{\omega}}$. Since $\mathcal{A}_o\Omega$ is dense in $\mathcal{H}_{\tilde{\omega}}$ we can find sequences A_n and B_n in $\mathcal{A}_o$ such that $A_n^*\Omega \to \Phi$ and $B_n\Omega \to \Psi$ strongly. Then $(\Phi, U_{\tilde{\omega}}(t)\Psi)$ is the limit of $\tilde{\omega}(A_n\alpha_t B_n)$ $= (A_n^*\Omega, U_{\tilde{\omega}}(t)B_n\Omega)$ and is therefore a bounded measureable function of t. Hence $U_{\tilde{\omega}}(t)$ is a weakly measurable unitary group on $\mathcal{H}_{\tilde{\omega}}$, thus it is strongly continuous. [21]

Let $f(t)$ be a smooth function having a Fourier transform $\tilde{f}$

with support on $t < 0$. Since $\omega_l(A\alpha_t^l B)$ is analytic and uniformly bounded for $\operatorname{Im} t > 0$, we have that $\int \omega_{l_n}(A\alpha_t^{l_n} B) f(t) dt = 0$. Moreover for $n \to \infty$ this integral converges by dominated convergense to $\int \tilde{\omega}(A\alpha_t B) f(t) dt$. Hence $H_{\tilde{\omega}} \geq 0$ and this proves the theorem. ∎

<u>Remark</u>: Since the statespace of a C*-algebra is weakly compact, we know that weak limit points of $\{\omega_l\}$ exist.

Let $\mathcal{A}$ be the algebra of interacting quasi local observables defined as the C*-algebra generated by $\alpha_t(e^{i\varphi(f)})$, $f \in \mathcal{H}_{n-1}^{-\frac{1}{2}}$ with compact support. It follows from the invariance of $\mathcal{A}_o$ under α_t that $\mathcal{A} \subseteq \mathcal{A}_o$.

<u>Theorem 6.3</u> For $|\lambda| < C^{-1} e^{-2B-1}$ we have that the restriction of any weak limit point of $\{\omega_l\}$ to $\mathcal{A}$ is unique, and ω_l converges weakly on $\mathcal{A}$ to a unique state ω on $\mathcal{A}$. Moreover ω is invariant under α_t and space translations, and α_t is implemented by a unitary group e^{itH_ω} on the representation space $\mathcal{H}_\omega$ of $\mathcal{A}$ contstructed from ω . One has $H_\omega \geq 0$. ω depends analytically on λ for $|\lambda| < C^{-1} e^{-2B-1}$.

Proof: We know that

$$\begin{aligned} &\omega_l(\alpha_{t_1}^l(e^{i\varphi(f_1)}) \ldots \alpha_{t_n}^l(e^{i\varphi(f_n)})) \\ &= (\Omega_l, e^{i\varphi(f_1)} e^{i(t_2-t_1)\bar{H}_l} \ldots e^{i(t_n-t_{n-1})\bar{H}_l} e^{i\varphi(f_n)} \Omega_l) \end{aligned} \tag{6.2}$$

are analytic in the upper half plane for $t_j - t_{j-1}$. For all $t_j - t_{j-1}$ on the positive imaginary axis, we see by (5.4) that (6.2) is given in terms of the G_l's . By theorem (5.1) the G_l's converge as $l \to \infty$, which implies that (6.2) converges as $l \to \infty$ for all $t_j - t_{j-1}$ positive imaginary. Since (6.2) is uniformly bounded in l and analytic in $\mathrm{Im}(t_j - t_{j-1}) > 0$, we get that there is a unique limit function. By theorem 6.1 it is of the form $\tilde{\omega}(\alpha_{t_1}(e^{i\varphi(f_1)}) \ldots \alpha_{t_n}(e^{i\varphi(f_n)}))$, where $\tilde{\omega}$ is a weak limit point of $\{\omega_l\}$. Since $\alpha_t(e^{i\varphi(f)})$ generates $\mathcal{A}$, we get that the restriction of any limit point to $\mathcal{A}$ is unique and that ω_l converges weakly to ω on $\mathcal{A}$. The rest of the theorem follows from theorem 6.2, the fact that $\mathcal{A}$ is invariant under α_t and the fact that the G's are translation invariant functions, analytically dependent on λ. ∎

