QUADRATIC HEDGING IN A FUEL AND ELECTRICITY FORWARD MARKET
-based on a structural spot price model-

by

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Contents

Introduction

Electricity is a paramount good for households and a key-element in nearly all production or commercial processes. Before actually being consumed, electricity must be generated, transported across a transmission network and finally distributed to users. Electricity is produced by transforming the energy stored by fuels (i.e. hydro, nuclear, fossil or other renewable resources) into electric power. This is done in power stations and then is transported through an electricity infrastructure. This can be owned by the state or by other independent transmission system operators (in deregulated markets). Afterwards, electricity is supplied to users by supply companies.

In order to liberalize the electricity trade, there have been included bilateral contracts and gross pool systems. The main characters in this type of pool systems are either those with proper facilities for production or consumption (system operators, producers, distributors) or traders, brokers, clearing companies, financial analysts etc. For instance, Nord Pool is the Nordic and Baltic power market. On this type of market, standard contracts of electricity are traded in an auction system with physical delivery (the real time and the day-ahead markets), while there is also the possibility to establish financial contracts for investment purposes.

On the one hand, the spot price is decided hourly upon balancing supply and demand. On the day-ahead market, for example, sellers and buyers equally submit their bids one day ahead and a system computes, roughly speaking, the intersection between sell and buy price curves.

On the other hand, the forward or futures contracts have the particularity of guaranteeing the delivery of electricity over a period of time instead of a fixed date in the future.

The relationship between forward and spot prices is important in managing the risk and in establishing an efficient energy market. However, it is difficult to have a tight connection spot-forward price. Since electricity is difficult to store, forward prices would have to be determined by balancing supply and demand for the precise delivery period. Fluctuations over time can lead to quite a significant difference between current spot and forward prices due to the impossibility of product inventories. Also, natural factors such as temperature or rain can trigger variations in demand, thus rendering spot and forward prices highly volatile by themselves. These disparities can lead to situations like contango [1] or normal backwardation [2].

These aspect being considered, one may claim that not only are electricity spot and forward prices volatile, but also their relationship. From the price determination point

[1] forward prices being above the expected spot prices and thus hedging future price risk being costly for suppliers

[2] forward prices being below the expected spot prices and thus hedging future price risk becoming costly for providers

of view, the forward price depends on expectations of supply and demand at delivery period, while the current spot price depends on current supply and demand. Hence, current prices can only enhance limited information on the cost of entering a forward contract.

As no direct correlation can be made, one may determine them separately, either the spot price from the forward price by aid of HJM model, for example ([2]) or the forward price from the spot price under a risk-neutral probability, for example ([1]).

Electricity is considered a "flow commodity", meaning that it is only useful when delivered during precise periods of time, hence their prices are averages over certain time intervals. It is a non-storable commodity, which makes it compulsory to generate it, transmit it and supply it to the users when needed. This makes it impossible to store it and to yield it conveniently, thus hedging becomes a more complex endeavor. These features render the electricity market an incomplete one. Mathematically speaking, this means that in such a market not all the claims can be attainable, meaning that they cannot be perfectly replicated by a self-financing portfolio. In this case, one can try to find trading strategies with final payoff as close as possible to the initial claim (for example, quadratic hedging strategies).

The present thesis is concerned with determining two types of quadratic hedging strategies (locally risk minimizing and mean-variance) for a particular model introduced by Aïd et al. (2013)[1]. This provides a spot price model which considers generation costs per fuel, capacity of production and demand. From this model we derive forward prices of fuels and electricity and we solve the hedging problem in a market where we trade on fuels and electricity contracts both in the calendar time and in a time-changed setting.

The novelty brought by this work resides in the way we deal with quadratic hedging (local-risk minimization, in particular) in a time-changed setup of our model. More precisely, we find the local risk-minimizing strategy as a solution to a BSDE problem and, in parallel, by adapting the Fölmer-Schweizer decomposition strategy. The thesis is structured as follows.

Chapter 1 introduces the spot price model and some notation on its components. Thereafter, the forward prices of fuels and electricity are derived. Chapter 2 settles the preliminaries on which quadratic hedging is based. Here, a way of constructing the minimal martingale and the variance-optimal measures is presented. Then it is explained in all its generality the connection between the Galtchouk-Kunita-Watanabe, the Föllmer Schweizer decomposition and the solution for the local risk minimizing and the mean-variance hedging problems.

Chapter 3 deals with the actual problem of hedging in the model of Aïd et al.(2013)[1]. The minimal martingale measure and the Föllmer Schweizer decomposition are given constructively. The solution of the local-risk minimization problem is given both in a closed form and as a solution to a backwards stochastic differential equation. Similarly, the mean-variance hedging strategy is given.

The time scale for the electricity and fuels forward contracts can be reset from the actual calendar time to the business time. Chapter 4 introduces the concept of time-change of a Lévy process. In Chapter 5 we proceed to changing the time scale for all the prices discussed so far. In addition, the quadratic hedging strategies are re-analyzed in this context. A new way of tackling the hedging problem is developed via BSDEs and, in parallel, via a structural argument.

All things considered, the present thesis aims at giving different perspectives on quadratic hedging in an incomplete market as the electricity market. It lays its basis on the spot price model of Aïd et al. (2013)[1] from where tractable ways of hedging are determined. This is done in a closed and a BSDE form equally with respect to the calendar time and under an absolutely continuous time-change.

Chapter 1

A model for spot and forward prices

As a commodity, electricity is somehow different. It is considered a flow-commodity because it is only useful when is delivered during a precise period of time. In addition, it is not storable, thus the spot electricity is not tradable and the market is incomplete. The actual tradable assets are future/forward contracts with delivery time over a time horizon $[0, T]$, rather than over a fixed date in time.

In this paper, we will be referring to Aïd et al. (2013) [1] as working approach. This provides us with a model of the spot price, where generation costs per fuel, capacity of production and demand come into place. Then, forward and futures prices are deducted. Further on, local risk minimization and mean-variance hedging will be applied to the subsequent hedging problem.

1.1 Spot price

Definition 1. A *spot price* is the current price at which the commodity can be bought or sold at a particular time.

Particularly, the electricity spot price is the half-hourly price of wholesale market electricity. The latter employs the spot prices for each trading period in order to schedule the available mean of generation such that the lower-cost would be dispatched first.

In order to introduce the spot price model of Aïd et al. (2013) [1], we must establish the following notations.

Notation

- P_t: electricity spot price at time t
- n: number of available fuels for generating electricity
- $S_t := (S_t^1, S_t^2 \dots S_t^n)$: fuel prices at time t
- $(h_1, h_2 \dots h_n)$: heating coefficients for the corresponding fuel
- C_t^i: capacity of the i-th technology to produce energy at time t
- $\bar{C}_t^{\,i} := \sum_{j \leq i} C_t^i$: total capacity for the first i fuels at time t

- $I_t^1 := (-\infty, C_t^1), I_t^j := [\bar{C_t}^{j-1}, \bar{C_t}^j], I_t^n := [\bar{C_t}^{n-1}, \infty)$: production intervals
- D_t : demand at time t
- r: the spot interest rate assumed to be a positive constant

Electricity generation costs are ranked according to a fixed and known relation:

$$h_1 S_t^1 \leq \ldots \leq h_n S_t^n$$

Consider the maximum production capacity as:

$$C_t^{max} = \bar{C_t}^n = \sum_{j \leq n} C_t^j$$

If $D_t \in I_t^i$, then the electricity will be produced with the first i fuels.

$C_t^{max} - D_t$ represents *the margin capacity* of the electricity production system and indicates the tension in the system.

Previous to introducing the spot price model, let us define the *scarcity function*:

$$g(x) = \min(M, \frac{\gamma}{x^{\nu}}) Id_{\{x \geq 0\}} + M Id_{\{x \leq 0\}} \tag{1.1}$$

where $\gamma, M, \nu > 0$ are constant parameters estimated from the data sets of the market. $g(C_t^{max} - D_t)$ models the effect of the scarcity upon the prices. Also, g as it is defined is said to give quite a good calibration of the price spikes occurring on the electricity market.

According to Aïd et al. (2013) [1], we define the electricity spot price as follows:

$$P_t(C_t, D_t, S_t, M, \gamma, \nu) = g(C_t^{max} - D_t) \sum_{i=1}^{n} h_i S_t^i Id_{\{D_t \in I_t^i\}} \tag{1.2}$$

Note that in this model the margin capacity comes into place via the scarcity function g.

1.2 Futures and forward contracts

Definition 2. A *futures contract* is an agreement to sell or buy an asset at some future date in time at a specified price which is determined by the law of supply and demand. The futures‘s prices are settled on a daily basis and made public in the financial press. Futures contracts include both a daily cash settlement (reflecting the difference between an agreed price and the variations on the market) and a financial cash settlement when the contract expires.

Definition 3. A *forward contract* is an agreement to sell or buy an asset for a pre-established price K. The contract is signed at time 0 for a certain future time T (called delivery date or maturity). In the case of an electricity market, the forward contract locks in the price and the time period of delivery.

As similar as these two types of contracts might seem, there are several differences between them that we shall point out.

- **The underlying instruments** are well stated in a futures contract, whereas in the forward case these are numerous and every one has its own negotiated price.
- **The delivery** of a futures is done at a standard date of the year at an pre-established location, while for the forward contract this is negotiated by the two parties and it can occur anywhere and at any time.
- **The price** for a futures are established with a daily price limit upon an open auction and they are made public. In the case of a forward contract, the price may vary and they is unbounded. They are established directly by the buyer and the seller.
- **The trading setting** for a futures is settled at the exchange floor at exact times, whereas for a forward the trading can take place by way of direct negotiation between the two parties worldwide and at any time.
- **The cash settlements** and the daily revaluations of the positions are handled by a central clearing house which assumes the risk, in the case of a futures contract. On the contrary, for a forward contract, there is no adjustment for daily fluctuations, hence no central clearing house. Here parties handle the risk themselves.
- **The volume of the transactions** is made public in the case a futures contract, while for a forward, this is not available.
- **The market liquidity** is high on the market of futures contracts, whereas due to variable contract terms and conditions, the forward contract market has a limited liquidity.

Following this distinction between forward and futures contracts, we shall further set on deriving a pricing formula for futures and forward contracts starting from Eq (1.2). As our interest rate r is assumed to be constant, futures and forwards are identical in our setup.

1.3 The dynamics of fuels prices, capacity and demand

We proceed to modeling the processes given in the spot price (1.2), namely capacity, demand and fuel prices. Only the latter component is tradable, hence our market will be incomplete.

Assumption 1. *Consider the filtered probability space* $(\Omega, \mathcal{F}, \mathbf{P})$*, where* $\mathbf{P}$ *denotes the physical probability measure. Assume that the market filtration is* $\mathcal{F}_t := \mathcal{F}_t^S \vee \mathcal{F}_t^C \vee \mathcal{F}_t^D$*, where* $\mathcal{F}_t^S$*,* $\mathcal{F}_t^C$ *and* $\mathcal{F}_t^D$ *are the natural filtrations generated by the mutually independent Brownian motions governing the dynamics of fuel prices* ($W^{S,i}$)*, capacity* ($W^{C,i}$)*, respectively demand* (W^D)*.*

Notation

- The superscript $\tilde{\cdot}$ denotes discounting.
- $E[\cdot]$ denotes the expectation under the physical probability $\mathbf{P}$

1.3.1 Production capacity

The capacity process C^i for fuel i, with $1 \leq i \leq n$ is adapted to the filtration $\mathcal{F}^C$ generated by the n-dimensional Brownian motion $(W^{C,i})_{1\leq k\leq n}$. The **P**-dynamics of capacity is assumed to be:

$$dC_t^i = \alpha_i(t, C_t^i)dt + \beta_i(t, C_t^i)dW_t^{C,i} \tag{1.3}$$

where $\alpha_., \beta_. : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ are measurable functions such that (1.3) has a unique strong solution[1].

1.3.2 Electricity demand

Assume that the dynamics of demand process adapted to $\mathcal{F}^D$ is given by:

$$dD_t = a(t, D_t)dt + b(t, D_t)dW_t^D \tag{1.4}$$

where $a_., b_. : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ are measurable functions such that (1.3) has a unique strong solution[2].

1.3.3 Fuel prices

Let $Y_t^i = h_i S_t^i - h_{i-1} S_t^{i-1}$ be the spread process on fuels, with $1 \leq i \leq n$ and S^0 is assumed to be equally null. Then we have that

$$S_t^i = \frac{1}{h_i} \sum_{j\leq i} Y_t^j \tag{1.5}$$

The **P**-dynamics of the fuel spreads are:

$$dY_t^i = \mu_i Y_t^i dt + \sigma_i Y_t^i dW_t^i \tag{1.6}$$

where W is an n-dimensional Brownian motion and $\mu_. \in \mathbf{R}^n, \sigma_. \in \mathbf{R}_+^n$ are fixed.

By making use of (1.5) and applying Itô‘s formula, we get the dynamics of S_t^i:

$$dS_t^i = \frac{1}{h_i} \sum_{j\leq i} (\mu_j Y_t^j dt + \sigma_j Y_t^j dW_t^j) \tag{1.7}$$

which can further be written as

$$dS_t^i = \mu_i^{S,i} S_t^i dt + \sigma_t^{S,i} S_t^i dW_t^{S,i} \tag{1.8}$$

In the latter expression, we make the notations:

$$\begin{aligned}
\mu_t^{S,i} &= \sum_{j\leq i} \frac{Y_t^j}{h_i S_t^i} \mu_j \\
\sigma_t^{S,i} &= \sqrt{\sum_{j\leq i} (\frac{Y_t^j}{h_i S_t^i})^2 \sigma_j^2} \\
dW_t^{S,i} &= \frac{1}{\sigma_t^{S,i}} \sum_{j\leq i} \frac{Y_t^j}{h_i S_t^i} dW_t^j
\end{aligned} \tag{1.9}$$

[1]see Appendix A.1

[2]see Appendix A.1

Note that the filtrations generated by W^S and W are the same. We prefer working under the form (1.7) for discounting reasons.

The $\mathbf{P}$-dynamics of the discounted fuel price process $\tilde{S}_t^i = e^{-rt}S_t^i$ are given by:

$$d\tilde{S}_t^i = e^{-rt}\Big[-rS_t^i + \frac{1}{h_i}\sum_{j\le i}\mu_j Y_t^j\Big]dt + \frac{e^{-rt}}{h_i}\sum_{j\le i}\sigma_j Y_t^j dW_t^j \tag{1.10}$$

Now, let $\mathbf{Q}^i$ be the (unique) equivalent martingale measure for each Y_t^i. Call $\lambda_i := \frac{\mu_i - r}{\sigma_i}$ the market price of risk and $\tilde{Y}_t^i = e^{-rt}Y_t^i$ the discounted spread of i-th fuel. By Itô, the $\mathbf{Q}^i$-dynamics of $\tilde{Y}_t^i$ is:

$$d\tilde{Y}_t^i = e^{-rt}\sigma_i Y_t^i d\hat{W}_t^i \tag{1.11}$$

where, by Girsanov's theorem, $\hat{W}$ is an n-dimensional Brownian motion given by $d\hat{W}_t^i = \lambda_i dt + dW_t^i, \;\; i = 1..n$.

The $\mathbf{Q}^i$-dynamics of the discounted fuel prices $\tilde{S}_t^i$ are:

$$d\tilde{S}_t^i = \frac{e^{-rt}}{h_i}\sum_{j\le i}\sigma_j Y_t^j d\hat{W}_t^j \tag{1.12}$$

Forward price of fuels

The fuel forward with maturity at time T (in practice, a period of time) is

$$F_t^i(T) = e^{r(T-t)}S_t^i \tag{1.13}$$

By Itô's formula, we get the dynamics of fuel forward prices with maturity $T > 0$

$$dF_t^i(T) = e^{r(T-t)}(-rS_t^i + \frac{1}{h_i}\sum_{j\le i}\mu_j Y_t^j)dt + e^{r(T-t)}\frac{1}{h_i}\sum_{j\le i}\sigma_j Y_t^j dW_t^j \tag{1.14}$$

Electricity forward price

Call the filtration generated by production capacity(C) and demand(D) $\mathcal{F}^{D,C} = \mathcal{F}^D \vee \mathcal{F}^C$. Then the price of a forward contract on electricity derived from (1.2) is given by the expectation under a risk-neutral measure $\mathbf{Q}$.

$$F_t^e(T) = E_{\mathbf{Q}}[e^{-r(T-t)}P_T|\mathcal{F}_t]$$

meaning

$$F_t^e(T) = \sum_{i=1}^{n} h_i F_t^i(T) E_{\mathbf{Q}}[g(C_T^{max} - D_t)\mathbf{1}_{\{D_T\in I_T^i\}}|\mathcal{F}_t^{D,C}] \tag{1.15}$$

This pinpoints the advantage to work on an extended market, where we can trade on fuels and electricity altogether.

We define the *conditional expectation of scarcity function (CES)* as

$$G_i^T(t, C_t, D_t) := E_{\mathbf{Q}}[g(C_T^{max} - D_t)\mathbf{1}_{\{D_T\in I_T^i\}}|\mathcal{F}_t^{D,C}] \tag{1.16}$$

And by replacing (1.16) into (1.15), we obtain the price of a forward contract

$$F_t^e(T) = \sum_{i=1}^{n} h_i G_i^T(t, C_t, D_t) F_t^i(T) \tag{1.17}$$

as a basket of fuel forwards.

In the sequel we work with $\mathbf{Q} = \hat{\mathbf{P}}$, where $\hat{\mathbf{P}}$ is defined in (3.1). This measure satisfies that $\hat{\mathbf{P}} = \mathbf{P}$ as it is proved further in (3.6). Hence, the above expectations will be under the physical measure.

Focusing on deriving the $\mathbf{P}$-dynamics of $F_t^e(T)$, we should note first that as fuels are independent of production capacity and demand we can write

$$dG_i^T(t, C_t, D_t) = \sum_{k=1}^{n} \frac{\partial G_i^T}{\partial c_k}(t, C_t, D_t)\beta_k(t, C_t^k) dW_t^{C,k} + \frac{\partial G_i^T}{\partial d}(t, C_t, D_t) b(t, D_t) dW_t^D \tag{1.18}$$

By making use of (1.17) and Itô's formula, we obtain the $\mathbf{P}$-dynamics of $F_t^e(T)$ as follows

$$dF_t^e(T) = \sum_{i=1}^{n} h_i [G_i^T(t, C_t, D_t) dF_t^i(T) + F_t^i(T) dG_i^T(t, C_t, D_t)]$$

As a final step, insert (1.18) into the dynamics of $F_t^e(T)$ and get

$$\begin{aligned} dF_t^e(T) = & \underbrace{e^{r(T-t)} \sum_{i=1}^{n} (\sum_{k=1}^{n} G_k^T(t, C_t, D_t)(\mu_i - r) Y_t^i \, dt}_{\theta^{drift}} \\ & \underbrace{+ e^{r(T-t)} \sum_{i=1}^{n} (\sum_{k=1}^{n} G_k^T(t, C_t, D_t) \sigma_i Y_t^i \, dW_t^i}_{\theta^S} \\ & \underbrace{+ \sum_{i=1}^{n} h_i F_t^i(T) \frac{\partial G_i^T}{\partial d}(t, C_t, D_t) b(t, D_t) \, dW_t^D}_{\theta^D} + \\ & \underbrace{+ \sum_{i=1}^{n} h_i F_t^i(T) \sum_{k=1}^{n} \frac{\partial G_i^T}{\partial c_k}(t, C_t, D_t) \beta_k(t, C_t^k) \, dW_t^{C,k}}_{\theta^C} \end{aligned} \tag{1.19}$$

which can be written as:

$$dF_t^e(T) = \theta_t^{drift} dt + \theta_t^S \cdot dW_t + \theta_t^C \cdot dW_t^C + \theta_t^D dW_t^D$$

considering $\cdot$ as the dot product.

