A NOTE ON SOME APPROXIMATION PROBLEMS

Let $D = \{z : |z|<1\}$, $T = \{z : |z| = 1\}$ and $V \subset T$ be open relative to T. $H_V^\infty(D)$ denotes all bounded continuous functions on $D \cup V$ being analytic in D. The following two results are known to be true for $H_V^\infty(D)$.

1 <u>There exists an open set $O \supset D \cup V$ such that each $f \in H_V^\infty(D)$ can be uniformly approximated on D by functions analytic in O.</u>

It is of course not possible to choose the approximating functions bounded in O but the following is true:

2 <u>Let $K \subset D \cup T$ be compact such that $K \cap V = \emptyset$. Given $\varepsilon > o$ there exists a constant C and an open set $O \supset D \cup V$ such that for each $f \in H_V^\infty(D)$ there exists g analytic in O such that: $|f(z) - g(z)| < \varepsilon$ if $z \in K$ and $\sup_{z \in O} |g(z)| \leq C \sup_{z \in D} |f(z)|$</u>

The first of these results appeared in [S 1].

The second is contained in the proof of Theorem 3.1 in [S 2] . In this note we give some new examples of spaces of analytic functions satisfying 1) and 2). We shall use the technique from [S 1] together with some classical results of Hardy - Littlewood and M. Riez. It should be mentioned that 1) has been considerable generalized. See [R.1].

Let u be a real valued integrable function on T. We define

$$H_u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u(\theta)\, d\theta \quad \text{if } z \text{ is not in the}$$

closes support of u. In the following A shall be a space of functions on $D \cup T$ such that:

i) A is a Banach algebra under some norm $\| \|_A$ and contains the constants and the identity map.

ii) The function $f_r(z) = f(rz)$ is in A whenever $f \in A$ and $0<r<1$ and $\|f - f_r\|_A \to 0$ as $r \to 1$.

iii) If u is continuously differentiable with compact support on V and $f \in A$ then $u \cdot \mathrm{Re} f = \mathrm{Re} h$ on T for some $h \in A$ such that $h(z) = H_{(\mathrm{Re} h)}(z)$ if $z \in D$.

iv) $D \cup T$ is the maximal ideal space of A.

THEOREM 1

<u>Assume $\emptyset \neq V \neq T$. Given $\varepsilon > 0$ and $f \in A$ there exists g analytic in $\mathbb{C} \setminus (T \setminus V)$ such that $g|_{D \cup T} \in A$ and $\|f - g|_{T \cup D}\|_A < \varepsilon$.</u>

<u>Remark</u>: Theorem 1 applies to the following examples:

i) The space B_p of all continuous functions f on $D \cup T$ analytic in D such that $f'(e^{i\theta}) = \frac{d}{d\theta}(f(e^{i\theta}))$ belongs to the Hardy space H^p. (Here $1 \leq p \leq \infty$). We have $\|f\|_{B_p} = \|f\|_\infty + \|f'\|_p$.

ii) The space of all analytic functions in D with a Taylor series about the origin being absolutely convergent.

For B_p we have the following if K, V is as above and $\emptyset \neq V \neq T$:

THEOREM 2

Assume $1<p<\infty$ and $\varepsilon>0$ is given. Then there exists a constant C depending only on p and an open set $O \supset D \cup V$ such that for each $f \in B_p$ there exists g analytic in O satisfying:

i) $|f(z) - g(z)| < \varepsilon \cdot \|f\|$ if $z \in K \cap D$

ii) $g|_{D \cup T} \in B_p$ and $\|g|_{D \cup T}\| + \sup_{z \in O} |g(z)| \leq C \|f\|$

PROOF OF THEOREM 1

Write $V = \bigcup_1^\infty V_j$ where $\text{dist}(T \setminus V, V_j) > 0$ for each j and $V_j \cap V_k = \emptyset$ if $|j-k|>1$. We also assume $K \cap V_j \neq \emptyset$ only for a finite number of the V_j's if $K \subset V$ is compact.

Choose continuously differentiable nonnegative functions u_j supported on V_j such that $\Sigma u_j(z) = 1$ if $z \in V$. If $f \in A$ put $f_j = H_{(u_j \, \text{Re} f)}$ and $h_j = (f_j)_{r_j}$ where r_j is choosen such that (*): $\|f_j - h_j\|_A < \varepsilon \cdot 2^{-j}$ for j = 1, 2 ....

We now choose rational functions g_j with poles only at the singularities of h_j approximating h_j good near $D \cup T$ such that (*) holds if we replace h_j by g_j for j = 1.2 .... .