<u>Theorem 6.4</u> The functions $G_l^k(x_1,\ldots,x_k)$ are analytic functions of $\xi_1 = x_2^o - x_1^o, \ldots, \xi_{k-1} = x_k^o - x_{k-1}^o$ for $\mathrm{Re}\,\xi_i > 0$ and the boundary values on the imaginary axis $W_l^k(x_1,\ldots,x_k)$ are the Wightman functions for the space cut-off interaction. Moreover for $|\lambda| < C^{-1}e^{-2B-1}$ the $W_l^k(x_1,\ldots,x_k)$ converge for $l \to \infty$ in the sense of distributions [22] to unique limits $W^k(x_1,\ldots,x_k)$, which depend analytically on λ and are the Wightman functions given by ω and α_t. $W^k(x_1,\ldots,x_k)$ is translation invariant in space and time. Moreover it is rotational invariant in space if $\chi_\varepsilon(\vec{x})$ is taken rotational invariant.

Proof: From (6.2) and (5.4) we get the identification of the Wightman functions with the boundary values of $G_l^k(x_1,\ldots,x_k)$. The existence of the limit as $l \to \infty$ follows from the existence of the limit for G_l^k and the same goes for the analyticity in λ. That the limits are the Wightman functions defined by ω and α_t follows from the previous theorems. The invariance of W^k follows from the invariance of G^k as proved in theorem 5.3. ∎

Theorem 6.5 For $|\lambda| < C^{-1}e^{-2B-1}$ let the vacuum $\Omega \in \mathcal{H}_\omega$ be a vector that corresponds to the infinite volume vacuum state ω. Then Ω is the only translation invariant state of $\mathcal{H}_\omega$, moreover zero is a simple eigenvalue of H_ω with eigenvector Ω. Furthermore the Wightman functions have the clusterproperties with respect to space translations

$$W^{k+1}(x_1+a,\ldots,x_k+a,y_1,\ldots,y_l) \longrightarrow W^k(x_1,\ldots,x_k)W^l(y_1,\ldots,y_l)$$

in the sense of distributions [22] as $|\vec{a}| \to \infty$, with $a = (0;\vec{a})$.

Proof: From the fact that the formula obtained from (5.4) by taking the limit $l \to \infty$ holds, we get e^{-tH_ω} expressed in terms of the infinite volume G-functions. The cluster properties of the G-functions as given in theorem 5.2 then prove that Ω is the only eigenvector with eigenvalue zero of H_ω. This comes from the cluster-properties in the time direction. The cluster-properties in the space direction give that Ω is the only vector invariant under space translations. The cluster-properties of the Wightman functions are a direct consequence of the cluster properties of the G-functions in the space direction. This proves the theorem. ∎

Remark: Since $|B_l|^{-1}E_l \to \tilde{\varepsilon}$ by theorem 5.4 for $|\lambda| < C^{-1}e^{-2B-1}$ and the interaction V_l is bounded in norm by a constant times $|B_l|$, we have that

$$|(\Omega_l, H_o\Omega_l)| \leq C_1|B_l| .$$

This inequality and the fact that ω_l tends, as $l \to \infty$, to the translation invariant state ω, can be used to prove, along the lines of [17], that ω is locally Fock.

Footnotes

1) See e.g. [1] and the references given therein.

2) See e.g. [1], [2] and the references given therein.

3) See e.g. [1], [2], [3]. See also footnote 5) below.

4) This has been proven also for the two-dimensional Yukawa interaction [5].

5) After completion of this paper we learned in a private communication from J. Glimm that for the polynomial interactions in two space-time dimensions without cut-offs he and collaborators (J. Dimock and D. Spencer) have solved the problem of the uniqueness of the vacuum for small coupling constants. As far as we know this has been done by methods different from the one we use in the present paper.

6) These models are related to the bounded interaction models studied in [6]. They are, in a sense, an Hamiltonian version of certain "non polynomial interactions" studied in recent years from other points of view. See e.g. [7].