Chapter 2

Quadratic hedging preliminaries

The idea of hedging was developed according to game strategies. It consists of constructing a portfolio that would replicate at best the asset‘s terminal payoff at time T. We denote the *portfolio strategy* by $\varphi = (V_t, \theta_t)$, where $(V_t)_{0\leq t\leq T}$ represents the value process and $(\theta_t)_{0\leq t\leq T}$ the number of risky assets. We say that a contingent claim is perfectly replicated by a portfolio $\varphi = (V, \theta)$ when $V_T = H$. In complete markets, this is possible for any contingent claim.

Albeit, in incomplete markets like electricity ones, a typical claim will carry intrinsic risk and the problem will be that of finding the portfolio strategy that minimizes the risk. This thesis is concerned with two types of such portfolio strategies, namely **locally risk minimizing (LRM)** and **mean-variance hedging (MVH)**. For this purpose, we will study a general case of semimartingales, their Galtchouk-Kunita-Watanabe and Föllmer-Schweizer decompositions and the connection between these two. We refer to Pham (2000)[28] for all the related notations and theoretical framework employed.

In this chapter we will work under the same probability space defined in Assumption 1.

Let H $\in L^2(\mathbf{P})$ be a *contingent claim*, meaning a payoff at time T of some derivative security. For this chapter, assume that H denotes a discounted contingent claim, S are the discounted prices and that V is the discounted value process.

Definition 4. The process $C_t(\varphi) = V_t - \int_0^t \theta_u dS_u$ is called the **cost process** of the portfolio strategy φ.

Definition 5. A portfolio strategy φ is **self-financing** if $C(\varphi)$ is constant $\mathbf{P}$-a.s.

Definition 6. A portfolio strategy φ is **mean self-financing** if $C(\varphi)$ is a $\mathbf{P}$-martingale.

Notation We denote by $\mathcal{A}$ the set of self-financing strategies and by $\mathcal{A}_m$ that of mean self-financing strategies.

Definition 7. H is **attainable** if $H = H_0 + \int_0^T \theta_t^H dS_t$, where H_0 is constant.

The following step is introducing the processes, measures and decompositions involved in our quadratic hedging problem.

2.1 Martingales

Definition 8. A **martingale** is a stochastic process $(M_t)_{0\leq t}$ such that:
(a). M_t is $\mathcal{F}_t$-measurable
(b). $E[|M_t|] < \infty, \forall t$
(c). $E[M_s|\mathcal{F}_t] = M_t$, if $t \leq s$

Definition 9. A **stopping time** is a random time (i.e. a measurable function defined as $T : \Omega \to [0, \infty])$ such that $\{T \leq t\} \in \mathcal{F}_t, \forall t \in [0, T]$.

Definition 10. A stochastic process $(M_t)_t \geq 0$ is a **local martingale** with respect to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ if there is a sequence of stopping times $(T_n)_n \geq 0$ such that:
(a). T_n is increasing and $lim_{n\to\infty} T_n = \infty$ almost surely
(b). $M_{t\wedge T_n} = \begin{cases} M_t, t \leq T_n \\ M_{T_n}, t \geq T_n \end{cases}$ is an $\mathcal{F}_t$-martingale, $\forall n \geq 0$.

Definition 11. We say that $(S_t)_{t\geq 0}$ is a **semimartingale** with respect to the filtration $(\mathcal{F}_t)_{0\leq t}$ if S_t may be written as $S_t = S_0 + A_t + M_t$, where S_0 is finite and $\mathcal{F}_0$-measurable, $(A_t)_{0\leq t\leq T}$ is a process of bounded variation and $(M_t)_{0\leq t\leq T}$ is a continuous local $\mathbf{P}$-martingale such that $M_0 = 0$. If it exists, this decomposition is unique.

2.2 Measures

2.2.1 Equivalent martingale measure (EMM)

Pricing assets as accurately as possible depends on how the investor perceives risk and how this is encompassed by the pricing model. Under a measure $\mathbf{Q}$ called *equivalent martingale measure or risk-neutral measure*, all discounted processes into martingales.

Definition 12. Two measures $\mathbf{P}, \mathbf{Q}$ are **equivalent**($\mathbf{P} \sim \mathbf{Q}$) if $\forall A \in \mathcal{F}$, we have $\mathbf{P}(A) > 0 \iff \mathbf{Q}(A) > 0$.

One way to construct equivalent probability measures is by Girsanov's theorem.

Theorem 1. *Let θ_t be an $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted stochastic process and define the process $dW_t = \theta_t dt + dB_t$. Put*

$$M_t = \exp(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds)$$

where $(B_t)_{0\leq t\leq T}$ is a Brownian motion. Assume that the process $(M_t)_{0\leq t\leq T}$ is an $\mathcal{F}$-martingale wrt $\mathbf{P}$. Define $\mathbf{Q}$ by $d\mathbf{Q} = M_T d\mathbf{P}$. Then the process $(Y_t)_{0\leq t\leq T}$ is a $\mathbf{Q}$-Brownian motion.

Note so far that this is an equivalent probability measure. In order for it to be risk neutral, it must satisfy that the discounted prices are martingales. In the case of incomplete markets nonetheless, there are infinitely many choices of equivalent martingale measures each of which producing a non-arbitrage price.

2.2.2 Minimal martingale measure (MMM)

Consider, for instance, that the discounted d-dimensional price process of a traded asset is given by a **P**-semimartingale $S = S_0 + A + M$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Assume that the **P**-dynamics of prices are

$$dS_t^i = \mu_t^{S,i} S_t^i dt + \sigma_t^{S,i} S_t^i dW_t^i \tag{2.1}$$

where $W = (W_t^i)_{i=1..d}$ is the Brownian motion generating $\mathcal{F}$.

Define the processes M, A in the decomposition of S. M is a locally square integrable local martingale and A is a predictable finite variation process with $M_0 = A_0 = 0$. In our framework, they have the form:

$$\begin{aligned} M_t &= \int_0^t \sigma_u^S S_u dW_u \\ A_t &= \int_0^t \mu_u^S S_u du \end{aligned} \tag{2.2}$$

We assume that our market is incomplete due to the presence of other stochastic inputs on the prices which are not directly traded. However, these are modeled by continuous processes with dynamics driven by independent Brownian motions.

As our market is incomplete, we have numerous equivalent martingale measures. Among all these, the **minimal martingale measure** for S will turn this process into a local martingale. Moreover, this new measure will preserve martingality and orthogonality structures (see Definition 20). This measure is a key-feature for determining local risk-minimizing strategies. Prior to defining the minimal martingale measure, we shall introduce some properties.

Definition 13. The quadratic variation process of $(M_t)_{0\le t\le T}$ is of the form

$$[M, M]_t = \lim_{\sup_k |t_{k+1}-t_k|\to 0} \sum_{k=1}^n (M_{t_k} - M_{t_{k-1}})^2$$

where the limit is in the sense of uniform convergence in probability.

Definition 14. The quadratic co-variation process of two L^2-local martingale processes $(M_t)_{0\le t\le T}, (L_t)_{0\le t\le T}$ is of the form

$$[M, L]_t = \lim_{\sup_k |t_{k+1}-t_k|\to 0} \sum_{k=1}^n (M_{t_k} - M_{t_{k-1}})(L_{t_k} - L_{t_{k-1}})$$

where the limit is in the sense of uniform convergence in probability.

Definition 15. Two square integrable martingales L, M are **strongly orthogonal** if $M \cdot L$ is a (uniformly integrable) martingale. This is equivalent to having $[M, L]$ be a (uniformly integrable) martingale.

Example 1. Take B^1, B^2 two L^2-valued independent Brownian motions which are martingales. We can apply directly Definition 15 to conclude that B^1, B^2 are orthogonal.

Remark 1. In the case of local martingales, the definition of orthogonality is slightly different: K is orthogonal to L if the process $[L, K]$ is a local martingale.

Example 2. Take B an $\mathbf{R}^d$-valued Brownian motion and let X, Y be two orthogonal $\mathbf{R}^d$-valued square-integrable martingales satisfying that $X_s \cdot Y_s = 0, \forall s > 0$ (where $\cdot$ denotes the scalar product). Then, the processes L, K which can be written as

$$L_t = L_0 + \int_0^t X_s dB_s, \qquad K_t = K_0 + \int_0^t Y_s dB_s$$

are orthogonal because $[L, K]_t = L_0 K_0 + \int_0^t X_s Y_s ds = L_0 K_0$ is a martingale.

Notation Hereafter, $\prime$ will denote the transpose and $< . >$ the *sharp bracket process.*

Definition 16. We say that the semimartingale $(S_t)_{0 \le t \le T}$ satisfies the **structure condition(SC)** if, given the decomposition in Definition 11, M is a locally $\mathbf{P}$-square integrable martingale (i.e $E[\sup_{t \le T_n} |M_t|^2 dt] < \infty$, for a sequence of stopping times $\{T_n\}_n$) and the finite variation part A is absolutely continuous with respect to the quadratic variation of M (i.e $A = \int d < M >_s \lambda_s$, for a predictable process λ such that the increasing process $\hat{K} = \int \lambda \prime d < M > \lambda < \infty$).

Definition 17. The process $\hat{K} = \int \lambda \prime d < M > \lambda$ is called the **mean-variance tradeoff process**.

Definition 18. We call **compensator** of a stochastic process $(M_t)_{0 \le t \le T}$ the unique adapted locally integrable process $(A_t)_{0 \le t \le T}$ with $A_0 = 0$ which makes $M_t - A_t$ a local martingale.

Definition 19. The **sharp bracket process** $< M >_t$ of $(M_t)_{0 \le t \le T}$ is the compensator of the quadratic variation process $[M, M]$.

Remark 2. For continuous local martingales, the sharp bracket process exists and coincides with the quadratic variation process. Therefore, in the remaining part of the thesis, we will use invariantly the sharp bracket process as the quadratic variation process.

In our framework, the sharp bracket process of M defined by (2.2) is

$$< M >_t = \int_0^t \sigma_u^S \sigma_u^{S,\prime} S_u^2 du$$

In the spirit of the structure condition, we can write the process A defined by (2.2) as

$$A_t = \int_0^t \lambda_u d < M >_u$$

where $\lambda_t^i := \frac{\mu_t^{S,i}}{(\sigma_t^{S,i})^2 S_t^i}, \;\; i = 1..d$. Now, the mean-variance tradeoff process is

$$\hat{K}_t = \int_0^t \frac{\mu_u^S \mu_u^{S,\prime}}{\sigma_u^S \sigma_u^{S,\prime}} du$$

With this explicit form of the semimartingale $(S_t)_{0 \le t \le T}$, we can proceed to introducing the measure we are looking for.

Definition 20. Suppose (S_t) satisfies (SC). An equivalent local martingale measure $\hat{\mathbf{P}}$ for S with $\mathbf{P}$-squared integrable density $\frac{d\hat{\mathbf{P}}}{d\mathbf{P}}$ is called **minimal martingale measure** for S if $\hat{\mathbf{P}} = \mathbf{P}$ on $\mathcal{F}_0$ and if every local $\mathbf{P}$-martingale L which is locally $\mathbf{P}$-square integrable and strongly $\mathbf{P}$-orthogonal to M is also a local $\hat{\mathbf{P}}$-martingale.

Now, we are looking for the density of such a measure. For that, we introduce the concept of *stochastic exponential or Doléans-Dade exponential.*

Definition 21. The **stochastic exponential or Doléans-Dade exponential** of the semimartingale process $(X_t)_{0\leq t\leq T}$ is the unique solution of the equation

$$\varepsilon_t(X) = 1 + \int_0^t \varepsilon_{s-}(X)dX_s$$

and is given by

$$\varepsilon_t(X) = \exp(X_t - X_0 - \frac{1}{2} < X^c >_t) \prod_{0<s\leq t} (1 + \Delta X_s)e^{-\Delta X_s}$$

where X^c is the continuous part of the process X and $\Delta X_s = X_s - X_{s-}$

In our case, we work with continuous processes, hence the Doléans-Dade exponential will have a simplified form.

Definition 22. The stochastic exponential or Doléans-Dade exponential of the continuous semimartingale X is defined by

$$\varepsilon(X) = \exp(X_t - X_0 - \frac{1}{2}d < X >_t), t \geq 0$$

Assume $(S_t)_{0\leq t\leq T}$ satisfies the (SC) for all $\mathbf{Q}$ equivalent local martingale measures with square integrable *martingale density* (i.e a $\mathbf{P}$-martingale with $Z_0^{\mathbf{Q}} = 1$ and $E[Z_T^{\mathbf{Q}}] = 1)$. The density processes of these measures are defined as:

$$Z_t^{\mathbf{Q}} = \frac{d\mathbf{Q}}{d\mathbf{P}}|_{\mathcal{F}_t} = \varepsilon(\int_0^t \lambda dM + L_t^{\mathbf{Q}}), \quad t \in [0, T] \tag{2.3}$$

with $(Z_t^{\mathbf{Q}})_{0\leq t\leq T}$ a local $\mathbf{P}$-martingale, $(SZ^{\mathbf{Q}})$ a $\mathbf{P}$-martingale and where $L^{\mathbf{Q}}$ is a locally $\mathbf{P}$-square integrable local $\mathbf{P}$-martingale.

If, among these (equivalent local martingale) measures, there exists the minimal martingale measure $\hat{\mathbf{P}}$, then its density process will be a martingale density also where $L^{\hat{\mathbf{P}}} = 0$. Its density process is given by

$$\hat{Z}_t = \varepsilon(-\int_0^t \lambda_t dM_t) = \exp(\int_0^t \lambda\prime_s dM_s - \frac{1}{2}\int_0^t \lambda_s\prime d < M >_s \lambda_s), \quad t \in [0, T] \tag{2.4}$$

2.2.3 Variance optimal martingale measure (VOMM)

This type of measure is of key importance for the mean hedging approach. We will see that for an EMM $\mathbf{Q}$ with square integrable density $\frac{d\mathbf{Q}}{d\mathbf{P}} \in L^2(\mathbf{P})$, the *variance optimal martingale measure* $\tilde{\mathbf{P}}$ is the one for which $\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}$ has a minimal $L^2(\mathbf{P})$-norm.

Notation

- $\mathcal{P}_2(\mathbf{P}) = \{\mathbf{Q} << \mathbf{P} | S$ is a $\mathbf{Q}$-local martingale$\}$
- $\mathcal{P}_2^e(\mathbf{P}) = \{\mathbf{Q} \sim \mathbf{P} | S$ is a $\mathbf{Q}$-local martingale$\}$
- $\Theta = \{\theta \in \mathbf{R}^d$predictable process$|\int_0^T \theta_t dS_t \in L^2(\mathbf{P})$and $\int \theta dS$ is a $\mathbf{Q}$-martingale, $\forall \mathbf{Q} \in \mathcal{P}_2^e(\mathbf{P})\}$

We aim at finding $\tilde{\mathbf{P}} \in \mathcal{P}_2^e(\mathbf{P})$ of smallest L^2-norm, i.e $D(\mathbf{Q}, \mathbf{P}) = \min_{\mathbf{Q}} ||\frac{d\mathbf{Q}}{d\mathbf{P}}||_{L^2(\mathbf{P})} = \min_{\mathbf{Q}} \sqrt{Var(\frac{d\mathbf{Q}}{d\mathbf{P}}) + 1}$ and the solution of

$$\min_{\mathbf{Q} \in \mathcal{P}_2(\mathbf{P})} E\left[\left(\frac{d\mathbf{Q}}{d\mathbf{P}}\right)^2\right] \tag{2.5}$$

For this, define $K_0 = \{f \in L^\infty(\Omega, \mathcal{F}, \mathbf{P}) | f = h\prime(S_{T_2} - S_{T_1})\}$ the set of simple stochastic integrals, with $T_1 \leq T_2$ stopping times wrt the filtration $\mathcal{F}$. Here, h denotes a simple $\mathcal{F}_{T_1}$-measurable process and is of the form $h_t = \sum_{k=1}^n h_k \mathbf{1}_{(T_k, T_{k-1}]}(t)$, for a series of stopping times $0 \leq T_1 < T_2 \cdots < T_n$.

Definition 23. The set of **signed local martingale measures** for S is the affine subspace $\mathcal{P}_2^s(\mathbf{P}) \subseteq L^1(\mathbf{P})$ defined as follows: $\mathcal{P}_2^s(\mathbf{P}) = \{\mathbf{Q} \mid \mathbf{Q}(A) = E[\mathbf{1}_A g], \forall A \in \mathcal{F}_T,$ with $g \in L^1(\mathbf{P})$ and $E[gf] = 0,$ for $f \in K_0$ and $E[g] = 1\}$

Notation We will call $\tilde{\mathcal{P}_2^s}(\mathbf{P})$ the **variance optimal signed local martingale measure**.

Remark 3. If $\mathcal{P}_2^s \bigcap L^2(\mathbf{P}) \neq \emptyset$, then this is a closed space in L^2-norm. Moreover, this is a convex set. A nonempty closed, convex subset of L^2 has a unique element of minimal norm. Hence, $\tilde{\mathcal{P}_2^s}(\mathbf{P})$ in this case is a singleton and $\tilde{\mathcal{P}_2^s}(\mathbf{P}) = \{\tilde{\mathbf{P}}\}$.

We know by Main Theorem 1.3 in Delbaen and Schachermayer ([9]) that if it exists, $\tilde{\mathbf{P}}$ is a measure which is equivalent to $\mathbf{P}$, hence $\tilde{\mathbf{P}} \in \mathcal{P}_2^e(\mathbf{P})$. Suppose that $\tilde{\mathbf{P}}$ is our VOMM (i.e the solution to (2.5)). Define the $\tilde{\mathbf{P}}$-martingale:

$$\tilde{Z}_t = \frac{E_{\tilde{\mathbf{P}}}[\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}|\mathcal{F}_t]}{E[\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}]^2}, \quad t \in [0, T] \tag{2.6}$$

This martingale will serve as a link between the VOMM and the value of the portfolio as we shall see in the mean-variance approach.

2.3 Local risk minimization (LRM)

The existence of a locally risk minimizing strategy φ is related to the existence of the Föllmer-Schweizer (hereafter FS) decomposition, which can be viewed as an extension of Galtchouk-Kunita-Watanabe (hereafter GKW) decomposition. Local risk minimization is used in the case of the (incomplete) electricity market and is based on the fact that FS and GKW decompositions are related to each other under the minimal martingale measure. These results will apply to all the cases where the discounted price process of the risky asset is continuous, as we have for our model described by (2.1).

Introduce some space notations that will add up to the previous ones from Section 2.2.3.

Notation

- $\mathcal{M}^2(\mathbf{P}) = \{\text{M martingale} \mid E_{\mathbf{P}}[\sup_t |M_t|^2] < \infty\}$,
- $\mathcal{M}^2_{loc}(\mathbf{P}) = \{\text{M locally square integrable martingale} | E_{\mathbf{P}}[\sup_t |M_t|^2] < \infty\}$
- $\mathcal{M}^2_0$ denote those sets of martingales defined as previously, but with initial value 0.

Remark 4. $\int \theta dM \in \mathcal{M}^2_0(\mathbf{P}) \iff \theta \in L^2(M)$ and $||\theta||_{L^2(M)} = ||\int_0^T \theta_t dM_t||_{L^2(\mathbf{P})}$

Theorem(Galtchouk-Kunita-Watanabe decomposition) 1. *Let $N, M \in \mathbf{R}^d$ be local martingales. If $N \in \mathcal{M}^2_{loc}(\mathbf{P})$ and $M \in \mathcal{M}^2_{loc}(\mathbf{P})$, then we can write:*

$$N_t = N_0 + \int_0^t \theta_u dM_u + L_t \tag{2.7}$$

for some $\theta \in \Theta, L \in \mathcal{M}^2_{0,loc}(\mathbf{P})$ orthogonal to M.

Our risk minimizing problem refers to finding a portfolio strategy $\varphi^* = (V^*, \theta^*)$ with $V^*_T = H$ that reduces the risk. We consider as a measure for risk:

$$R_t(\varphi) = E[(C_T(\varphi) - C_t(\varphi))^2 | \mathcal{F}_t)] \tag{2.8}$$

and we want to solve the problem

$$\min_{\varphi \in \mathcal{A}} R_t(\varphi), \forall 0 \leq t \leq T \tag{2.9}$$

As shown by Lemma 4.1 in Pham([28]), the solution to (2.9) lies de facto in $\mathcal{A}_m$(the set of mean self-financing cost processes) for processes which are **P**-local martingales.