The function (1) $h = f - \sum_1^\infty (f_j - g_j)$ satisfies the theorem except for the poles of the g_j's. (We can clearly arrange it such that the series (1) converges on compact subsets of $\mathbb{C} \setminus (T \setminus V)$ away from the poles of the g_j's).

But for k = 1, 2, ... we can define rational functions $\{g_{jk}\}_{j=1}^\infty$ with their poles contained in

$\{z : |z| > 1 + \frac{1}{k}$ & $\text{dist}(T \setminus V_j, z) < 1 + \frac{2}{k}\}$ such that

$\|g_j - g_{jk}\|$ is as small as we please.

It is easy now to see that g can be defined by

$$g = \lim_{k \to \infty} [f - \sum_j (f_j - g_{kj})] .$$

PROOF OF THEOREM 2:

Let U_j and V_j be as above and define $O_j = \{z: 1-t_j < |z| < 1+t_j$ & $\frac{z}{|z|} \in V_j\}$ where $\{t_j\} \subset (0, \frac{1}{2})$ is a sequence converging to zero.

We can arrange it such that $\text{dist}(T \setminus V, O_j) > 0$ for each j. Choose $f \in B_p$ with $\|f\| \le 1$ and define $f_j = H(u_j \cdot \text{Re} f')$. Let $h_j = (f_j)_{r_j}$. We can choose the numbers r_j depending only on $\text{dist}(O_j, \text{supp } u_j)$ such that

$|h_j(z) - f_j(z)| \le \varepsilon 2^{-j} \|f'\|_p \le \varepsilon 2^{-j}$ if $z \in \mathbb{C} \setminus O_j$. Now consider $b = f' + \sum_j (h_j - f_j)$. Fix $z' \in V$. We can find a closed disc $\Delta_{z'}$ and numbers j_1 and j_2 such that $u_{j_1} + u_{j_2} = 1$ near $\Delta_{z'}$ and $\Delta_{z'} \cap O_j = \emptyset$ if $j \ne j_1, j_2$. We write

(2): $b = (f' - f_{j_1} - f_{j_2}) + (h_{j_1} + h_{j_2}) + \sum_{j \ne j_1, j_2} (h_j - f_j) = F_1 + F_2 + F_3$. say.

If w is an interior point of $\Delta_{z'}$ or in D, we define $b_1(w) = \int^w b(z)dz + f(0)$. By a well known theorem of M. Riez [Du. Thm. 4.1. p. 54] we have $\|F_1\|_p + \|F_2\|_p \le K_p \|f'\|_p \le K_p$ where K_p depends only on p. Since $\text{Re} F_1 \equiv 0$ near $\Delta_{z'} \cap T$ we have $F_1(z_o) = \overline{F_1(z_o^*)}$ if z_o and z_o^* are conjugate points w.r.t. T. Integrating along a straight line from o to w we get $\int_o^w |F_1'(z)| dz \le 2 \int_{-w_o}^{w_o} |F_1(z)| dz \le \|F_1\|_p$ where $w_o = \frac{w}{|w|}$ and the last inequality follows from the Fejer-Riez inequality [Du, Thm 3.13 p. 46]. We can assume that the radius of $\Delta_{z'}$ is so small that h_{j_1} and h_{j_2} are analytic near $\Delta_{z'}$. As for F_1 we now get

$|\int_o^w F_2(z)dz| \le C_p \|f'\|_p \le C_p$ where C_p depends only on p. Since $|F_3| \le \varepsilon$ on the line segment from o to w we have $|b_1(z)| \le M_p \|f'\|_p \le M_p$ where M_p depends only on p.

Choosing a locally finite covering $\{\Delta_i\}$ of discs like $\Delta_{z'}$ containing V we put $O = D \cup (\bigcup_i (\Delta_i^o)$ where Δ_i^o is the interior of Δ_i^o. On O we have $|b_1(z)| \leq M_p \|f'\|_p \leq M_p$ where M_p depends only on p. Finally we consider $\|b_1\| \leq M_p + \|b\|_p$. Choose $z \in \Delta_{z'} \cap V$. We have $|\operatorname{Re} b(z)| \leq \varepsilon + 2 |\operatorname{Re} f'(z)| + F(z)$ where $F(z) = \sup_{r<1} f'(rz)$. By a theorem of Hardy - Littlewood $\| F \|_p \leq C'_p \|f'\|_p$ where C'_p depends only on p. But then $\|b\|_p \leq C''_p \|f'\|_p$ (C''_p depending only on p).

Q.E.D.

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