7) Note that, due to the presence of the ultraviolet cut-off, no Wick ordering of the interaction is required. In fact our interactions

$$\int e^{is\varphi_\varepsilon(\vec{x})} d\nu(s)d\vec{x}$$

and the correspondent Wick-ordered ones

$$\int :e^{is\varphi_\varepsilon(\vec{x})}: d\nu_1(s)d\vec{x}$$

can be made to coincide by choosing $d\nu_1(s) = \exp(\frac{1}{2}s^2K)d\nu(s)$, where K is a constant (equal to the value for $x = 0$ of the propagator $G_\varepsilon(x)$ defined below).

8) This analogy has been exploited from a different point of view particularly in the references [8] (and references quoted therein) and [7b,c].

9) See also [10], where a euclidean Markov field theoretical

relation is exploited to prove the uniqueness of the vacuum energy density and the van Hove phenomenon for two-dimensional polynomial interactions. For further results on this infinite volume behaviour, see [11]. For references concerning work previous to Nelson's one, see [8].

10) E_1 and Ω_1 are obtained in [6c] as the unique (norm) limits of the lowest eigenvalues and respective eigenvectors of suitable approximating Hamiltonians ("piecewise constant momentum approximation").

11) See e.g. [12].

12) The correlation functions of similar "euclidean gases of charged particles" associated with field theoretical models have been introduced, in another context, in references [7b] and [7c].

13) These spaces have been introduced in classical statistical mechanics by Ruelle in order to study the infinite volume limit of the correlation functions in the grand canonical ensemble. Here and in the rest of this section we shall follow closely the lines of classical statistical mechanics as given in Ruelle's book, Ref. [15], Ch.4. This reference contains also bibliographical notes on previous work on the infinite volume limit of correlation functions.

14) Using the analogy with classical statistical mechanics this can be interpreted as the fact that the total interaction of our gas in $\mathbb{R}^n$ satisfies the "stability condition" of [15].

15) This can be interpreted as the fact that the interaction of our gas in $\mathbb{R}^n$ satisfies the "regularity condition" of [15].

16) Their expansions in powers of λ are given as Liouville-Neumann series with known kernel: see Remark at the end of the section 4.

17) For all $|\lambda| < C^{-1}e^{-2B-1}$ (and all $d\nu$) the expansions of ρ^1 and $(-\tilde{\varepsilon})$ in powers of λ are the Mayer series $\Sigma(nb_n)(-\lambda)^n$ resp. $\Sigma b_n(-\lambda)^n$ for the "density" respectively

"pressure" of our "gas" in $\mathbb{R}^n$. Hence information on the expansion coefficients is readily available (see e.g. [15], p.84-86).
Note that the well known virial expansion (of the pressure in powers of the density) corresponds in our case to an expansion of $-\tilde{\varepsilon}$ in powers of ρ^1, expansion which can be obtained by inverting the expansion of ρ^1 in powers of λ in a neighborhood of $\lambda = 0$ (which is possible since $\lim_{\lambda \to 0} \frac{\rho^1}{\lambda} = 1$).

18) The coefficients in the expansion of $G^k(x_1,\dots,x_k)$ in powers of λ can be obtained from those of the expansions of the ρ's in powers of λ, using (5.5). The latter are known and given by (4.14) (see also footnote 17)).

19) The expansion for $(-\tilde{\varepsilon}(\lambda))$ is the Mayer power series expansion $\Sigma b_n(-\lambda)^n$ for the "pressure" of our gas as a function of $(-\lambda)$. Hence the coefficient are the quantities b_n of footnote 17)), which can be computed explicitely.

20) For additional results on the α_t^1 see [6b], Lemma 4.

21) See e.g. [16], §138, p.382-383.

22) E.g. in $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n)$ being the Schwartz'space of distributions over $\mathcal{D}(\mathbb{R}^n) \equiv C_0^\infty(\mathbb{R}^n)$. But the test function space can also be chosen to be more general, as can be seen from the preceding proofs.