In our framework given by the semimartingale $(S_t)_{0 \leq t \leq T}$ defined by (2.1) having the decomposition (2.2), a risk minimizing strategy does not necessarily exist[1]. We refer to Pham (2000)[28] for all the upcoming results.

Definition 24. Let φ be an H-admissible portfolio strategy. We call φ **local risk minimizing** if the cost process $C(\varphi) \in M^2(\mathbf{P})$ and $C(\varphi)$ is orthogonal to M under **P**.

The following result involves the *FS decomposition* and establishes the connection between this and the existence of a risk minimizing strategy.

[1] see Remarks 5 and 6

Proposition 1. *There exists* $\varphi^H = (V^H, \theta^H)$ *a risk minimizing strategy* $\iff$ *H admits the Fölmer-Schweizer decomposition under measure* $\mathbf{P}$*i.e:*

$$H = H_0 + \int_0^T \theta_t^H dS_t + L_T^H \tag{2.10}$$

with $H_0 \in \mathbf{R}, \theta^H \in \Theta$ *and* $L^H \in \mathcal{M}^2(\mathbf{P})$ *orthogonal to* M.

The value process is then $V_t^H = H_0 + \int_0^t \theta_u^H dS_u + L_t^H$.

Proof. $\Longleftarrow$ If H is decomposed as in (2.10), then the cost process is given by $C_t(\varphi^H) = H_0 + L_t^H$ and thus φ^H is locally risk minimizing.
$\Longrightarrow$ Let φ be a locally risk minimizing strategy. Then $H = C_T(\varphi) + \int_0^T \theta_t dS_t = C_0(\varphi) + \int_0^T \theta_t dS_t + (C_T(\varphi) - C_0(\varphi))$. By taking $H_0 = C_0(\varphi)$, $\theta^H = \theta$ and $L^H = C_T(\varphi) - C_0(\varphi)$ we get (2.10).

□

Thus, the local risk minimization approach boils down to finding a FS for H under the minimal martingale measure described in Section 2.2.2. The natural question whether the FS decomposition exists for all contingent claims is answered by the next obervations.

Remark 5. Monat and Stricker (1995) [24] prove that a sufficient condition for the FS to exist is that the mean-variance tradeoff process $\hat{K}$ is uniformly bounded.

Remark 6. Delbaen et al. (1997) [10] prove that the existence of the FS decomposition for every contingent claim H is equivalent to assuming the *reverse Hölder inequality* $R_2(\mathbf{P})$ on the density process $(\hat{Z}_t)_{0\leq t\leq T}$ of the minimal martingale measure given by (2.4).

Remark 7. The uniformly integrable process $(\hat{Z}_t)_{0\leq t\leq T}$ having $\hat{Z}_0 = 1$ and $\hat{Z}_T > 0$ $\mathbf{P}$-a.s satisfies $R_2(\mathbf{P}) \iff \exists$ C constant such that $\forall t \in [0, T]$,

$$E\left[\left(\frac{\hat{Z}_T}{\hat{Z}_t}\right)^2 |\mathcal{F}_t\right] \leq C$$

Finally, Theorem 4.2 in Pham (2000) [28] relates the FS decomposition with actually finding the portfolio strategy that minimizes the risk.

Theorem 2. *Let* $(S_t)_{0\leq t\leq T}$ *be a continuous semimartingale and* $\hat{K}$ [2] *be uniformly bounded. Then there exists a unique locally risk minimizing strategy given by*

$$V_t^H = V_t^* := E_{\hat{P}}[H|\mathcal{F}_t] \tag{2.11}$$

$$\theta_t^* = \theta_t^H \tag{2.12}$$

with θ^H *being the integrand from the FS decomposition (2.10) and* θ^* *the integrand in the GKW decomposition of* V^* *under* $\hat{\mathbf{P}}$*, where* V^* *is a* $\hat{\mathbf{P}}$*-martingale and* S *is a continous* $\hat{\mathbf{P}}$*-local martingale.*

[2] see Definition 17

Proof. By Remark 5, there exists a FS decomposition of H of the type

$$H = H_0 + \int_0^T \theta_u^H dS_u + L_T^H$$

with $H_0 \in \mathbf{R}, \theta^H \in \Theta, L^H \in \mathcal{M}^2(\mathbf{P})$ and L^H orthogonal to M.

We have that $H \in L^1(\mathbf{P})$ since H and $\hat{Z}$ are $\mathbf{P}$-square integrable. Hence the value process (2.11) is well defined and is a $\hat{\mathbf{P}}$-martingale.

The stochastic integral $\int \theta_u dS_u$ satisfies that

$$E[\sup_{0\leq t\leq T} |\int_0^t \theta_u^H dS_u|]^2 < \infty$$

Additionally, by Hölder inequality, the latter is satisfied under any equivalent martingale measure $\mathbf{Q} \in \mathcal{P}_2^e$. In particular, it is given under the MMM $\hat{\mathbf{P}}$. So $\int \theta_u^H dS_u$ is a $\hat{\mathbf{P}}$-martingale.

Furthermore, from L^H being orthogonal to M, $L^H$is a $\hat{\mathbf{P}}$-martingale. This is proved in Proposition 4.3 by Pham (2000) [28].

Then, going back to the FS decomposition of H and taking condition expectation under $\hat{\mathbf{P}}$, we get the KW projection of $V^\star$ into S:

$$V_t^\star = H_0 + \int_0^t \theta_u^H dS_u + L_t^H$$

Indeed $\theta^\star = \theta^H$ and the LRM strategy is unique because the of the unique of the FS decomposition.

□

2.4 Mean-variance hedging (MVH)

In complete markets, the conditions of self-financing and attainability of contingent claims are a fact. In contrast, in incomplete markets, some contingent claims can be non-attainable. The mean-variance hedging approach keeps the condition of self-financing and aims at minimizing the difference between the respective contingent claim and the value of the portfolio at time T (2.13).

$$(\mathcal{H}) \qquad I = \inf_{(V,\theta)\in\mathcal{A}} E[H - V_T]^2 \tag{2.13}$$

Moreover, if we consider $V_0 = x$, $\theta \in \Theta$ and $\varphi = (V, \theta)$ being self-financing, we can write

$$V_t^{x,\theta} = x + \int_0^t \theta_u dS_u$$

where θ_u represents the number of units of risky assets and we can write

$$(\mathcal{H}) \qquad I = \inf_{x\in\mathbf{R},\theta\in\Theta} E[H - V_T^{x,\theta}]^2 \tag{2.14}$$

When the contingent claim is attainable (i.e $H = H_0 + \int_0^T \theta_t^H dS_t$), the solution to (2.13) is $\begin{cases} V_t^\star = H_0 + \int_0^t \theta_u^H dS_u \\ \theta_t^\star = \theta_t^H \end{cases}$ and $x^\star = E_{\mathbf{Q}}[H], \forall \mathbf{Q} \in \mathcal{P}_2^e(\mathbf{P})$

If we define the set $\mathcal{A}(x) = \{\varphi = (V, \theta) | V_0 = x\}$, then we can re-write our problem as

$$(\mathcal{H}(x)) \qquad J(x) = \inf_{(V,\theta)\in\mathcal{A}(x)} E[H - V_T]^2 = \inf_{\theta\in\Theta} E[H - V_T^{x,\theta}]^2 \tag{2.15}$$

Let $G_T(\Theta) = \{\int_0^T \theta_t dS_t | \theta \in \Theta\}$ be a (linear) subspace of $L^2(\mathbf{P})$. $(\mathcal{H}(x))$ is an $L^2(\mathbf{P})$ projection on $G_T(\theta)$. Hence, the solution to $(\mathcal{H}(x))$ is reduced to weather $G_T(\Theta)$ is closed in $L^2(\mathbf{P})$.

If S is a martingale $G_T(\Theta)$ is automatically closed since the stochastic integral defines an isometry. Then we can use the KW projection of H into S and make $t = T$: $H = E[H] + \int_0^T \theta_t^H dS_t + L_T^H$, where $\theta \in \Theta$ and $L^H \in \mathcal{M}_0^2(\mathbf{P})$ is orthogonal to S. Proposition 5.1 in Pham (2000) ([28]) gives the MVH solution:

Proposition 2. *Assume $S \in \mathcal{M}_{loc}^2(\mathbf{P})$. Then $G_T(\Theta)$ is closed in $L^2(\mathbf{P})$ and there exists a unique solution to $\mathcal{H}(x)$(2.15), $\forall x \in \mathbf{R}$ given by $(V^{x,\theta^H}, \theta^H)$, with $J(x) = (E[H] - x)^2 + E[L_T^H]^2$. Furthermore, $x^\star = E[H]$ and the unique solution to $(\mathcal{H})$(2.13) is $(V^{x^\star,\theta^H}, \theta^H)$ and $I = E[L_T^H]^2$*

Proof. By the isometry argument, $G_T(\Theta)$ is closed. Using KW decomposition for H, we get:

$$E[H - x - \int_0^T \theta_t dS_t]^2 = (E[H] - x)^2 + E[\int_0^T \theta_t^H - \theta_t dS_t]^2 + E[L_T^H]^2]$$

Thus, the solution for (2.15) is θ^H (from the KW decomposition) and that $J(x) = (E[H] - x)^2 + E[L_T^H]^2$. Also, the solution to (2.13) is $x^\star = E[H]$ and $\theta^\star = \theta^H$.

□

If S is a continuous semimartingale following Gouriéroux et al (1998), we consider the extension of our problem to the space

$$\Theta_2 = \{\theta \in \mathbf{R}^d \text{predictable process} | \int_0^T \theta_t dS_t \in L^2(\mathbf{P}) \text{and} \int \theta dS \text{ is a } \mathbf{Q}\text{-martingale}, \forall \mathbf{Q} \in \mathcal{P}_2^e(\mathbf{P})\}$$

Then the optimization problems become

$$J_2(x) = \inf_{\theta\in\Theta_2} = E[H - x - \int_0^T \theta_t dS_t]^2 \tag{2.16}$$

$$I_2 = \inf_{x\in\mathbf{R},\theta\in\Theta_2} E[H - x - \int_0^T \theta_t dS_t]^2 \tag{2.17}$$

Pham(2000) ([28]) proves in Proposition 5.2 that $G_T(\Theta_2)$ is closed in $L^2(\mathbf{P})$, hence there is a solution for (2.16) and (2.17) which come down to finding the variance-optimal martingale measure (VOMM).

Theorem 5.1 in Pham (2000)([28]) provides a link between the VOMM and the value of the portfolio via the $\tilde{\mathbf{P}}$-martingale $\tilde{Z}$ defined by (2.6).

Theorem 3. *There exists $\tilde{\theta} \in \Theta_2$ such that $\tilde{Z}_t = V_t^{1,\tilde{\theta}}$, $\mathbf{P}$-a.s, $0 \leq t \leq T$, where $\tilde{\theta}$ solves the problem*

$$\min_{\theta \in \Theta_2} E[V_T^{1,\theta}]^2 \tag{2.18}$$

Proof. (2.18) admits a solution since the linear subspace $G_T(\Theta_2)$ is closed in $L^2(\mathbf{P})$. Also, $V_t^{1,\tilde{\theta}} \geq 0$ $\mathbf{P}$-a.s. What is more, by the optimality for (2.18) and the linearity argument,

$$E[V_T^{1,\tilde{\theta}} Y] = 0, \forall Y \in G_T(\Theta_2)$$

Also, $E[V_T^{1,\tilde{\theta}}] = E[V_T^{1,\tilde{\theta}}]^2 > 0$. Assuming that $\mathcal{P}_2^e(\mathbf{P}) \neq \emptyset$, $E_{\mathbf{Q}}[V_T^{1,\tilde{\theta}}] = 1, \forall \mathbf{Q} \in \mathcal{P}_2^e(\mathbf{P})$. Define a new probability measure $\bar{\mathbf{P}} << \mathbf{P}$ (i.e $\bar{\mathbf{P}} \in \mathcal{P}_2(\mathbf{P})$):

$$\frac{d\bar{\mathbf{P}}}{d\mathbf{P}} = \frac{V_T^{1,\tilde{\theta}}}{E[V_T^{1,\tilde{\theta}}]^2} \in L^2(\mathbf{P})$$

But $E_{\mathbf{Q}}[V_T^{1,\tilde{\theta}}] = 1, \forall \mathbf{Q} \in \mathcal{P}_2^e(\mathbf{P})$ and since $\mathcal{P}_2^e \subset \mathcal{P}_2 \subset L^2(\mathbf{P})$ are subsequently dense in one another, we have that

$$E[\frac{d\bar{\mathbf{P}}}{d\mathbf{P}}(\frac{d\bar{\mathbf{P}}}{d\mathbf{P}} - \frac{d\mathbf{Q}}{d\mathbf{P}})] = 0, \forall \mathbf{Q} \in \mathcal{P}_2(\mathbf{P})$$

Thus, $\bar{\mathbf{P}}$ is the solution to (2.5), hence it is the VOMM $\bar{\mathbf{P}} = \tilde{\mathbf{P}}$. We obtain the conclusion by the definition of density of $\bar{\mathbf{P}}$ and by taking $\tilde{\mathbf{P}}$ conditional expectation. □

In order to establish the strategy for mean-variance hedging, we use the fact that $\tilde{Z}$ defined by (2.6), with $\tilde{Z}_0 = 1$ is a strictly positive $\mathbf{Q}$-martingale, $\forall \mathbf{Q} \in \mathcal{P}_2^e(\mathbf{P})$. To each such measure $\mathbf{Q}$, we will associate another measure $\tilde{\mathbf{Q}}$ such that $\frac{d\tilde{\mathbf{Q}}}{d\mathbf{Q}}|_{\mathcal{F}_t} = \tilde{Z}_t$. The set of measures of this type is denoted by $\tilde{\mathcal{P}_2^e}(\mathbf{P})$). Then, to $\tilde{\mathbf{P}}$-solution of the optimization problem (2.5), we associate $\tilde{\tilde{\mathbf{P}}} \in \tilde{\mathcal{P}_2^e}(\mathbf{P})$. The Radon-Nikodym derivative of it will be

$$\frac{d\tilde{\tilde{\mathbf{P}}}}{d\mathbf{P}} = E[\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}]^2 \tilde{Z}_T^2 \tag{2.19}$$

Define a process $\tilde{X} \in \mathbf{R}^{d+1}, \tilde{X}^0 = \frac{1}{\tilde{Z}}, \tilde{X}^i = \frac{S^i}{\tilde{Z}}$. This is a continuous local martingale under any measure $\tilde{\mathbf{Q}} \in \tilde{\mathcal{P}_2^e}$ (as shown by Lemma 5.1 in Pham (2000)([28])). Define

$$\begin{aligned} L^2(\tilde{X}, \tilde{\tilde{\mathbf{P}}}) \quad &= \quad \{\phi \in \mathbf{R}^{d+1} \text{predictable process} | \int_0^T \phi_t d\tilde{X}_t \in L^2(\mathbf{P}), \\ &\text{and} \quad \int \phi d\tilde{X} \text{ is a } \tilde{\mathbf{Q}} - \text{martingale}, \forall \tilde{\mathbf{Q}} \in \tilde{\mathcal{P}_2^e}(\mathbf{P})\} \end{aligned} \tag{2.20}$$

The next result is used in the approach of Gouriéroux et al (1998) and gives the connection between the density $\tilde{Z}$ and our optimization problem (2.15).

Theorem 4.

$$\{V_T^{x,\theta}|\theta \in \Theta_2\} = \{\tilde{Z}_T(x + \int_0^T \phi_t d\tilde{X}_t)|\phi \in L^2(\tilde{X}, \tilde{\tilde{X}})\} \tag{2.21}$$

Where the connection between θ and ϕ is given by $\phi^0 = V^{x,\theta} - \theta^t S$ and $\phi^i = \theta^i$ and the number of units invested in the risky assets is

$$\theta^i = \phi^i + \tilde{\theta}^i(x + \int \phi d\tilde{X} - \phi^t \tilde{X}) \tag{2.22}$$

By the Radon-Nikodym derivative of the measure $\tilde{\tilde{\mathbf{P}}}$ with respect to $\mathbf{P}$, we have that $\frac{H}{\tilde{Z}_T} \in L^2(\tilde{\tilde{\mathbf{P}}})$. We can apply the KW decomposition to the process $E_{\tilde{\tilde{\mathbf{P}}}}[\frac{H}{\tilde{Z}_T}|\mathcal{F}_t]$:

$$\frac{H}{\tilde{Z}_T} = E_{\tilde{\tilde{\mathbf{P}}}}[\frac{H}{\tilde{Z}_T}] + \int_0^T \tilde{\phi}_t^H d\tilde{X}_t + \tilde{\tilde{L}}_T^H$$

with $\tilde{\phi}^H \in L^2(\tilde{X}, \tilde{\tilde{\mathbf{P}}})$ and $\tilde{\tilde{L}} \in \mathcal{M}^2(\tilde{\tilde{\mathbf{P}}})$ is orthogonal to $\tilde{X}$.

Theorem 5.4 in Pham (2000) ([28]) gives a solution to (2.15).

Theorem 5. *For all $x \in \mathbf{R}$, there exists a unique solution to the optimization problem (2.16) :*

$$\theta^\star(x)^i = (\tilde{\phi}^H)^i + \tilde{\theta}^i(x + \int \tilde{\phi}^H d\tilde{X} - (\tilde{\phi}^H)^i \tilde{X})$$

and

$$J_2(x) = \frac{(E_{\tilde{\mathbf{P}}}[H] - x)^2 + E_{\tilde{\tilde{\mathbf{P}}}}[\tilde{\tilde{L}}_T^H]^2}{E[\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}]^2}$$

Also, there is a unique solution to the optimization problem (2.17) :

$$(x^\star, \theta^\star) = (E_{\tilde{\mathbf{P}}}, \theta^\star(x^\star))$$

and

$$I_2 = \frac{E_{\tilde{\tilde{\mathbf{P}}}}[\tilde{\tilde{L}}_T^H]^2}{E[\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}]^2}$$

If $G_T(\Theta)$ is closed, then $\Theta = \Theta_2$ and the unique solution to (2.14) is $\theta^\star(x)$. $J(x) = J_2(x)$. The solution to (2.14) is $(V^{x^\star,\theta^\star(x)}, \theta^\star(x))$ and $x^\star = E_{\tilde{\mathbf{P}}}[H]$.

Equivalently, Pham and Schweizer (1998) [34] give another characterization of the solution by way of the mean-variance tradeoff process $\hat{K} = \int \hat{\lambda}' d < M > \hat{\lambda}$. It is known that if $\hat{K}$ is bounded, then the density $\hat{Z}_T \in L^2(\mathbf{P})$ and then there exists the FS decomposition

$$\hat{Z}_T := \frac{d\hat{\mathbf{P}}}{d\mathbf{P}} = E[\hat{Z}_T^2] - E[\hat{Z}_T \hat{L}_T] + \int_0^T \hat{\xi}_s dX_s + \hat{L}_T \tag{2.23}$$

where $\int \hat{\xi}$ and $\hat{L}$ are both $\hat{\mathbf{P}}$-martingales.

Theorem 7 in Pham and Schweizer (1998) [34] claims that:

Theorem 6. *If S is continuous, $\hat{K}$ is bounded and $\hat{L}_T = 0$ in (2.23), then the solution to the MVH problem can be written in the following feedback form:*

$$\theta_t = \xi_t^H - \frac{\hat{\xi}_t}{\hat{Z}_t^0}(\hat{V}_t - x - \int_0^t \theta_s dS_s) \tag{2.24}$$

where

$$\hat{Z}_t^0 := \hat{E}[\hat{Z}_T|\mathcal{F}_t] = E[\hat{Z}_T^2] + \int \hat{\xi}_s dS_s \tag{2.25}$$

$$\hat{V}_t = \hat{E}[H|\mathcal{F}_t] = H_0 + \int_0^t \xi_s^H dS_s + L_t^H \tag{2.26}$$

Chapter 3

Quadratic hedging in the forward market

In what follows, we connect our quadratic hedging preliminaries discussed so far with the spot price model and all the processes introduced in Chapter 1.

Assumption 2. *Assume, for this, like in Chapter 1, that we are working in the probability space* $(\Omega, \mathbf{P}, \mathcal{F})$*, where* $\mathbf{P}$ *is the physical measure and* $\mathcal{F} = \mathcal{F}^S \vee \mathcal{F}^C \vee \mathcal{F}^D$ *is the market filtration generated by the mutually independent Brownian motions corresponding to the dynamics of fuels, capacity and demand.*

3.1 Local risk minimization in our forward market

As mentioned before, the main idea of LRM resides in finding the GKW decomposition of the contingent claim under the minimal martingale measure. In the latter, we will be able to identify the integral part (headgeable) and the orthogonal part (unheadgeable) of our contingent claim.

3.1.1 The minimal martingale measure in our model

Consider back the dynamics under $\mathbf{P}$ of the discounted fuel price process (1.10). We also consider the measure $\mathbf{Q}^i$ of the fuel spreads process and its dynamics (1.11), (1.12) depicted in Section 1.3.3.

Recall that $\lambda_i = \frac{\mu_i - r}{\sigma_i}$ the market price of risk of the i-th spread Y_t^i.

We construct a measure $\hat{\mathbf{P}}$ as the product of all the $\mathbf{Q}^i$'s with the following Radon-Nikodym derivative:

$$\frac{d\hat{\mathbf{P}}}{d\mathbf{P}} = \prod_{i=1}^{n} \frac{d\mathbf{Q}^i}{d\mathbf{P}} = \prod_{i=1}^{n} e^{-\lambda_i W_T^i - \frac{\lambda_i^2}{2} T} \tag{3.1}$$

and we define its corresponding n-dimensional Brownian motion by Girsanov's theorem, as

$$\hat{W}_t^i = W_t^i + \lambda_i t \tag{3.2}$$

Notation $\hat{E}[\cdot]$ denotes the expectation under the measure $\hat{\mathbf{P}}$.

Proposition 3. *The measure $\hat{\mathbf{P}}$ defined by (3.1) is the minimal martingale measure.*

Proof. By the Remark 6 and by linking Theorems A, B and C in Delbaen et al. (1997)[10], we prove that $\hat{\mathbf{P}}$ is our minimal martingale measure if we can show that its Radon-Nikodym derivative $\hat{Z}_t = E[\frac{d\hat{\mathbf{P}}}{d\mathbf{P}}|\mathcal{F}_t]$ satisfies the reverse Hölder inequality $R_2(P)$ described in Remark 7. Indeed, using Hölder‘s inequality and making use of the fact that Brownian motion has independent normally distributed increments we obtain:

$$E\left[\left(\frac{\hat{Z}_T}{\hat{Z}_t}\right)^2|\mathcal{F}_t\right] = E\left[\prod_{i=1}^n e^{-2\lambda_i(W_T^i - W_t^i) - \lambda_i^2(T-t)}|\mathcal{F}_t\right] = \prod_{i=1}^n e^{-\lambda_i^2(T-t)} E\left[\prod_{i=1}^n e^{-2\lambda_i(W_T^i - W_t^i)}\right] \leq$$

$$\leq \prod_{i=1}^n e^{-\lambda_i^2(T-t)} \prod_{i=1}^n e^{\frac{1}{2}4\lambda_i^2(T-t)} = \prod_{i=1}^n e^{\lambda_i^2(T-t)} \leq \prod_{i=1}^n e^{\lambda_i T} \quad ,\forall t \in [0,T]$$

Take $C = \prod_{i=1}^n e^{\lambda_i T}$ and we get the reverse Hölder inequality. Hence, $\hat{\mathbf{P}}$ is the MMM. □

Assumption 3. *Set $\frac{d\hat{\mathbf{Q}}^i}{d\hat{\mathbf{P}}} = e^{-rT}\frac{S_T^i}{S_0^i}$, for all $i = 1..n$. In other words, the discounting factor of prices is absorbed by the minimal martingale measure.*

Now, let us determine the $\hat{\mathbf{P}}$-dynamics of the i-th discounted fuel price. This can be easily done by introducing (3.2) into the $\mathbf{P}$-dynamics (1.10). We get

$$dS_t^i = \frac{e^{-rt}}{h_i} \sum_{j \leq i} \sigma_j Y_t^j d\hat{W}_t^j \tag{3.3}$$

which is the $\mathbf{Q}^i$-dynamics of $\tilde{S}^i$ determined in (1.12).

Also, provided that $dF_t^i(T) = e^{r(T-t)} dS_t^i$, we can compute the $\hat{\mathbf{P}}$-dynamics of the forward price on fuels

$$dF_t^i(T) = e^{r(T-t)} \frac{1}{h_i} \sum_{j \leq i} Y_t^j \sigma_j d\hat{W}_t^j \tag{3.4}$$

Consequently, we can derive the $\hat{\mathbf{P}}$-dynamics of the forward price on electricity $F_t^e(T)$ defined in (1.15). This can be done easily by replacing the change in Brownian motion ($d\hat{W}_t^i = dW_t^i + \lambda_i dt$) into the $\mathbf{P}$-dynamics (1.19):

$$\begin{aligned} dF_t^e(T) = e^{r(T-t)} & \underbrace{\sum_{i=1}^n (\sum_{k=1}^n G_k^T(t, C_t, D_t) \sigma_i Y_t^i \, d\hat{W}_t^i}_{\theta^S} \\ & + \underbrace{\sum_{i=1}^n h_i F_t^i(T) \frac{\partial G_i^T}{\partial d}(t, C_t, D_t) b(t, D_t) \, dW_t^D}_{\theta^D} + \\ & + \underbrace{\sum_{i=1}^n h_i F_t^i(T) \sum_{k=1}^n \frac{\partial G_i^T}{\partial c_k}(t, C_t, D_t) \beta_k(t, C_t^k) \, dW_t^{C,k}}_{\theta^C} \end{aligned} \tag{3.5}$$

which can be written as:

$$dF_t^e(T) = \theta_t^S \cdot d\hat{W}_t + \theta_t^C \cdot dW_t^C + \theta_t^D dW_t^D$$

considering $\cdot$ as the dot product.

Under the filtration $\mathcal{F}^{D,C}$ defined in Section(1.2), we have that $\hat{\mathbf{P}} = \mathbf{P}$. Indeed, $\forall A \in \mathcal{F}^{D,C}$,

$$\begin{aligned}
\hat{\mathbf{P}}(A) &= \hat{E}[\mathbf{1}_A] = E[\frac{d\hat{\mathbf{P}}}{d\mathbf{P}}\mathbf{1}_A] = E[E[\frac{d\hat{\mathbf{P}}}{d\mathbf{P}}\mathbf{1}_A|\mathcal{F}^{D,C}]] = E[E[\prod_{i=1}^n e^{-\lambda_i W_T^i - \frac{\lambda_i^2}{2}T}\mathbf{1}_A|\mathcal{F}^{D,C}]] = \\
&= E[\prod_{i=1}^n e^{-\frac{\lambda_i^2}{2}T}\mathbf{1}_A E[\prod_{i=1}^n e^{-\lambda_i W_T^i}|\mathcal{F}^{D,C}]] = E[\prod_{i=1}^n e^{-\frac{\lambda_i^2}{2}T}\mathbf{1}_A \prod_{i=1}^n e^{+\frac{\lambda_i^2}{2}T}] = \\
&= \mathbf{P}(A). \qquad (3.6)
\end{aligned}$$

Even for a larger market where we trade fuels and electricity alike (S, F^e), $\hat{\mathbf{P}}$ remains our minimal martingale measure. This result is proven by Aïd et al. (2013) [1] in Proposition 3.8 by making use of the property of orthogonality of S and $F^e(T)$.

3.1.2 The LRM strategy in our model

As mentioned before, the local risk minimization hedging technique resides in the finding the GKW decomposition under the minimal martingale measure. We want to build a replicating portfolio of a European contingent claim $H \in L^2(\hat{\mathbf{P}}) \cap L^2(\mathbf{P})$ written on forward contracts on fuels and electricity, on capacity and demand on the a market (S, F^e) of fuels and electricity.

Assumption 4. *We consider $H \in L^2(\hat{\mathbf{P}})$ of the form*

$$H = \rho(F_T^e(T^\star), F_T^i(T^\star), C_T, D_T)$$

where $T^\star$ is any positive maturity time.

Since we can only trade on fuels and electricity forwards, we are looking for a GKW decomposition under $\hat{\mathbf{P}}$ of the type:

$$\hat{E}[H|\mathcal{F}_t] = \hat{E}[H] + \int_0^t \xi_s \cdot dF_s(T^\star) + \int_0^t \xi_s^e dF_s^e(T^\star) + L_t^H \qquad (3.7)$$

with L^H orthogonal to the two local martingales $F(T^\star)$ and $F^e(T^\star)$ in the sense of the Definition 15. Note that (3.7) is an application of Theorem (Galtchouk-Kunita-Watanabe decomposition) 1 , where $\theta_t = (\xi_t, \xi^e)$ resides in the space Θ for $d = n+1$ (see Notation in Section 2.2.3).

Remark 8. By markovianity of the process $(F(T^\star), C, D)$, we can assume that the value process $V_t^H = \hat{E}[H|\mathcal{F}_t]$ and it can be written as

$$V_t^* = V_t^H := \phi(t, F_t(T^\star), C_t, D_t))$$

for $\phi \in \mathcal{C}^{1,2,2,2}$. Note that ϕ is a function of only 4 variables, as we have seen in (1.15) that $F_t^e(T^\star)$ is a function itself of $(t, F_t(T^\star), C_t, D_t)$.

Now replacing this into (3.7) and assuming that the mean-variance tradeoff process $\hat{K}$ is absolutely bounded, then by Remark 5, there exists the FS decomposition in the sense of Proposition 1. This is given by:

$$V_t^H = \hat{E}[H] + \int_0^t \xi_s \cdot dF_s(T^\star) + \int_0^t \xi_s^e dF_s^e(T^\star) + L_t^H$$

where L^H is a $\mathbf{P}$-martingale orthogonal to $F(T^\star), F^e(T^\star)$ in the sense of Definition 15.

Furthermore, Proposition 3.13 in Aïd et al. (2013) [1] gives explicitly the components of the FS decomposition.

Proposition 4. *Under the Assumption 4 on H contingent claim and for $T \leq T^\star$, the local risk minimizing strategy is given by*

$$\xi_t^e = \frac{1}{||(\theta_t^C, \theta_t^D)||^2}[\sum_{i=1}^n \theta_t^{C,i} \frac{\partial \phi}{\partial c_i} \beta_i(t, C_t^i) + \theta_t^D \frac{\partial \phi}{\partial d} b(t, D_t)] \tag{3.8}$$

$$\xi_t^i = \frac{\partial \phi}{\partial y_i} + \frac{h_i G_i^{T^\star}(t, C_t, D_t)}{||(\theta_t^C, \theta_t^D)||^2}[\sum_{i=1}^n \theta_t^{C,i} \frac{\partial \phi}{\partial c_i} \beta_i(t, C_t^i) + \theta_t^D \frac{\partial \phi}{\partial d} b(t, D_t)] \tag{3.9}$$

$$dL_t^H = \sum_{i=1}^n (\frac{\partial \phi}{\partial c_i} \beta_i(t, C_t^i) - \frac{\sum_{i=1}^n \theta_t^{C,i} \frac{\partial \phi}{\partial c_i} \beta_i(t, C_t^i) + \theta_t^D \frac{\partial \phi}{\partial d} b(t, D_t)}{||(\theta_t^C, \theta_t^D)||^2} \theta_t^{C,i} \mathbf{1}_{\{||(\theta_t^C, \theta_t^D)||^2 > 0\}}) dW_t^{C,i} - \\ (\frac{\partial \phi}{\partial d} b(t, D_t) - \frac{\theta_t^D \frac{\partial \phi}{\partial d} b(t, D_t)}{||(\theta_t^C, \theta_t^D)||^2} \mathbf{1}_{\{||(\theta_t^C, \theta_t^D)||^2 > 0\}}) dW_t^D \tag{3.10}$$

Proof. See Appendix A.2. □

In order to complete the LRM, it only remains to determine ϕ, $\frac{\partial \phi}{\partial c^k}$, $\frac{\partial \phi}{\partial d}$, $\frac{\partial \phi}{\partial y}$.These will depend on G_t^i. ϕ will be the solution to the following PDE:

$$\begin{cases} \frac{\partial \phi}{\partial t} + \sum_{i=1}^n \frac{\partial \phi}{\partial c_i} \alpha_i(t, c_i) + \frac{\partial \phi}{\partial d} a(t, d) + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 \phi}{\partial y_i^2} [\sum_{j \leq i} (y_j - y_{j-1})^2 \sigma_j^2] + \\ \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 \phi}{\partial c_i^2} \beta_i(t, c_i)^2 + \frac{1}{2} \frac{\partial^2 \phi}{\partial d^2} b(t, d)^2 = 0 \\ \\ \phi(T, y, c, d) = \rho(y, c, d) \end{cases} \tag{3.11}$$

In what follows, we will determine ϕ in some particular cases.

3.1.3 The value of the portfolio

Suppose we want to price an energy derivative, for instance, an option on electricity forward contracts in the case when we have two fuels available (i.e $n = 2$). As a notation, let f_X and $\hat{f}_X$ be density functions at time T of a process X under measures $\mathbf{P}$, respectively, $\hat{\mathbf{P}}$. Also, we shall denote by $BS_t(\sigma, K)$ the Black-Scholes formula corresponding to the price at time t of an option with volatility σ and strike K.

Consider a contingent claim H of a European call option on forward contract on electricity with instantaneous delivery period (i.e $T^\star > T$) and with payoff $H = \varphi(F^e_T(T^\star))$, where $\varphi(x) = (x-K)^+$.

By the Proposition 3.10 in Aïd et al. [1], the price of the discounted contingent claim H is given by

$$\hat{E}[e^{-r(T-t)}H|\mathcal{F}_t] = \hat{E}[\psi(t, F_t(T^\star), C_T, D_T)|\mathcal{F}^{D,C}]$$

with

$$\psi(t, F_t(T^\star), C_T, D_T) = e^{-r(T-t)}\hat{E}[(h_1 G_1^{T^\star}(T, C_T, D_T)F^1_T(T^\star) + h_2 G_2^{T^\star}(T, C_T, D_T)F^2_T(T^\star))^+|\mathcal{F}^W_t] \tag{3.12}$$

W and (W^D, W^C) are independent, therefore one can compute the expectations separately. To compute (3.12) further, we denote by $w_i = e^{-r(T^\star - T)}G_i^{T^\star}(T, C_T, D_T)$. We use the fact that $F^i_T(T^\star) = e^{r(T^\star - T)}S^i_T$ and $h_1 S^1_t = Y^1_t$, $h_1 S^1_t + h_2 S^2_t = Y^1_t + Y^2_t$.

Then (3.12) in $t = 0$ will be:

$$\psi(0) = e^{-rT}\hat{E}[(w_1 h_1 S^1_T + w_2 h_2 S^2_T - K)^+] = e^{-rT}\hat{E}[((w_1 + w_2)Y^1_T - (K - w_2 Y^2_T))^+] \tag{3.13}$$

and we will compute the expression inside the expected value depending on the positivity of the strike.

$$\psi(0) = \begin{cases} (w_1+w_2)\int_0^{\frac{K}{w_2}} \hat{f}_{Y^2_T}(y)\, \underbrace{e^{-rT}\hat{E}[(Y^1_T - \frac{K - w_2 y}{w_1 + w_2})^+]}_{BS_0(\sigma_1, \frac{K - w_2 y}{w_1 + w_2})}\, dy & \text{, if } Y^2_T \le \frac{K}{w_2} \\ (w_1+w_2)\underbrace{\hat{E}[e^{-rT}Y^1_T \mathbf{1}_{\{Y^2_T > \frac{K}{w_2}\}}]}_{Y^1_0 \hat{\mathbf{Q}}(Y^2_T > \frac{K}{w_2})} + w_2 \underbrace{\hat{E}[e^{-rT}(Y^2_T - \frac{K}{w_2})\mathbf{1}_{\{Y^2_T > \frac{K}{w_2}\}}]}_{BS_0(\sigma_2, \frac{K}{w_2})} & \text{, if } Y^2_T > \frac{K}{w_2} \end{cases} \tag{3.14}$$

To sum up, the price at time $t = 0$ of a European call option written on a $T^\star$-forward contract on electricity is given by

$$\pi_0 = \int_{\mathbf{R}} f_{D_T}(z) \int_{\mathbf{R}^2} f_{C^1_T}(c_1) f_{C^2_T}(c_2)\psi_0(c_1, c_2, z)dc_2 dc_2 dz$$

where $\psi_0(c_1, c_2, z)$ is binding (3.14) together:

$$\psi_0(c_1, c_2, z) = (w_1 + w_2)\{\int_0^{\frac{K}{w_2}} \hat{f}_{Y^2_T}(y) BS_0(\sigma_1, \frac{K - w_2 y}{w_1 + w_2})dy + Y^1_0 \Phi(d)\} + w_2 BS_0(\sigma_2, \frac{K}{w_2})$$

with Φ being the cumulative distribution function of a standard normal random variable and $d = \frac{(r - \frac{\sigma_2^2}{2}) - ln(\frac{K}{w_2})}{\sigma_2\sqrt{T}}$.

3.2 A BSDE approach to local-risk minimization in our forward market

The problem of finding quadratic hedging strategies such as LRM can also be tackled via backwards stochastic differential equations (BSDEs)[1]. The advantage of taking this approach is that it allows us to determine the locally risk minimizing strategy without the need to determine the minimal martingale measure. This is addressed, among others, by by Jeanblanc et al. (2012) [17] and Di Nunno et al. (2015) [11]. In the sequel, we are relating the LRM strategy problem for our spot price model introduced in Section 3.1 to the BSDEs setup in Di Nunno et al.(2015)[11].

Consider the same assumptions on the complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ as in Assumption 2.

Notation

- L^2_T denotes the space of $\mathcal{F}_T$-measurable random variables $X : \Omega \to \mathbf{R}$ satisfying that $||X||^2 = \hat{E}[X^2] < \infty$.

- H^2_T denotes the space of $\mathcal{F}_T$-predictable processes $\varphi : \Omega \times [0, T] \to \mathbf{R}$ satisfying that $||\varphi||^2_{H^2_T} = E[\int_0^T |\varphi(t)|^2 dt] < \infty$.

To set the grounds, the following results are proved by Kunita and Watanabe (1967) [20].

To begin with, Di Nunno and Eide (2010) [12] give the representation with explicit integrands in terms of non-anticipating derivatives. In our setup, this could be given as follows.

Theorem 7. *Every random variable $X \in L^2_T$ has a unique representation of the form:*

$$X = X^0 + \sum_{k=1}^{3} \int_0^T \rho_k(t) \mu_k(dt)$$

where the stochastic integrators

$$\mu_1(dt) = W(dt) \qquad \mu_2(dt) = W^C(dt) \qquad \mu_3(dt, dz) = W^D(dt)$$

are orthogonal martingale random variables on $[0, T]$ and the stochastic integrands $\rho_i \in H^2_T$, for $i = 1, 2, 3$. Moreover, $X^0 = E[X]$.

This result can be read as the KW integral representation for orthogonal martingales as noises.