References

[1] J. Glimm - A. Jaffe, Quantum Field Theory Models, in Statistical Mechanics and Quantum Field Theory, Les Houches Summer School, 1970, Ed. by De Witt and Stora, Gordon & Breach, 1971.

Also: J. Glimm - A. Jaffe, Boson Quantum Field Models, 1971 London Lectures, Ed. by R. Streater (to appear).

[2] J. Glimm - A. Jaffe, The $\lambda \Phi_2^4$ Quantum Field Theory without Cutoffs IV. Perturbation of the Hamiltonian, Preprint 1972.

[3] R. Streater, On the connection between spectrum condition and Lorentz invariance, Commun.Math.Phys. 26, 109-120 (1972).

[4] R. Høegh-Krohn, A general class of quantum fields without cut-offs in two space-time dimensions, Commun.math.Phys. 21, 244-251 (1971).

[5] R. Schrader, Yukawa quantum field theory in two space-time dimensions without cut-offs, Annals of Physics, 70, 412-457 (1972).

[6] a) R. Høegh-Krohn, Boson fields under a general class of cut-off interactions, Commun.math.Phys. 12, 216-225 (1969).

b) R. Høegh-Krohn, Boson fields under a general class of local relativistic invariant interactions, Commun.math.Phys. 14, 171-184 (1969).

c) R. Høegh-Krohn, Boson fields with bounded interaction densities, Commun.math.Phus. 17, 179-193 (1970).

d) R.F. Streater - I.F. Wilde, The time evolution of quantized fields with bounded quasi-local interaction density, Commun,math.Phys. 17, 21-32 (1970).

[7] a) G.V. Efimov, Essentially non linear interaction Lagrangians and non localized quantum field theory, Theor. and Mathem. Phys. 2, 26-40 (1970).

b) D. Ya Petrina - V.I. Skripnik, Kirkwood-Salzburg equations for the coefficient functions of the scattering matrix, Theor. and Mathem. Phys., 8, 896-903 (1971).

c) D. Fivel, Construction of unitary, covariant S matrices defined by convergent perturbation series, Phys.Rev. D, 4, 1653-1662 (1971).

[8] K. Symanzik, Lectures on euclidean quantum field theory, in Local Quantum Theory, International School of Physics "Enrico Fermi",Varenna, Ed. by R. Jost, Academic, N.Y.,(1969).

[9] a) E. Nelson, Lectures on probability theory and applications to quantum mechanical problems given at Princeton University, 1970-71.

b) E. Nelson, Quantum fields and Markov fields, Proc. 1971 AMS Summer Conference; and papers in preparation.

[10] F. Guerra, Uniqueness of the vacuum energy density and Van Hove phenomenon in the infinite volume limit for two-dimensional self-coupled Bose fields, Phys.Rev.Lett. 28, 1213-1215 (1972).

[11] F. Guerra, L. Rosen and B. Simon, Nelson's symmetry and the infinite volume behaviour of the vacuum in $P(\Phi)_2$, Commun. math.Phys. 27, 10-22 (1972).

[12] I.M. Gelfand - N. Ya Vilenkin, Generalized Functions, Vol.4, Academic Press, New York (1964).

[13]a) I. Segal, Tensor algebras over Hilbert spaces I, Trans.Am. Mathem. Soc. 81, 106-134 (1956).

b) B. Simon - R. Høegh-Krohn, Hypercontractive semigroups and two-dimensional self-coupled Bose fields, J.Funct.Analys. 9, 121-180 (1972).

[14] R. Høegh-Krohn, Infinite dimensional analysis with applications to self-interacting boson fields in two space time dimensions, Talk given at Aarhus Conference on Functional Analysis, Spring 1972.

[15] D. Ruelle, Statistical Mechanics, Rigorous Results, W.A. Benjamin, Inc. New York, (1969). Especially ch.4.

[16] F. Riesz - B.Sz Nagy, Leçons d'analyse fonctionnelle, III éd., Ac. des Sciences de Hongrie (1955).

[17] J. Glimm - A. Jaffe, The $\lambda(\Phi^4)_2$ quantum field theory without cutoffs. III. The physical vacuum, Acta Mathem., 125, 203-267 (1970).