Now, turning to our hedging problem, assume as before that we want to find the LRM strategy for hedging the contingent claim $H = \varphi(F^i_T(T^\star), F^e_T(T^\star), C_T, D_T)$. Our hedging instruments represent the forward contracts on fuels and on electricity given by the $\mathbf{P}$-dynamics developed in Section 1.3.3:

[1]see Appendix B for definition and basic results on BSDEs

$$dF_t^i(T) = e^{r(T-t)}\Big(- rS_t^i + \frac{1}{h_i}\sum_{j\le i}\mu_j Y_t^j\Big)dt + e^{r(T-t)}\frac{1}{h_i}\sum_{j\le i}\sigma_j Y_t^j dW_t^j \tag{3.15}$$

$$dF_t^e(T) = \theta_t^{drift}dt + \theta_t^S \cdot dW_t + \theta_t^D dW_t^D + \theta_t^C \cdot dW_t^C \tag{3.16}$$

By definition of semimartingales, we can decompose $F_t^i(T^\star), F_t^e(T^\star)$ into locally square integrable local martingales $M = (M^i)_{i=1..n}$ and M^e

$$M_t^i = \int_0^t e^{T^\star - s}\frac{1}{h_i}\sum_{j\le i} Y_s^j \sigma_j dW_s^j \tag{3.17}$$

$$M_t^e = \int_0^t \theta_s^S \cdot dW_s + \int_0^t \theta_s^C \cdot dW_s^C + \int_0^t \theta_s^D dW_s^D$$

with $M_0 = 0, M_0^e = 0$ and predictable finite variables $A = (A^i)_{i=..n}$ and A^e

$$\begin{aligned} A_t^i &= \int_0^t e^{r(T^\star-s)}\Big(- rS_s^i + \frac{1}{h_i}\sum_{j\le i}\mu_j Y_s^j\Big)ds \\ A_t^e &= \int_0^t \theta_s^{drift}ds \end{aligned} \tag{3.18}$$

The sharp bracket processes of M, M^e are given by:

$$\begin{aligned} < M^i >_t &= \int_0^t e^{2r(T^\star-s)}\frac{1}{h_i}\sum_{j\le i}(Y_s^j\sigma_j)^2 ds \\ < M^e >_t &= \int_0^t \{\theta_s^S\theta_s^S\prime + \theta_s^C\theta_s^C\prime + (\theta_s^D)^2\}ds \end{aligned} \tag{3.19}$$

In the spirit of the structure condition (SC), the processes $(A^i)_{i=1..n}$, A^e can be represented as

$$\begin{aligned} A_t^i &= \int_0^t \omega_s^i d < M^i >_s \\ A_t^e &= \int_0^t \omega_s^e d < M^e >_s \end{aligned} \tag{3.20}$$

provided that

$$\begin{aligned} \omega_t^i &:= \frac{-rS_t^i + \frac{1}{h_i}\sum_{j\le i}\mu_j Y_t^j}{\frac{1}{h_i}\sum_{j\le i}(\sigma_j Y_t^j)^2} \\ \omega_t^e &:= \frac{\theta_t^{drift}}{\theta_t^S\theta_t^S\prime + \theta_t^C\theta_t^C\prime + (\theta_t^D)^2} \end{aligned} \tag{3.21}$$

By definition of mean-variance tradeoff process, we have that $\hat{K}_t = \int_0^t \omega_s d < M >_s \omega_s'$ and $\hat{K}_t^e = \int_0^t \omega_s^e d < M^e >_s \omega_s^e\prime$.

Notation

- $L(F(T^\star))$ (respectively $L(F(T^\star)))$ represent the space of $F(T^\star)$- (respectively $F^e(T^\star)$)-integrable processes.
- $\Theta = \{\xi \in L(F(T^\star)) \ | \ E[\int_0^T \xi_s d < M >_s \xi_s' + (\int_0^T |\xi_s \cdot dA_s|)^2] < \infty\}$
- $\Theta^e = \{\xi^e \in L(F^e(T^\star)) \ | \ \hat{E}[\int_0^T (\xi_s^e)^2 d < M^e >_s + (\int_0^T |\xi_s^e dA_s^e|)^2] < \infty\}$

Take $\xi \in \Theta$ and $\xi^e \in \Theta^e$ and $\tilde{H} = e^{-rT}H$ the discounted value of our contingent claim. Then define the discounted process

$$\tilde{V}_t^H = E[\tilde{H} - \int_t^T \xi_s \cdot dA_s - \int_t^T \xi_s^e dA_s^e | \mathcal{F}_t]$$

Here, ξ_s, ξ_s^e are assumed to contain the discounting factor $e^{-r(T-s)}$.

Applying the GKW decomposition of the process $U_T := \tilde{H} - \int_t^T \xi_s \cdot dA_s - \int_t^T \xi_s^e dA_s^e$ under $\mathbf{P}$, we get that there exist $\tilde{\xi} \in L(F(T^\star))$ and $\tilde{\xi}^e \in L(F^e(T^\star))$ such that:

$$U_T = E[\tilde{H} - \int_0^T \xi_s \cdot dA_s - \int_0^T \xi_s^e dA_s^e] + \int_0^T \tilde{\xi}_s \cdot dM_s + \int_0^T \tilde{\xi}_s^e dM_s^e + L_T^H \tag{3.22}$$

where L^H is orthogonal to M and M^e.

Taking conditional expectation over (3.22) and knowing that $E[U_T|\mathcal{F}_t] = \tilde{V}_t^H, \forall 0 \leq t \leq T$, we get:

$$\tilde{V}_t^H = V_0^H + \int_0^t \xi_s \cdot dA_s + \int_0^t \tilde{\xi}_s \cdot dM_s + \int_0^t \xi_s^e dA_s^e + \int_0^t \tilde{\xi}_s^e dM_s^e + L_t^H \tag{3.23}$$

Remark 9. A result of Schweizer (1994)[33] claims that $\xi = \tilde{\xi}$ in $L^2(M)$ and $\xi^e = \tilde{\xi}^e$ in $L^2(M^e)$ if the mean-variance tradeoff processes $\hat{K}$, $\hat{K}^e$ are absolutely bounded. Hence, the following inequalities hold for some positive contstants $C, C^e, \forall 0 \leq t \leq T$:

$$\frac{|-rS_t^i + \frac{1}{h_i}\sum_{j\leq i} \mu_j Y_t^j|}{\sqrt{\frac{e^{r(T-t)}}{h_i}\sum_{j\leq i}(\sigma_j Y_t^j)^2}} \leq C \tag{3.24}$$

$$\frac{|\theta_t^{drift}|}{\sqrt{\theta_t^S \cdot \theta_t^S{\prime} + \theta_t^C \cdot \theta_t^C{\prime} + (\theta_t^D)^2}} \leq C^e \tag{3.25}$$

Under the previous result, we have that $\xi = \tilde{\xi}$ and $\xi^e = \tilde{\xi}^e$ in the decomposition (3.23). In addition, we have that $dF(T^\star) = dM + dA, \ \ dF^e(T^\star) = dM^e + dA^e$ since the forward price processes are semimartingales. Then we can re-write the last decomposition as the actual FS decomposition:

$$\tilde{V}_t^H = \tilde{V}_0^H + \int_0^t \xi_s \cdot dF_s(T^\star) + \int_0^t \xi_s^e dF_s(T^\star)^e + L_t^H \tag{3.26}$$

where L^H is orthogonal to $F(T^\star)$ and $F^e(T^\star)$.

The FS decomposition (3.26) provides a direct path to the LRM strategy that we are looking for. Namely $\tilde{V}_t^H$ is the value of the portfolio at time t; ξ, ξ^e are the number of forward contracts on fuel and electricity to enter at time t and $L^H + \tilde{V}_0^H$ stands for cost process in the LRM strategy. These components can be identified by solving the BSDE:

$$\begin{aligned} d\tilde{V}_t^H &= \xi_t \cdot dF_t(T^\star) + \xi_t^e dF_t^e(T^\star) + dL_t^H \\ \tilde{V}_T^H &= \tilde{H} \end{aligned} \tag{3.27}$$

Furthermore, as $L_T^H \in L_T^2$ (i.e is $\mathcal{F}_T$-measurable square integrable), we can apply the initial Theorem 7 to decompose it in the following manner:

$$L_t^H = E[L_T^H] + \int_0^t \alpha_s \cdot dW_s + \int_0^t \beta_s \cdot dW_s^C + \int_0^t \gamma_s dW_s^D \tag{3.28}$$

But the expected value of a martingale is equal to the expected value of its initial value, so $E[L_T^H] = E[L_0^H] \underbrace{=}_{L_0^H = 0 \text{ from } (3.26)} 0$. Hence:

$$L_t^H = \int_0^t \alpha_s \cdot dW_s + \int_0^t \beta_s \cdot dW_s^C + \int_0^t \gamma_s dW_s^D \tag{3.29}$$

with $\alpha = (\alpha^i)_{i=1..n}, \quad \beta = (\beta^k)_{k=1..n}$

Finally, plugging the decomposition (3.29) and the $\mathbf{P}$-dynamics of $F_t^i(T^\star)$ (3.15) , respectively of $F_t^e(T^\star)$(3.16) into the BSDE (3.27) we have that:

$$\begin{aligned} d\tilde{V}_t^H &= \Big\{ \sum_{i=1}^n \xi_t^i e^{r(T^\star - t)} \Big(-rS_t^i + \frac{1}{h_i} \sum_{j \le i} Y_t^j \mu_j \Big) + \xi_t^e \cdot \theta_t^{drift} \Big\} dt + \\ &+ \Big\{ \sum_{i=1}^n \xi_t^i \frac{e^{r(T^\star - t)}}{h_i} \sum_{j \le i} \sigma_j Y_t^j dW_t^j \Big\} + (\xi_t^e \cdot \theta_t^S + \sigma_t) \cdot dW_t + \\ &+ (\xi_t^e \cdot \theta_t^C + \beta_t) \cdot dW_t^C + (\xi_t^e \cdot \theta_t^D + \gamma_t) \cdot dW_t^D \\ \tilde{V}_T^H &= \tilde{H} \end{aligned} \tag{3.30}$$

Last but not least, we should add the orthogonality conditions given by L^H being orthogonal to $F(T^\star), F^e(T^\star)$. See Appendix B.2 for this.

Therefore, under this approach, we have managed to bridge the gap between the GKW decomposition of the contingent claim and the locally risk minimizing strategy. The solution to the BSDE (3.30) gives the locally risk minimizing strategy in our spot price model.

3.3 Mean-variance hedging in our forward market

The key-point in determining the mean-variance hedging strategy is to obtain first the VOMM, then the expectation under this measure of the density $\tilde{Z}$ of it, describing the change of measure.

3.3.1 The variance optimal martingale measure in our model

We know about the VOMM that it should satisfy (2.5). There are several results linking MMM with VOMM. Vandaele (2009) [36] claims that if the mean-variance tradeoff process $\hat{K} = \int \lambda' d < S > \lambda$ is deterministic, then the VOMM and MMM coincide. Also, if this process is bounded, then the density $\hat{Z}_T \in L^2(\mathbf{P})$ and there exists a FS decomposition

$$\hat{Z}_T := \frac{d\hat{\mathbf{P}}}{d\mathbf{P}} = E[\hat{Z}_T^2] - E[\hat{Z}_T \hat{L}_T] + \int_0^T \hat{\xi}_s dS_s + \hat{L}_T \tag{3.31}$$

where $\int \hat{\xi}$ and $\hat{L}$ are both $\hat{\mathbf{P}}$-martingales.

Delbaen and Schachermayer (1996) ([9]) prove that MMM coincides with VOMM if $\hat{L}_T = 0$ in (2.23).

In what follows, we will assume that $\hat{K}$ is deterministic and bounded, hence that our VOMM is the measure $\hat{\mathbf{P}}$ determined by (3.1).

3.3.2 The mean-variance solution

We will hedge risk with forward contracts on electricity and forward contracts on fuels. Consider the same contingent claim $H = \varphi(F_T^e(T^\star), F_T(T^\star), C_T, D_T)$. We have obtained in the LRM approach the GKW decomposition of H (3.7). Now we will identify each element from Theorem 6 with the aid of ξ, ξ^e, L^H giving the GKW of H.

$$V_t^H = H_0 + \int_0^t \xi_s dF_s(T^\star) + \int_0^t \xi_s^e dF_s^e(T^\star) + L_t^H$$

is completely determined by (3.8), (3.9), (3.10).

By the definition of MMM density in our setup,

$$\hat{Z}_t = \epsilon(-\int_0^t \omega_s \cdot dM_s - \int_0^t \omega_s^e dM_s^e)_t$$

where $M = (M_t)_{0\le t\le T}^{i=1..n}$ and $(M_t^e)_{0\le t\le T}$ are the local martingales defined by (3.17) and $(\omega_t^i)_{0\le t\le T}, i = 1..n$, $(\omega_t^e)_{0\le t\le T}$ are defined by (3.21). This can be further written as:

$$\hat{Z}_t = \epsilon(\int_0^t \omega_s \cdot dF_s(T^\star) + \int_0^t \omega_s^e dF_s^e(T^\star) + \hat{K}_t + \hat{K}_t^e) = e^{\hat{K}_t + \hat{K}_t^e} \epsilon(\int_0^t \omega_s \cdot dF_s(T^\star) + \int_0^t \omega_s^e dF_s^e(T^\star))_t$$

We also have that:

$$\hat{Z}_T = e^{\hat{K}_T + \hat{K}_T^e} \epsilon(\int_0^T \omega_s \cdot dF_s(T^\star) + \int_0^T \omega_s^e dF_s^e(T^\star))_T$$

Assuming that $\hat{K}_T$ and $\hat{K}_T^e$ are deterministic allows us to write that

$$\hat{Z}_T = e^{\hat{K}_T + \hat{K}_T^e} + \int_0^T e^{-\hat{K}_T} \epsilon(-\int_0^t \omega_s \cdot dF_s(T^\star))_t \omega_t dF(T^\star)_T + \int_0^T e^{-\hat{K}_T^e} \epsilon(-\int_0^t \omega_s^e dF_s^e(T^\star))_t \omega_t^e dF^e(T^\star)_T$$

In order to apply Theorem 6, we need to look for $\hat{\xi}, \hat{\xi}^e$, giving

$$\hat{Z}_t^0 = E[\hat{Z}_T^2] + \int_0^t \hat{\xi}_s dF_s(T^\star) + \int_0^t \hat{\xi}_s^e dF_s^e(T^\star)$$

Now, as $\epsilon(-\int_0^t \omega_s \cdot dF_s(T^\star))_t$ and $\epsilon(-\int_0^t {\omega^e}_s dF_s^e(T^\star))_t$ are $\hat{\mathbf{P}}$-martingales, we have:

$$\hat{Z}_t^0 = e^{\hat{K}_T + \hat{K}_T^e}\epsilon(-\int_0^t \omega_s \cdot dF_s(T^\star) - \int_0^t \omega_s^e dF_s^e(T^\star))_t$$

Then we get

$$-\frac{\hat{\xi}_t^i}{\hat{Z}_t^0} = \omega_t^i, \quad i = 1..n$$

$$-\frac{\hat{\xi}_t^e}{\hat{Z}_t^0} = \omega_t^e$$

Then, our solution to the MVH problem, following Theorem 6, is:

$$\theta_t^{\star,i} = \xi_t^i + \omega_t^i\bigg(V_t^H + x - \int_0^t \theta_s^{\star,i} dF_s^i(T^\star)\bigg), i = 1..n \tag{3.32}$$

$$(\theta_t^e)^\star = \xi_t^e + \omega_t^e\bigg(V_t^H + x - \int_0^t (\theta_s^e)^\star dF_s^e(T^\star)\bigg) \tag{3.33}$$

where

$$\begin{aligned}
\xi_t^e &= \frac{1}{||(\theta_t^C, \theta_t^D)||^2}[\sum_{i=1}^n \theta_t^{C,i}\frac{\partial\phi}{\partial c_i}\beta_i(t, C_t^i) + \theta_t^D\frac{\partial\phi}{\partial d}b(t, D_t)] \\
\xi_t^i &= \frac{\partial\phi}{\partial y_i} + \frac{h_i G_i^{T^\star}(t, C_t, D_t)}{||(\theta_t^C, \theta_t^D)||^2}[\sum_{i=1}^n \theta_t^{C,i}\frac{\partial\phi}{\partial c_i}\beta_i(t, C_t^i) + \theta_t^D\frac{\partial\phi}{\partial d}b(t, D_t)] \\
\omega_t^i &= \frac{-rS_t^i + \frac{1}{h_i}\sum_{j\leq i}\mu_j Y_t^j}{\frac{1}{h_i}\sum_{j\leq i}(\sigma_j Y_t^j)^2} \\
\omega_t^e &= \frac{\theta_t^{drift}}{\theta_t^S\theta_t^S{\prime} + \theta_t^C\theta_t^C{\prime} + (\theta_t^D)^2}
\end{aligned} \tag{3.34}$$

Chapter 4

Preliminaries on time-change of Lévy processes

In the remaining part of the thesis we aim at making a further development of the spot model of Aïd et al. (2013)[1] by introducing a random time-change that captures stochastic volatility. We will see how the future prices change under this procedure, as well as the quadratic hedging that we discussed in the previous chapter. The theoretical framework is settled by Carr and Wu (2004)[6]. The novelty of their paper consists in the methodology they develop for tackling the new time-changed Lévy process. They introduce a complex-valued measure that absorbs all the possible correlation between the initial Lévy process and the time-changing process[1]. With the aid of this measure, the characteristic function of the time-changed process is computed and contingent claims can be priced by FFT(Fast Fourier Transform). Since we will assume independence between the subordinated Brownian motion and the time-changing process, there is no leverage effect and no need for a change of measure and we can directly proceed to valuing contingent claims.

Assumption 2 and a time horizon $T \in [0, \infty]$ still apply to the upcoming results. Also, we call a *càdlàg* (fr. *continue à droite, limité à gauche*) function one which is continuous to the right and has a limit to the left.

4.1 Lévy processes

Definition 25. A Lévy process $L = (L_t)_{0 \leq t \leq T}$ is a càdlàg adapted, real-valued stochastic process with $L_0 = 0$ $\mathbf{P}$-a.s. which satisfies the following:
(a). L has *independent increments*: $L_t - L_s$ independent of $\mathcal{F}_s$, $0 \leq s \leq t \leq T$
(b). L has *stationary increments*: the distribution of $L_{t+s} - L_t$ does not depend on $t, \forall 0 \leq s, t \leq T$
(c). L is *stochastically continuous*: $\lim_{s \to t} \mathbf{P}(|L_t - L_s| > \varepsilon) = 0$, $\forall 0 \leq t \leq T, \forall \varepsilon > 0$

Note that the first two properties are common with the definition of Brownian motion, while the last property insures that there are no jumps at fixed times, but only at random times.

[1]The negative correlation between the initial process and

Assumption 5. *Let* $(X_t)_{0\le t\le T}$*, with* $X_0 = 0$ *be a d-dimensional Lévy process with real values. As any such process,* X *is characterized by the triplet* (μ, Σ, Π)*, called Lévy characteristics and given by the Lévy-Itô decomposition*[2]*. Here,* $\mu \in \mathbf{R}^d$ *is a vector,* $\Sigma \in \mathcal{M}_d(\mathbf{R})$ *is a positive semi-definite d-dimensional matrix and* $\Pi \in \mathbf{R}^d\backslash\{0\} = \mathbf{R}^{d\star}$ *is the Lévy measure, which describes the arrival rates for jumps of every size for each component of* X*.* Π *is formally defined as*

$$\Pi(A) = E[\#\{t \in [0,1] | \Delta X_t \neq 0, \Delta X_t \in A\}], \quad \forall A \in \mathcal{B}(\mathbf{R}^d) \tag{4.1}$$

satisfying that

$$\int (|x|^2 \wedge 1)\Pi(dx) < \infty \tag{4.2}$$

The Lévy-Khinchin representation theorem connects the Lévy components and the characteristic function of a Lévy process:

Theorem(Lévy-Khincin decomposition) 1. *Let* $(X_t)_{0\le t\le T}$ *be a* $\mathbf{R}^d$*-valued Lévy process with characteristics* (μ, Σ, Π)*. Then*

$$E[e^{iz'X_t}] = e^{t\psi(z)}, \quad \forall z \in \mathbf{R}^d, \tag{4.3}$$

$$\Psi(z) = -\frac{1}{2}z'\Sigma z + i\mu' z + \int_{\mathbf{R}^d} (e^{iz'x} - 1 - iz'x\mathbf{1}_{|x|\le 1})\Pi(dx) \tag{4.4}$$

In other words, the characteristic function of our process X_t evaluated in $\theta \in \mathbf{R}^d$ is given by (4.3) in the Lévy-Khintchine Theorem:

$$\phi_{X_t}(\theta) = E[e^{i\theta' X_t}] = e^{-t\Psi_X(\theta)}, \quad t \ge 0 \tag{4.5}$$

where ψ represents the characteristic exponent of X and is given by (4.4):

$$\Psi_X(\theta) = -i\mu'\theta + \frac{1}{2}\theta'\Sigma\theta + \int_{\mathbf{R}^{d\star}} (1 - e^{i\theta' x} + i\theta' x\mathbf{1}_{|x|<1})\Pi(dx) \tag{4.6}$$

The domain of the characteristic function ϕ is extended to the complex plane: $\mathcal{D} \subset \mathbf{C}^d$, where $\mathcal{D}$ is such that the expectation in (4.5) is well defined.

4.2 The time-change process

Let $(X_t)_{0\le t\le T}$ be the Lévy process as previously defined (also called base process) and consider a non-negative, non-decreasing stochastic process $(T_t)_{0\le t\le T}$ not necessarily independent of $(X_t)_{0\le t\le T}$. The time-changed process $Y_t = X_{T_t}, \ 0 \le t \le T$ is a random variable with respect to the new filtration $\tilde{\mathcal{F}}_t$ where:

$$\tilde{\mathcal{F}}_t := \sigma\{Y_s, T_s, s \le t\} \vee \mathcal{N}, \tag{4.7}$$

$\mathcal{N}$ being the $\mathbf{P}$-null sets. In the coming results, we will refer to this definition as our new choice of filtration.

The process T_t described above is a *random time change* and can represent the change from calendar time to business time.

[2]see Appendix A.4

Remark 10. If the time-change T_t is normalized such that $E[T_t] = t$, then this will represent the actual calendar time.

Generally, we will deal with time-changed Lévy processes, but later on we shall narrow the study down to time-changed Brownian motion. Indeed, if we were to deal with some more specific processes (continuous local martingales, semimartingales), the following results would still narrow the problem down to time-changed Brownian motion.

Dubins-Schwarz theorem 1. *Every continuous local martingale* $(M_t)_{t\geq 0}$ *can be written as a time-changed Brownian motion* $(B_{<M>_t})_{0\leq t\leq T}$*, where* $(<M>_t)_{0\leq t\leq T}$ *represents the continuous quadratic variation process (sharp bracket process) of M.*

Remark 11. In the framework of Dubins-Schwarz theorem, take $M_t = \int_0^t \sigma_s dB_s$ and consider that $(\sigma_s)_{0\leq s\leq T}$ is independent non-negative with càdlàg sample paths. Then the quadratic variation of the time-changing process is $<M>_t = \int_0^t \sigma_s^2 ds$.

This result can be interpreted as a scaling property of the Brownian motion. It means that Brownian motion translates spatial scaling ($\sigma_t B_t$) into time-scaling ($B_{\sigma_t^2 t}$). The result applies to semimartingales to a certain extent.

Monroe theorem 1. *Every (càdlàg) semimartingale* $(S_t)_{0\leq t\leq T}$ *can be written as a time-changed Brownian motion* $(B_{T_t})_{0\leq t\leq T}$*, for a (càdlàg) family of stopping times* $(T_t)_{0\leq t\leq T}$ *on a suitably extended probability space.*

The latter result means that every arbitrage-free model can be expressed in terms of time-changed Brownian motion.

Furthermore, in terms of the types of time-change processes $(T_t)_{0\leq t\leq T}$, we consider two types: **subordinators** and **absolutely continuous time-changes**.

4.2.1 Subordinators

Definition 26. A **subordinator** is an $\mathbf{R}$-valued non-decreasing Lévy process $(T_t)_{0\leq t\leq T}$ which takes only positive values.

The following proposition gives an equivalent definition for subordinators in terms of Lévy triplet.

Proposition 5. *A Lévy process* $(T_t)_{0\leq t\leq T}$ *on* $\mathbf{R}$ *is a subordinator if and only if its Lévy triplet is given by* $(b, 0, \rho)$*, where* $\rho((-\infty, 0)) = 0$ *and* $\int_0^\infty (1 \wedge |x|)\rho(dx) < \infty$.

In order to determine the time-changed process $(Y_t = X_{T_t})_{0\leq t\leq T}$ by its Lévy triplet, we need to determine first the Lévy triplet of the subordinator itself (T_t), then to study the connection between the two Lévy triplets.

Firstly, we shall compute the subordinator‘s characteristic function and exponent by applying the Lévy-Khincin decomposition 1 :

$$\Psi_{T_t}(u) = -ibu + \int_0^\infty [1 - e^{izu} + izu\mathbf{1}_{(0,1)}(z)\rho(dz)], \forall u \in \mathbf{R} \tag{4.8}$$

$$\phi_{T_t}(u) = E[e^{iuT_t}] = e^{-t\Psi_{T_t}(u)} = e^{iaut - t\int_0^\infty (1-e^{izu})\rho(dz)} \tag{4.9}$$

where we made the notation

$$a := b - \int_0^1 x\rho(dx)$$

Introducing the variable change $u := i\lambda, \lambda \geq 0$ into (4.8) and (4.9) we get

$$\phi_{T_t}(\lambda) = E[e^{-\lambda T_t}] = e^{-tl(\lambda)} \tag{4.10}$$

$$l(\lambda) = a\lambda + \int_0^\infty (1 - e^{-\lambda z})\rho(dz) \tag{4.11}$$

Definition 27. The function $l(.)$ in (4.11) is called the **Laplace exponent** and the term a is called the **drift**.

Secondly, Theorem 4.2 in Cont and Tankov (2004) [7] gives the connection between the Lévy triplet of the subordinator discussed so far and that of the time-changed process $(Y_t = X_{T_t})_{0\leq t\leq T}$.

Theorem 8. *Let $(X_t)_{t\geq 0}$ be an $\mathbf{R}^d$-valued Lévy process with triplet (μ, Σ, Π) and characteristic exponent $\Psi_X(.)$ given by (4.6). Let $(T_t)_{t\geq 0}$ be a subordinator with triplet $(b, 0, \rho)$ and Laplace exponent $l(.)$ given by (4.11) independent from $(X_t)_{0\leq t\leq T}$. Then, the subordinate $(Y_t)_{t\geq 0}$ to the process $(X_t)_{t\geq 0}$ is a Lévy process with characteristic function*

$$\phi_{Y_t}(\theta) = e^{tl(\Psi_{X_t}(\theta))}, \forall\theta \in \mathbf{R}^d \tag{4.12}$$

and triplet (μ^Y, Σ^Y, Π^Y) given by:

$$\mu^Y = b\mu + \int_0^\infty \rho(ds) \int_{|x|\leq 1} x p_s^X(dx) \tag{4.13}$$

$$\Sigma^Y = b\Sigma \tag{4.14}$$

$$\Pi^Y(A) = b\Pi(A) + \int_0^\infty p_s^X(A)\rho(ds), \forall A \in \mathcal{B}(\mathbf{R}^d) \tag{4.15}$$

where $p_.^X$ is the probability distribution of $(X_t)_{t\geq 0}$.

Proof. For a complete proof of the expressions of the triplet, see Theorem 30.1 in Sato (1999)[30]. We will see that $(Y_t)_{t\geq 0}$ is a Lévy process.

- Independent increments: Take a sequence of times $t_0 < \cdots < t_n$ and applying the property of independent increments for the Lévy processes X and T, plus the Lévy Khintchine formula for X, we get that

$$E[\prod_{i=1}^n e^{iu_i(X_{T_{t_i}} - X_{T_{t_{i-1}}})}] \underbrace{=}_{\text{tower property}} E[E[\prod_{i=1}^n e^{iu_i(X_{T_{t_i}} - X_{T_{t_{i-1}}})}|\mathcal{F}_\infty]] \underbrace{=}_{\text{independent increments of } X}$$

$$= E[\prod_{i=1}^n E[e^{iu_i(X_{T_{t_i}} - X_{T_{t_{i-1}}})}|\mathcal{F}_\infty]] \underbrace{=}_{\text{Lévy-Khincin theorem}} E[\prod_{i=1}^n e^{(T_{t_i} - T_{t_{i-1}})\Psi_Y(u_i)}] \underbrace{=}_{\text{independent increments of } T}$$

$$= \prod_{i=1}^n E[e^{(T_{t_i} - T_{t_{i-1}})\Psi_Y(u_i)}] = \prod_{i=1}^n E[e^{iu_i(X_{T_{t_i}} - X_{T_{t_{i-1}}})}]$$

- Stationary increments: by analogous argument.
- Stochastically continuous: The Lévy processes have stationary increments, consequently we have that every Lévy process is uniformly continuous in probability. This means that $P(|X_{T_s} - X_{T_t} > \epsilon| \mid |T_s - T_t| < \delta) = 1, \forall \epsilon, \delta > 0$. Then we can write that $\forall \epsilon > 0, \delta > 0$,

$$P(|X_{T_s} - X_{T_t}| > \epsilon) \underbrace{\leq}_{P(A) \leq P(A|B)+1-P(B)} P(|X_{T_s} - X_{T_t} > \epsilon| \mid |T_s - T_t| < \delta) + P(|T_s - T_t| \geq \delta)$$

And the first term can be made small by making δ small, while the second term will go to zero when $s \to t$ because T itself is continuous in stochastically continuous. Therefore, $\lim_{s \to t} P(|X_{T_s} - X_{T_t}| > \epsilon) = 0$

□

Finally, we are going to narrow our study down to a subordinated Brownian motion.

Remark 12. Brownian motion time-changed by an independent subordinator always yields a Lévy process.

Example 3. If W_t is a Brownian motion with respect to a filtration $\mathcal{F}_t^W$ and T_t a subordinator, then $X_t = \sigma W_{T_t} + \mu T_t, \sigma, \mu \in \mathbf{R}$ is a new Brownian motion with respect to the new filtration $\tilde{\mathcal{F}}^W := \sigma\{W_s, T_s, s \leq t\}$. Moreover, if W_t and T_t are independent, then X_t is a Lévy process.

Theorem 4.3 in Cont and Tankov (2004) [7] describes the jump structure of a process which is a subordinated Brownian motion with drift.

Theorem 9. *Let* $\Pi \in \mathbf{R}$ *be a Lévy measure and* $\alpha \in \mathbf{R}$*. There exists a Lévy process* $(Z_t)_{t \geq 0}$ *with measure* Π *such that* $Z_t = W_{T_t} + \alpha T_t$ *for some subordinator* $(T_t)_{t \geq 0}$ *with Lévy triplet* $(b, 0, \rho)$ *and an* $\mathcal{F}$*-Brownian motion* $(W_t)_{t \geq 0}$ *independent from* $(T_t)_{t \geq 0}$ *if and only if:*
(i). Π *is absolutely continuous with density* $\Pi(x)$
(ii). $\Pi(x)e^{-\alpha x} = \Pi(-x)e^{\alpha x}, \forall x \in \mathbf{R}$
(iii). $\Pi(\sqrt{u})e^{-\alpha\sqrt{u}}$ *is a completely monotonic function on* $(0, \infty)$.

The result can be extended to multidimensional Brownian motions.

Remark 13. The measure $\Pi \in \mathbf{R}^d$ such is described by the Theorem 9 can be a Lévy measure when $Z_t = W_{T_t}$ (no drift) if and only if Π is symmetric and $\Pi(\sqrt{u}), \forall u \in \mathbf{R}^d$ is a completely monotonic function on $(0, \infty)$.

Example 4. Take $X = (W_t)_{0 \leq t \leq T}$ a d-dimensional Brownian motion with respect to the filtration $\mathcal{F}$. Assume its diffusion component is Σ, hence its Lévy triplet is $(0, \Sigma, 0)$. Take also a subordinator $(T_t)_{0 \leq t \leq T}$ as defined before, with triplet $(b, 0, \rho)$ and Laplace exponent given by (4.11). Then, by Theorem 8, we determine the triplet of $Y_t = W_{T_t}$. Hence,

$$\Psi_{X_t}(\theta) = -\frac{1}{2}\theta' A\theta \Longrightarrow \phi_{Y_t}(\theta) = e^{tl(\Psi_{X_t}(\theta))} = e^{tl(-\frac{1}{2}\theta' A\theta)}$$

$$\mu^Y = \int_0^\infty \rho(ds) \int_{|x| \leq 1} x p_s^W(ds)$$

$$A^Y = bA$$
$$\Pi^Y(B) = \int_0^\infty p_s^W(B)\rho(ds)$$

with $p_s^W(.)$ the probability distribution of W.

Generally, a subordinator can be modeled as the following semimartingale:

$$T_t = \alpha_t + \int_0^t \int_0^\infty y\nu(dy, ds) \tag{4.16}$$

Here, α_t is a non-negative process of bounded variation and ν represents the counting measures of the jumps in the process $(T_t)_{t\geq 0}$.

4.2.2 Absolutely continuous time changes

Definition 28. The **absolutely continuous time-changes** are processes of the type of (4.16) from where the jumps in time have been removed.

Hence, such a time-change is of the following form:

$$T_t = \int_0^t v(s)ds \tag{4.17}$$

where v represents the *instantaneous activity rate* and is a positive process. Note that $v(.)$ might have jumps, but $(T_t)_{t\geq 0}$ remains a continuous process.

Next, we want to determine the characteristic function of the new time-changed process $Y_t = X_{T_t}$. By definition, this will be the expected value over two sources of randomness:

$$\phi_{Y_t}(\theta) := E[e^{i\theta' X_{T_t}}] = E[E[e^{i\theta' X_u}|T_t = u]] \tag{4.18}$$

If we consider that the time-change T_t and the process X_t are independent, then we can take out part of the expectation and apply the result in (4.5):

$$\phi_{Y_t}(\theta) = E[e^{-T_t\Psi_X(\theta)}] \tag{4.19}$$

Note that the last term is of the form of a Laplace transform of the function T_t evaluated in $\Psi_X(\theta)$. Then, we can write:

$$\phi_{Y_t}(\theta) = \mathscr{L}_{T_t}(\Psi_X(\theta)) \tag{4.20}$$

where we denote by $\mathscr{L}$ the Laplace transform with Laplace exponent $l(.)$ previously defined. In our setup, this is of the form:

$$\mathscr{L}_{T_t}(\lambda) = E[\exp(-\lambda \int_0^t v(s)ds)]$$

4.3 Affine activity rate processes

The determination of the characteristic function and Laplace transform goes forcedly through discussing the form of the activity rate model $v(.)$ itself. We can assume our initial process to be a Lévy jump-diffusion (Brownian motion and a compensated compound Poisson process). Many types of such processes, together with their characteristic exponent are depicted in Table 1 in Carr and Wu (2004)[6]. Assume, generally, that the instantaneous activity rate $v(.)$ is a function of a Markov process $(Z_t)_{0\leq t\leq T}$ which has the following dynamics:

$$\begin{aligned} dZ_t &= \mu(Z_t)dt + \sigma(Z_t)dW_t \\ Z_0 &= z_0 \end{aligned} \tag{4.21}$$

where the first term is a d-dimensional drift vector and the second one a diffusion matrix (d-dimensional), W a d-dimensional Brownian motion with respect to a filtration $(\mathcal{F})_{0\leq t\leq T}$

Definition 29. The instantaneous activity rate $v(.)$ of $T_t = \int_0^t v(s)ds$ is **affine** if it can be written as:

$$v(t) = b_v' Z_t + c_v$$

where $b_v \in \mathbf{R}^d$ and $c_v \in \mathbf{R}$.

Once the activity rate has such a form, the drift vector $\mu(Z)$ and the covariance matrix $\sigma(Z)'\sigma(Z)$ are all Z-affine, then, by Proposition 1 in Carr and Wu (2004)[6], the Laplace transform is exponential-affine in z_0 and is given by:

$$\mathscr{L}_{T_t}(\lambda) = E[e^{-\lambda T_t}] = \exp(-b(t)'z_0 - c(t)) \tag{4.22}$$

where $b(.) \in \mathbf{R}^d$ and $c(.)$ is a scalar function.

Now, determining $b(.), c(.)$ from (4.22) comes down to solving some differential equations.

First, we establish the initial assumptions:

$$\begin{cases} v(t) = b_v' Z_t + c_v \\ \mu(Z_t) = a - kZ_t \\ (\sigma(Z_t)\sigma(Z_t)')_{ii} = \alpha_i + \beta_i' Z_t \\ (\sigma(Z_t)\sigma(Z_t)')_{ij} = 0, i \neq j, \text{the diffusion covariance matrix is diagonal} \end{cases} \tag{4.23}$$

Then the differential equations governing $b(.), c(.)$ are:

$$\begin{cases} b(t)\prime = \lambda b_v - k'b(t) - \frac{\beta b(t)^2}{2} \\ b(0) = 0 \end{cases} \tag{4.24}$$

$$\begin{cases} c(t)\prime = \lambda c_v + b(t)'a - \frac{b(t)'\alpha b(t)}{2} \\ c(0) = 0 \end{cases} \tag{4.25}$$

where α is a diagonal matrix and β is a d×d-dimensional matrix with i-th column β_i and $b(t)^2$ is a vector.

Hence, the Laplace transform and thus the characteristic function of the time-changed Lévy process remain completely determined.

4.4 Valuing contingent claims

The pricing of contingent claims is done in Carr and Wu (2004)[6] with the method of fast Fourier transforms (FFT), given the generalized Fourier transform of, for instance, the state vector Y_t or the return s_t of an asset with a price process of the type $S_t = S_0 e^{\gamma' Y_t}$. In a more general setup, we consider a European contingent claim with payoff

$$\Pi_Y(k; , a, b, \gamma, c) = (a + be^{\gamma' Y_t})\mathbf{1}_{c'Y_t \le k} \tag{4.26}$$

where we have performed a time-change $Y_t = X_{T_t}$.

Example 5. Under this definition, a European call option with strike K whose underlying asset has a price process that is described in the previous lines will have as payoff $(S_0 e^{\gamma' Y_t} - K)_+$, meaning that

$$\Pi(-\ln \frac{K}{S_0}; -K, S_0, \gamma, \gamma)$$

Example 6. A European put option with strike K has payoff $(S_0 e^{\gamma' Y_t} - K)_+$, hence

$$\Pi(\ln \frac{K}{S_0}; -K, S_0, \gamma, \gamma)$$

Moreover, let $G(k; a, b, \gamma, c)$ be the initial price of a contingent claim with payoff as in (4.26). For determining the forward price of the claim at the initial time, one has to compute $G(k; a, b, \gamma, c) = E[\Pi_Y(k; a, b, \gamma, c)]$ under the forward measure [3]. For this purpose, Carr and Wu (2004)[6] introduced the FFT method. Denote by $\mathscr{G}(k; a, b, \gamma, c)$ the generalized transform of $G(k; a, b, \gamma, c)$.

$$\mathscr{G}(k; a, b, \gamma, c) = \int_{-\infty}^{\infty} e^{izk} G(k; a, b, \gamma, c) dk = \frac{i}{z}(a\phi_Y(zc) + b\phi_Y(zc - i\gamma)) \tag{4.27}$$

where z lies in a subspace of the complex plane where $\mathscr{G}(k; a, b, \gamma, c)$ is well defined.

Then, by the inversion formula, we obtain an approximation on finite interval:

$$G(k) = \frac{1}{2\pi} \int_{iz_i - \infty}^{iz_i + \infty} e^{-izk} \mathscr{G}(k; a, b, \gamma, c) \approx G(k)^\star = \frac{e^{z_i k}}{\pi} \sum_{k=0}^{N-1} e^{-iz_r(j)k} \varphi(z_r(j) + iz_i) \Delta z_r \tag{4.28}$$

where $z_r(j)$ are the nodes of $z_r = Re(z)$ and Δz_r is the spacing between the nodes.

[3]The measure under which the forward price is a martingale, i.e if r_t is an $\mathcal{F}_t$-adapted short-term interest rate process and N_t the numéraire process, then the derivative of the forward measure with respect to the risk-neutral measure is $\frac{d\mathbf{P}^\star}{d\mathbf{P}} = e^{\int_0^t r(s)ds} \frac{N_T}{N_0}$

Chapter 5

Hedging in the time-changed forward market

Suppose we intend to introduce a time-change process $(T_t)_{0\leq t\leq T}$ of the type defined in Section 4.3 in our spot model given by Aïd et al. (2013)[1]. We will adapt the theoretical framework settled by Carr and Wu(2004)[6] to our fuel, production and demand dynamics and we shall see what changes appear in the process of hedging.

Assume the random time-change process is given by the positive instantaneous activity rate v(.) and that $T_t = \int_0^t v(s)ds$ is an absolutely continuous process. Additionally, consider that we introduce the time-change to the n-dimensional Brownian motion $(W_t)_{0\leq t\leq T}$ describing the **P**-dynamics computed in Section 1.3.3 on the filtration generated by $\mathcal{F}_t = \mathcal{F}_t^S \vee \mathcal{F}_t^C \vee \mathcal{F}_t^D$. We will call the new time-changed process $\tilde{W}_t^i := W_{T_t}^i$. This is a *conditional Brownian motion* and a local martingale. We shorten, for the time being, the notation by suppressing the i superindice corresponding to the i-th fuel. Assume that T_t and W_t are independent and so the new process $\tilde{W}$ is a Lévy process.

With some abuse of notation, define the new filtration we will be working on again as

$$\mathcal{F} := \sigma\{W_s, T_s, s \leq t\} \vee \mathcal{F}^C \vee \mathcal{F}^D = \tilde{\mathcal{F}} \vee \mathcal{F}^C \vee \mathcal{F}^D$$

And consider the same probability space as in Assumption 1.

By the Dubins-Schwarz Theorem 1 and the Remark 11, we have the following equality in distribution:

$$d\tilde{W}_t^i = \sqrt{v(t)}dW_t^i \tag{5.1}$$

This allows us to determine the sharp-bracket process of $\tilde{W}^i$ for $i = 1..n$ as being

$$< \tilde{W}^i >_t = T_t \tag{5.2}$$

Indeed, as $\tilde{W}_t = \int_0^t \sqrt{v(s)}dW_s \Longrightarrow < \tilde{W} >_t = \int_0^t v(s)d < W >_s = \int_0^t v(s)ds = T_t$

Regarding the characteristic function of the time-changed process $\tilde{W}_t$, this can be determined by the Theorem **??** and by aid of the Table 1 in Carr and Wu (2004)[6]:

$$\phi_{\tilde{W}_t}(\theta) = E[e^{i\theta\tilde{W}_t}] = \mathscr{L}_{T_t}(\frac{1}{2}\theta^2) \tag{5.3}$$

Under $\mathbf{P}$, $v(t)$ satisfies the equations for the affine rates (4.24),(4.25) and our time-changed process has a characteristic function which is exponential-affine to v_0 of the form:

$$\phi_{\tilde{W}_t}(\theta) = \exp(-b(t)v_0 - c(t)) \tag{5.4}$$

5.1 A special case of affine activity rate

The closed-form of (5.4) can be obtained analytically only under special cases. This is due to the fact that the solutions of (4.24) and (4.25) can be analytically determined under further assumptions. Inspired by Heston(1993)[16], assume that $v(.)$ follows a mean-reverting square-root process i.e:

$$dv(t) = (a - kv(t))dt + \eta\sqrt{v(t)}dZ_t \tag{5.5}$$

The fact that we assumed independence between the subordinated Brownian motion (i.e W_t) and the time-changing process (T_t) translates into a zero-correlation between the Brownian motion driving the activity rate (i.e. Z_t) and the Brownian motion driving our process (i.e W_t) : $E[dZ_t d\hat{W}_t] = 0$. A negative correlation between these two processes would create a leverage effect that could be absorbed by a suitable change of measure (see Carr and Wu (2004) [6]).

Under the above conditions, one can actually solve the equations (4.24) and (4.25) in Section 4.3 giving the particular values $b_v = 1$, $c_v = 0$, $a^Q = a$, $\alpha = 0, \beta = \eta^2, \gamma = 0, \lambda = \Psi_W(\theta) = \frac{\theta^2}{2}$.

Therefore, we obtain:

$$\begin{cases} b(t) = \frac{e^{\delta t}(\delta-k)^2+(\delta+k)^2}{\eta^2\{-\delta-k+e^{t\delta}(\delta-k)\}} \\ \\ c(t) = ab(t) \\ \\ \delta^2 = k^2 + \theta^2\eta^2 \end{cases} \tag{5.6}$$

which leaves the characteristic function $\phi_{\tilde{W}_t}(\theta) = \exp\{-b(t)v_0 - c(t)\}$ completely determined.

5.2 The new model under time-change

Now, turning to our hedging problem, assume as before that we want to find hedging strategies for the contingent claim $H = \varphi(F_T^i(T^\star), F_T^e(T^\star), C_T, D_T)$. Our hedging instruments represent the forward contracts on fuels and on electricity. Let us proceed to studying how the initial model derived in Section 1.3.3 changes after time-changing the Brownian motion. The $\mathbf{P}$-dynamics of the discounted price of fuels (1.10), forward price on fuels (1.14) and forward electricity price (1.19) change under the time-changed Brownian motion $(\tilde{W}_t)_{0\leq t\leq T}$ in the following way:

$$d\tilde{S}_t^i = e^{-rt}\Big[-rS_t^i + \frac{1}{h_i}\sum_{j\le i}\mu_j Y_t^j\Big]dt + \frac{e^{-rt}}{h_i}\sum_{j\le i}\sigma_j Y_t^j d\tilde{W}_t^j \tag{5.7}$$

$$dF_t^i(T) = e^{r(T-t)}\Big(-rS_t^i + \frac{1}{h_i}\sum_{j\le i}\mu_j Y_t^j\Big)dt + e^{r(T-t)}\frac{1}{h_i}\sum_{j\le i}\sigma_j Y_t^j d\tilde{W}_t^j \tag{5.8}$$

And

$$\begin{aligned} dF_t^e(T) = & \underbrace{e^{r(T-t)}\sum_{i=1}^{n}(\sum_{k=1}^{n} G_k^T(t,C_t,D_t)(\mu_i - r)Y_t^i\,dt}_{\theta^{drift}} \\ & \underbrace{+ e^{r(T-t)}\sum_{i=1}^{n}(\sum_{k=1}^{n} G_k^T(t,C_t,D_t)\sigma_i Y_t^i\,d\tilde{W}_t^i}_{\theta^S} \\ & \underbrace{+\sum_{i=1}^{n} h_i F_t^i(T)\frac{\partial G_i^T}{\partial d}(t,C_t,D_t)b(t,D_t)\,dW_t^D}_{\theta^D} + \\ & \underbrace{+\sum_{i=1}^{n} h_i F_t^i(T)\sum_{k=1}^{n}\frac{\partial G_i^T}{\partial c_k}(t,C_t,D_t)\beta_k(t,C_t^k)\,dW_t^{C,k}}_{\theta^C} \end{aligned} \tag{5.9}$$

which can be written as:

$$dF_t^e(T) = \theta_t^{drift}dt + \theta_t^S\cdot d\tilde{W}_t + \theta_t^C\cdot dW_t^C + \theta_t^D dW_t^D \tag{5.10}$$

In this new context, our hedging instruments will have a stochastic volatility of the Heston type (see Section 5.1). In the sequel we will proceed to studying the local-risk minimizing strategy for this new model.

5.3 Local-risk minimization in the time-changed setting

As indicated above, we want to hedge against risk for a square integrable contingent claim $H = \rho(F_T^i(T^\star), F_T^e(T^\star), C_T, D_T)$. The procedure for deriving the LRM strategies is more complex than the one used in the previous setup due to the fact that our dynamics depend on a Lévy process $(d\tilde{W}_t)$. However, this difficulty is overcome by a structural argument in the same spirit of the LRM implementation done earlier in this thesis. Afterwards, finding the locally-risk minimizing strategy comes down to finding the GKW decomposition of our contingent claim under the (new) minimal martingale measure.

5.3.1 The minimal martingale measure for the new model

Our main focus is on finding the GKW decomposition of H under the minimal martingale measure. In our previous setup, this was the measure $\hat{\mathbf{P}}$, described by the density process

$$\hat{Z}_T = \frac{d\hat{\mathbf{P}}}{d\mathbf{P}} = \prod_{i=1}^{n} \frac{d\mathbf{Q}^i}{d\mathbf{P}} = \prod_{i=1}^{n} e^{-\lambda_i W_T^i - \frac{\lambda_i^2}{2}T} \tag{5.11}$$

Under time-change, the Brownian motion driving the fuel prices W_t would be changed by the Lévy process $\tilde{W}_t$. But it is not obvious that if we perform this time-change inside the density process $\hat{Z}_t$ (actually becoming $\hat{Z}_{T_t}$), $\hat{\mathbf{P}}$ will still be the minimal martingale measure.

However, Kassberger and Liebmann (2011) [19] prove in Proposition 3.4 and Corollary 3.5 by an argument related to minimal entropy that the minimal martingale measure as defined above is structure preserving through a continuous time-change process $(T_t)_{0\leq t\leq T}$. In our setting, this means that $\hat{\mathbf{P}}$ remains our minimal martingale measure and that its density process $(\hat{Z}_t^T = \hat{Z}_{T_t})_{0\leq t\leq T}$ with respect to the newly defined filtration $\mathcal{F}$ follows the same form defined in (2.4). This is to say:

$$\hat{Z}_t^T := \frac{d\hat{\mathbf{P}}}{d\mathbf{P}}|_{\mathcal{F}_t} = \prod_{i=1}^{n} e^{-\lambda_i \tilde{W}_t^i - \frac{\lambda_i^2}{2}T_t}$$

Now, adapting the corresponding n-dimensional Brownian motion change (3.2) from $\hat{W}_t^i = W_t^i + \lambda_i t$ to our new setup (in the sprit of Girsanov's theorem), we get the Lévy process:

$$R_t^i := \hat{W}_{T_t}^i = \tilde{W}_t^i + \lambda_i T_t \tag{5.12}$$

Remark 14. By Theorem 3.1 in Di Nunno and Karlsen (2015)[13], $(R_t^i)_{0\leq t\leq T}$ defined as in (5.12) is a time-changed $(\mathcal{F}, \hat{\mathbf{P}})$-Brownian motion.

In other words, we not only time-changed the initial Brownian motion driving the $\mathbf{P}$-dynamics of fuels, but also the new Brownian motion with drift describing the $\hat{\mathbf{P}}$-dynamics of fuels.

Then we can re-write the $\hat{\mathbf{P}}$-dynamics of fuel prices (3.3), forward price of fuels (3.4) and forward price of electricity (3.5) under the time-change process. We obtain a new model where:

$$d\tilde{S}_t^i = \frac{e^{-rt}}{h_i} \sum_{j\leq i} \sigma_j Y_t^j dR_t^j \tag{5.13}$$

$$dF_t^i(T) = e^{r(T-t)} \frac{1}{h_i} \sum_{j\leq i} \sigma_j Y_t^j dR_t^j \tag{5.14}$$

$$
\begin{aligned}
dF_t^e(T) &= \underbrace{e^{r(T-t)}\sum_{i=1}^{n}(\sum_{k=1}^{n} G_k^T(t,C_t,D_t)\sigma_i Y_t^i}_{\theta^S}\, dR_t^i + \\
&+ \underbrace{\sum_{i=1}^{n} h_i F_t^i(T)\frac{\partial G_i^T}{\partial d}(t,C_t,D_t)b(t,D_t)}_{\theta^D}\, dW_t^D + \\
&+ \underbrace{\sum_{i=1}^{n} h_i F_t^i(T)\sum_{k=1}^{n}\frac{\partial G_i^T}{\partial c_k}(t,C_t,D_t)\beta_k(t,C_t^k)}_{\theta^C}\, dW_t^{C,k} \qquad (5.15)
\end{aligned}
$$

which can equally be written as:

$$dF_t^e(T) = \theta_t^S \cdot dR_t + \theta_t^C \cdot dW_t^C + \theta_t^D dW_t^D$$

considering $\cdot$ as the dot product.

5.3.2 The local-risk minimizing strategy in the new model

As far as the hedging problem is concerned, the procedure is analogous to the one from Section 3.1.2. We will adapt the arguments to our new model by taking into account the following remark on the process $(R_t)_{0\leq t\leq T}$

Remark 15. The quadratic variation process (or sharp-bracket process, given continuity) of $(R_t)_{0\leq t\leq T}$ is

$$< R^i >_t = T_t, \quad \forall i = 1..n, t \in [0,T]$$

And consequently

$$d < R^i >_t = v(t)dt$$

Moving back to the LRM, we aim at finding the GKW decomposition under $\hat{\mathbf{P}}$ of our contingent claim $H \in L^2(\mathbf{P}) \cap L^2(\hat{\mathbf{P}})$:

$$\hat{E}[H|\mathcal{F}_t] = \hat{E}[H] + \int_0^t \xi_t dF_t(T^\star) + \int_0^t \xi_t^e dF_t^e(T^\star) + L_t^H$$

where L^H is orthogonal (in the sense of Definition 15) to both $F(T^\star)$ and $F^e(T^\star)$.

By Remark 8, there exists a function $\phi \in \mathcal{C}^{1,2,2,2}$ such that

$$V^* := V_t^H = \phi(t, F_t(T^\star), C_t, D_t)$$

We will see that in this model ϕ will have a slight variation from the one determined before.

Now, replacing this into the GKW decomposition of H and assuming the the mean-variance tradeoff process $\hat{K}$ is absolutely continuous, then there exists the FS decomposition and is written as:

$$V_t^H = \hat{E}[H] + \int_0^t \xi_s dF_s(T^\star) + \int_0^t \xi_s^e dF - s^e(t^\star) + L_t^H$$

where L^H is a **P**-martingale orthogonal to $F(T^\star)$ and $F^e(T^\star)$ in the sense of Definition 15.

Due to the slight variation from the model in Section 3.1.2, the solution to the LRM problem will be the one given by Proposition 4 but where ϕ is the solution of a different PDE.

Proposition 6. *The solution to the local-risk minimizing problem in the new time-changed model is given by*

$$
\begin{aligned}
\xi_t^e &= \frac{1}{||(\theta_t^C,\theta_t^D)||^2}[\sum_{i=1}^n \theta_t^{C,i}\frac{\partial\phi}{\partial c_i}\beta_i(t,C_t^i)+\theta_t^D\frac{\partial\phi}{\partial d}b(t,D_t)]\\
\xi_t^i &= \frac{\partial\phi}{\partial y_i}+\frac{h_iG_i^{T^\star}(t,C_t,D_t)}{||(\theta_t^C,\theta_t^D)||^2}[\sum_{i=1}^n \theta_t^{C,i}\frac{\partial\phi}{\partial c_i}\beta_i(t,C_t^i)+\theta_t^D\frac{\partial\phi}{\partial d}b(t,D_t)]\\
dL_t^H &= \sum_{i=1}^n(\frac{\partial\phi}{\partial c_i}\beta_i(t,C_t^i)-\frac{\sum_{i=1}^n\theta_t^{C,i}\frac{\partial\phi}{\partial c_i}\beta_i(t,C_t^i)+\theta_t^D\frac{\partial\phi}{\partial d}b(t,D_t)}{||(\theta_t^C,\theta_t^D)||^2}\theta_t^{C,i}\mathbf{1}_{\{||(\theta_t^C,\theta_t^D)||^2>0\}})dW_t^{C,i}-\\
&- (\frac{\partial\phi}{\partial d}b(t,D_t)-\frac{\theta_t^D\frac{\partial\phi}{\partial d}b(t,D_t)}{||(\theta_t^C,\theta_t^D)||^2}\mathbf{1}_{\{||(\theta_t^C,\theta_t^D)||^2>0\}})dW_t^D
\end{aligned}
\tag{5.16}
$$

Moreover, ϕ is the solution of the following PDE:

$$
\begin{cases}
\frac{\partial\phi}{\partial t}+\sum_{i=1}^n\frac{\partial\phi}{\partial c_i}\alpha_i(t,c_i)+\frac{\partial\phi}{\partial d}a(t,d)+\frac{1}{2}\sum_{i=1}^n\frac{\partial^2\phi}{\partial y_i^2}[\sum_{j\le i}(y_j-y_{j-1})^2\sigma_j^2v(t)]+\\
\frac{1}{2}\sum_{i=1}^n\frac{\partial^2\phi}{\partial c_i^2}\beta_i(t,c_i)^2+\frac{1}{2}\frac{\partial^2\phi}{\partial d^2}b(t,d)^2=0\\
\\
\phi(T,y,c,d)=\rho(y,c,d)
\end{cases}
\tag{5.17}
$$

Proof. Obtaining ξ^i,ξ^e,L^H is analogous to the previous case (see Appendix A.2), where one can write that $\theta_t^{S,i}dR_t^i=h_iG_i^{T^\star}(t,C_t,D_t)dF_t^i(T^\star)$.

In order to obtain ϕ as a solution to a PDE, we put the condition that $V_t^H=\phi(t,F_t(T^\star),C_t,D_t)$ is a $\hat{\mathbf{P}}$-martingale, hence that the dt-term of dV_t^H vanishes. The variation from the previous model (3.11) comes from the fact that we don't have that $(d\hat{W}_t^i)^2=dt$ anymore, but $(dR_t^i)^2=v(t)dt$. Indeed, by using Itô formula and Remark 15, we get the PDE (5.17). □

5.4 A BSDE approach to local-risk minimization in the time-changed setting

As in the original setup, our goal is to find LRM strategies for hedging a contingent claim like in Assumption 4 of the type $H=\rho(F_T^e(T^\star),F_T(T^\star),C_T,D_T)$, where the hedging instruments are the forward contracts on fuels and electricity. Note that we have made the time-change $\tilde{W}_t:=W_{T_t}$, where $T_t=\int_0^t v(s)ds$. The BSDE approach goes in the same lines described in Section 3.2 with some precautions in the way we construct our BSDE.

Both $F_t(T),F_t^e(T)$ are semimartingales. Just as before, we are interested in their decompositions as in Definition 11 in terms of square integrable local martingales M_t,M_t^e and bounded variation processes A_t,A_t^e. However, given the dynamics of the semimartingales ((5.8) and (5.10)), this is not as direct as in the initial framework (see Section 3.2).

The complexity arises from the fact that $\tilde{W}_t$ is no longer a Brownian motion and hence its sharp bracket process $< \tilde{W} >_t$ becomes $< \tilde{W} >_t = T_t$. In addition, our BSDE will be expressed in a semimartingale setting. To tackle these problems, we adapt the techniques developed by Carbone et al. (2008) [5] and Di Nunno and Karlsen (2015) [13] and obtain the LRM strategy as a solution of a more special BSDE.

The working strategy will be somehow in a feedback form, meaning that we will start by defining some new spaces and new measures, then we will add the decompositions of $F_t(T), F_t^e(T)$(as semimartingales) and then we will write the FS decomposition of the discounted value process $\tilde{V}_t^H$. Finally, a more sophisticated BSDE will be derived. Take $(\Omega, \mathbf{F}, \mathbf{P}))$ our initial probability space as in Assumption 2.

Notation

- $\mathcal{B}$: the Borel σ-algebra on [0,T]
- $\mathcal{T}$: a time interval in $\mathcal{B}$

Assume that the affine positive activity rate $v(\cdot)$ satisfies that

$$\lim_{h\to 0} P(|v(t+h) - v(t)| \geq \epsilon) = 0,$$

$\forall \epsilon > 0$ and almost all $t \in [0,T], E[\int_0^T v(t)dt] < \infty$

Our time-changing process $T_t = \int_0^t v(s)ds$ defined on the time interval $[0,T]$ can be viewed as a random measure. The filtration generated by it is defined as

$$\mathcal{F}^T = \sigma\{T(\mathcal{T}), \mathcal{T} \subset [0,T]\} \tag{5.18}$$

The following theorem proved by Serfozo, 1972 [35] and Grigelionis, 1975 [15] gives a connection between the distribution of the subordinated n-dimensional Brownian motion $(W_t)_{0\leq t\leq T}$ and that of $(\tilde{W}_t)_{0\leq t\leq T}$

Theorem 10. *Let* $(W_t)_{0\leq t\leq T}$ *be a n-dimensional Brownian motion independent from the process* $(T_t)_{0\leq t\leq T}$. *Then* $\tilde{W}_t = W_{T_t}$, $\forall t \in [0,T]$ *in the sense of equality in distribution* $\Longleftrightarrow$ $\tilde{W}$ *is a signed random measure on* $\mathcal{B}$ *satisfying:*
(i). $P(\tilde{W}(\mathcal{T}) \leq x|\mathcal{F}^T) = P(\tilde{W}(\mathcal{T})|T(\mathcal{T})) = \Phi(\frac{x}{\sqrt{\mathcal{T}}}), x \in \mathbf{R}, \mathcal{T} \subset [0,T]$
(ii). For any disjoint $\mathcal{T}_1, \mathcal{T}_2 \subset [0,T]$, $\tilde{W}(\mathcal{T}_1)$ *and* $\tilde{W}(\mathcal{T}_2)$ *are conditionally independent given* $\mathcal{F}^T$. *In particular, define the usual process* $\tilde{W}_t$ *as* $\tilde{W}([0,t])$.
(iii). $\tilde{W}$ *is conditionally independent given* $\mathcal{F}^T$.

The importance of these results reside in the application they give in view of finding the square integrable local martingales M, M^e from the decomposition of our semimartingales $F(T), F^e(T)$. They allow us to define an Itô type of non-anticipating stochastic integral from where a similar result to Itô representation theorem is derived. The following are applications given by Di Nunno and Karlsen (2015) [13].

Define $\mathcal{G}_t := \mathcal{F}^{\tilde{W}} \vee \mathcal{F}^T$, where $\mathcal{F}^T$ is defined by (5.18) and $\mathcal{F}^{\tilde{W}} = \sigma\{\tilde{W}(\mathcal{T}), \mathcal{T} \subset [0,T]\}$. Set the filtration

$$\tilde{\mathcal{F}}_t = \bigcap_{r>t} \mathcal{G}_r$$

Remark 16. $(\tilde{W}_t)_{t\in[0,T]}$ is a $\tilde{\mathcal{F}}$-martingale. Indeed, it is $\tilde{\mathcal{F}}$-measurable and -adapted and it satisfies the martingale property. Namely, by the property (ii) in Theorem 10, $\forall \mathcal{T} \subset (t,T]$ we have:

$$E[\tilde{W}(\mathcal{T})|\tilde{\mathcal{F}}_t] = E[\tilde{W}(\mathcal{T})|\bigcap_{r>t} \mathcal{G}_r] = E[\tilde{W}(\mathcal{T})|\bigcap_{r>t} \mathcal{F}_r^{\tilde{W}} \vee \mathcal{F}^T] = 0$$

Hereafter, in order to approach our hedging problem, we will be working on the new filtration $\tilde{\mathcal{F}} \vee \mathcal{F}^C \vee \mathcal{F}^D$, where $\mathcal{F}^C$ and $\mathcal{F}^D$ are the natural filtrations discussed earlier, generated by the Brownian motions driving capacities and demand. With a certain abuse of notation, we will call our new filtration again $\mathcal{F}$.

Define the space $\mathcal{I}_{\mathcal{F}} \subset L^2([0,T] \times \Omega, \mathcal{B} \times \mathcal{F}, T \times \mathbf{P})$ of random variables ν such that:

$$||\nu||_{\mathcal{I}_{\mathcal{F}}} := E[\int_0^T \nu(s)^2 v(s) ds]^{\frac{1}{2}} < \infty$$

For any such $\nu \in \mathcal{I}_{\mathcal{F}}$, the *non-anticipative stochastic integral* $I : \mathcal{I}_{\mathcal{F}} \to L^2(\Omega, \mathcal{F}, \mathbf{P})$ is defined by

$$I(\nu) := \int_0^T \nu(s) \cdot d\tilde{W}_s + \int_0^T \nu^C(s) \cdot dW_s^C + \int_0^T \nu^D(s) dW_s^D$$

Provided that $(\tilde{W}_t)_{0\le t\le T}$ is an $\tilde{\mathcal{F}}$-martingale, the following integral representation theorem holds[1] under the general filtration $\mathcal{F}$:

Theorem 11. *Let Ξ be a square integrable process in $(\Omega, \mathcal{F}, \mathbf{P})$. Then there exists a unique process $\nu \in \mathcal{I}_{\mathcal{F}}$ such that*

$$\Xi = \Xi^0 + \int_0^T \nu(s) \cdot d\tilde{W}_s + \int_0^T \nu^C(s) \cdot dW_s^C + \int_0^T \nu^D(s) dW_s^D \tag{5.19}$$

where $\Xi^0 \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ is orthogonal to the integral parts.

Remark 17. In the setting of the Theorem 11, Di Nunno and Eide (2010) [12] characterize Ξ^0 as $d\Xi^0 \equiv 0$.

Furthermore, we will apply the previous theorem in order to find the processes $M = (M_t^i)_{0\le t\le T}^{i=1..n}$, $(M_t^e)_{0\le t\le T}$ in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ and $A = (A_t^i)_{0\le t\le T}^{i=1..n}$ and $(A_t^e)_{0\le t\le T}$ with $M_0^i = M_0^e = A_0^i = A_0^e = 0$ that give the decompositions

$$F_t^i(T^\star) = F_0^i(T^\star) + A_t^i + M_t^i$$

$$F_t^e(T^\star) = F_0^e(T^\star) + A_t^e + M_t^e$$

[1] See Di Nunno and Eide (2010) [12]

By Theorem 11, we have that for all $i = 1..n$ there exists a unique processes $\nu^i, \nu^e \in \mathcal{I}_{\mathcal{F}}$ such that

$$M_t^i = \nu^{i,0} + \int_0^T \nu^i(s) \cdot d\tilde{W}_s + \int_0^T \nu^{i,C}(s) \cdot dW_s^C + \int_0^T \nu^{i,D}(s) dW_t^D$$

$$M_t^e = \nu^{e,0} + \int_0^T \nu^e(s) \cdot d\tilde{W}_s + \int_0^T \nu^{e,C}(s) \cdot dW_s^C + \int_0^T \nu^{e,D}(s) dW_t^D$$

But due to the uniqueness of these integral decompositions, we can identify the terms $\nu^i, \nu^{i,C}, \nu^{i,D}, \nu^e, \nu^{e,C}, \nu^{e,D}, \Xi^{0,i}, \Xi^{0,e}$ directly from the dynamics of $F_t^i(T^\star)$(5.8) and $F_t^e(T^\star)$(5.10):

$$\begin{aligned} \nu^i(t) : &= e^{r(T^\star - t)} \frac{1}{h_i} \sum_{j \le i} \sigma_j Y_t^j, \qquad \nu^{i,C} = \nu^{i,D} = \Xi^{0,i} = 0 \\ \nu^e(t) : &= \theta_t^S, \qquad \nu^{e,C} := \theta_t^C, \qquad \nu^{e,D}(t) := \theta_t^D, \qquad \Xi^{0,e} = 0 \end{aligned} \tag{5.20}$$

And according to Theorem 11 we have

$$\begin{aligned} M_t^i &= \int_0^t e^{r(T^\star - s)} \frac{1}{h_i} \sum_{j \le i} \sigma_j Y_t^j d\tilde{W}_s^j \\ M_t^e &= \int_0^t \theta_s^S \cdot d\tilde{W}_s + \int_0^t \theta_s^C \cdot dW_s^C + \int_0^t \theta_s^D dW_s^D \end{aligned} \tag{5.21}$$

Then, it comes out that our new bounded variation processes are

$$\begin{aligned} A_t^i &= \int_0^t e^{r(T^\star - s)} (-r S_s^i + \frac{1}{h_i} \sum_{j \le i} \mu_j Y_s^j) ds \\ A_t^e &= \int_0^t \theta_s^{drift} ds \end{aligned} \tag{5.22}$$

But, in the spirit of the structure condition, there exist for all $i = 1..n$ the processes $\tilde{\omega}_t^i, \tilde{\omega}_t^e$ such that

$$A_t^i := \int_0^t \tilde{\omega}_s^i d < M^i >_s = \int_0^t \tilde{\omega}_s^i e^{2r(T^\star - s)} \frac{1}{h_i} \sum_{j \le i} (\sigma_j Y_t^j)^2 v(s) ds \tag{5.23}$$

$$\begin{aligned} A_t^e &:= \int_0^t \tilde{\omega}_s^e d < M^e >_s = \int_0^t \tilde{\omega}^e{}_s \{ \theta_s^C \theta_s^{C\prime} + (\theta_s^D)^2 \} ds + \int_0^t \tilde{\omega}^e{}_s \theta_s^S \theta_s^{S\prime} \, d < \tilde{W} >_s = \\ &= \int_0^t \tilde{\omega}_s^e \{ \theta_s^S \theta_s^{S\prime} \, v(s) + \theta_s^C \theta_s^{C\prime} + (\theta_s^D)^2 \} ds \end{aligned} \tag{5.24}$$

where

$$\tilde{\omega}_t^i = \frac{-r S_t^i + \frac{1}{h_i} \sum_{j \le i} \mu_j Y_t^j}{e^{r(T^\star - t)} \frac{1}{h_i} \sum_{j \le i} (\sigma_j Y_t^j)^2 v(t)} \tag{5.25}$$

$$\tilde{\omega}_t^e = \frac{\theta_t^{drift}}{\theta_t^S \theta_t^{S\prime} \, v(t) + \theta_t^C \theta_t^{C\prime} + (\theta_t^D)^2} \tag{5.26}$$

And the mean-variance tradeoff processes are $\hat{K}^i_t = \int_0^t \tilde{\omega}^i_t d < M^i >_s \tilde{\omega}^{i\prime}_t \quad i = 1..n$ and $\hat{K}^e_t = \int_0^t \tilde{\omega}^e_t d < M^e >_s \tilde{\omega}^e_t$.

$$\hat{K}^i_t = \int_0^t \frac{\Big(- rS^i_t + \frac{1}{h_i}\sum_{j\leq i}\mu_j Y^j_t\Big)^2}{\frac{1}{h_i}\sum_{j\leq i}(\sigma_j Y^j_s)^2 v(s)} ds \tag{5.27}$$

$$\hat{K}^e_t = \int_0^t \frac{(\theta^{drift}_s)^2}{\theta^S_s \theta^{S\prime}_s v(s) + \theta^C_s \theta^{C\prime}_s + (\theta^D_s)^2} ds \tag{5.28}$$

The next stage is to study the portfolio strategy and to find the FS decomposition of it. The working strategy is similar to the one we employed in the initial BSDE approach in Section 3.2. We will employ the same notations for parallelism. Take $\xi \in \Theta$ and $\xi^e \in \Theta^e$ and the discounted contingent claim $\tilde{H}$ Then the discounted value process is

$$\tilde{V}^H_t = E[\tilde{H} - \int_t^T \xi_s dA_s - \int_t^T \xi^e_s dA^e_s | \mathcal{F}_t]$$

Applying the GKW decomposition of the process $U_T := \tilde{H} - \int_t^T \xi_s dA_s - \int_t^T \xi^e_s dA^e_s$, taking conditional expectation and replacing $E[U_T|\mathcal{F}_t] = \tilde{V}^H_t$ we get that there exist $\tilde{\xi} \in L(F(T^\star))$ and $\tilde{\xi}^e(F^e(T^\star))$ such that:

$$\tilde{V}^H_t = \tilde{V}^H_0 + \int_0^t \xi_s \cdot dA_s + \int_0^t \tilde{\xi}_s \cdot dM_s + \int_0^t \xi^e_s dA^e_s + \int_0^t \tilde{\xi}^e_s dM^e_s + L^H_t \tag{5.29}$$

where L^H and M, M^e are orthogonal.

Here, assuming that the mean-variance tradeoff process compound of $(\hat{K}^i)_{0\leq t\leq T}$, $(\hat{K}^e)_{0\leq t\leq T}$ is not necessarily absolutely bounded due to the stochastic term $v(\cdot)$ appearing in its form. Then we cannot say if $\xi = \tilde{\xi}$ in $L^2(M)$ and $\xi^e = \tilde{\xi}^e$ in $L^2(M)$.

Then, we have the following BSDE

$$\begin{aligned} d\tilde{V}^H_t &= \xi_t dA_t + \tilde{\xi}_t dM_t + \xi^e_t dA^e_t + \tilde{\xi}^e_t dM^e_t + dL^H_t \\ \tilde{V}^H_T &= \tilde{H} \end{aligned} \tag{5.30}$$

After replacing the dynamics of A, A^e (5.23) and M, M^e (5.21) and computing we get the semimartingale BSDE:

$$\begin{aligned} d\tilde{V}^H_t &= \sum_{i=1}^n \xi^i_t \, \tilde{\omega}^i_t \, e^{2r(T^\star - t)} \frac{1}{h_i} \sum_{j\leq i} (\sigma_j Y^j_t)^2 d < \tilde{W}^j >_t + \sum_{i=1}^n \xi^e_t \, \tilde{\omega}^e_t \, (\theta^{S,i}_t)^2 d < \tilde{W}^i >_t + \\ &+ \sum_{i=1}^n \xi^e_t \, (\theta^{C,i}_t)^2 d < W^{C,i} >_t + (\theta^D_t)^2 d < W^D >_t + \\ &+ \sum_{i=1}^n \tilde{\xi}^{\;i}_s \, e^{r(T^\star - t)} \frac{1}{h_i} \sum_{j\leq i} \sigma_j Y^j_t d\tilde{W}^j_t + \sum_{i=1}^n \tilde{\xi}^e_t \, \theta^{S,i}_t d\tilde{W}^i_t + \\ &+ \sum_{i=1}^n \tilde{\xi}^e_t \, \theta^{C,i}_t dW^{C,i}_t + \tilde{\xi}^e_t \, \theta^D_t dW^D_t + dL^H_t \\ \tilde{V}^H_T &= \tilde{H} \end{aligned} \tag{5.31}$$

5.5 Value of the portfolio in the time-changed setting

From the value of the portfolio standpoint, let us consider a discounted contingent claim written on electricity and fuels. Just like in the Assumption 4, consider the European-type contingent claim H written on a forward contract on electricity, fuels and on capacities and demand. Assume it is discounted. Additionally, shall assume that H is $\mathbf{Q}(\theta)$-square integrable.

$$H = \rho(F^e_T(T^\star), F_T(T^\star), C_T, D_T) \tag{5.32}$$

where $T^\star$ is any positive maturity time and the forward on fuels is an n-dimensional vector.

Then, we have that the value of our contingent claim at time t is

$$\tilde{G}(z; F^e_t(T^\star), F_t(T^\star), C_t, D_t) := \hat{E}[H|\mathcal{F}_t] \tag{5.33}$$

where $z \in \mathscr{C} \subset \mathbf{C}^n$ is the complex part.

Let $\mathscr{G}$ be the generalized Fourier transform of (5.33). Then, by (4.27), we have that

$$\mathscr{G}(z; F^e_t(T^\star), F_t(T^\star), C_t, D_t) = \int_{-\infty}^{\infty} e^{izk}\tilde{G}(k; F^e_t(T^\star), F_t(T^\star), C_t, D_t)dk$$

$$\mathscr{G}(z; F^e_t(T^\star), F_t(T^\star), C_t, D_t) = \frac{i}{z}(F^e_t(T^\star)\phi_{\tilde{W}}(zD_t) + F_t(T^\star)'\phi_{\tilde{W}}(zD_t - iC_t)) \tag{5.34}$$

where the vector of fuels is transpose.

Now, we make use of the inversion formula (4.28) and we are able to go back to the value of our contingent claim $\tilde{G}$.

As a final step in determining the value of our contingent claim H written on forward contracts on electricity and fuels and capacities and demand for energy, we apply the fast Fourier transform (FFT) to obtain a fairly accurate approximation (on a finite interval) of $\tilde{G}(k; F^e_t(T^\star), F_t(T^\star), C_t, D_t)$

Conclusions and further study

Through the present thesis we believe to have tackled the quadratic hedging problem (local risk minimization) in a calendar time and a time-changed framework in an incomplete market where one can trade on electricity and fuel forward contracts. The starting point was the spot price model (1.2) built by Aïd et al. (2013) [1]. From there, the dynamics of fuel prices, forward fuel prices and forward electricity contracts are derived. The incompleteness of the market arises precisely from the different noises coming into place in our price dynamics, as could be seen from (1.7), (1.14), (1.19) . Then, the problem of finding local-risk minimizing strategies is solved in a twofold approach.

On the one hand, this is done in a classical manner, where the property of the minimal martingale measure of preservation of orthogonality is employed in order to go from the Galtchouk-Kunita-Watanabe to the Fölmer-Schweizer decomposition. On the other hand, a BSDE approach is adapted to our setup and the solution of the local-risk minimization problem is expressed as a solution of a BSDE.

Furthermore, an absolutely continuous time-change (T_t) was applied to the Brownian motion driving the fuel prices and a new model was built. We assumed independence between the Brownian motion driving the fuel prices and the one driving the activity rate $(v(\cdot))$, hence no possibility for leverage effect. We chose a popular activity rate, namely the Heston (1993) [16] type. The local-risk minimization problem arising from the new model was equally tackled in a classical and a BSDE manner.

The former method relied on the fact that in our setup the minimal martingale measure was preserved through the time-change. And therefore an analogous reasoning was developed.

The latter method again expressed the hedging solution as a solution of a BSDE. However, in this case, a more special type of BSDE was developed since it was not driven by Brownian motions anymore, but by general martingales.

Needless to say, quadratic hedging in an incomplete market is a very complex subject and there are many directions that the present thesis could be extended to.

We chose to apply the time-change only to the Brownian motion driving the fuels (W^i), but this could be applied to the other two Brownian motions alike (W^C, W^D). Also, the time changing process (T_t) was chosen to be of absolutely continous type, but this could have the more general form of a subordinator (see Section 4.2.1). In addition, the form of the affine activity rate was chosen to be of the Heston type, but it could equally be, among others, of Stein and Stein or Hull-White type. Were we to assume any negative correlation between the time-changed process and the Brownian motion driving the activity rate, then the leverage effect should have been considered and a measure change in the complex domain should have been applied (Carr and Wu (2004)[6]).

All things considered, the present thesis brought a new approach on local-risk minimization in an incomplete market when there is a time-change.

Appendix A

Additional proofs and results

A.1 Unique strong solution to a SDE

Theorem 12. *A stochastic differential equation of the form*

$$\begin{aligned} dX_t &= \alpha(t, X_t)dt + \beta(t, X_t)dW_t \\ X_0 &= x_0 \end{aligned} \tag{A.1}$$

with $\alpha, \beta : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ *measurable in the probability space* $(\Omega, \mathcal{F}, \mathbf{P}))$ *has a unique strong solution if* α, β *are locally Lipschitz continuous, i.e.* $\forall n > 0$, *there exists* $K_n > 0$ *such that* $\forall t > 0$ *and* $\forall x, y \in \mathbf{R}$ *with* $||x|| \leq n, ||y|| \leq n$

$$||\alpha(t, x) - \alpha(t, y)|| + ||\beta(t, x) - \beta(t, y)|| \leq K_n ||x - y||$$

A.2 The explicit LRM strategy

Proof of Proposition 4

Assume $\hat{E}[H] = 0$ and ϕ regular, defined as in (2.11). Applying Itô's formula, we get:

$$\tilde{V}_t^H = \int_0^t \sum_{i=1}^n \frac{\partial \phi}{\partial y_i} dF_s^i + \int_0^t \sum_{i=1}^n \frac{\partial \phi}{\partial c_i} \beta_i(s, C_s^i) dW_s^{C,i} + \int_0^t \frac{\partial \phi}{\partial d} b(s, D_s) dW_s^D \tag{A.2}$$

Note that the dt-term is vanishing, as $\tilde{V}_t^H$ is a $\hat{\mathbf{P}}$-martingale.

By (3.5), $dF_t^e = \theta_t^S d\hat{W}_t + \theta_t^C dW_t^C + \theta_t^D dW_t^D$ and we can split from here the part that cannot be hedged using the fuels, i.e $dF_t^{C,D} = \theta_t^C dW_t^C + \theta_t^D dW_t^D$.

Define a new Brownian motion

$$W_t^{C,D} := \int_0^t \frac{\theta_s^C dW_s^C + \theta_s^D dW_s^D}{||(\theta_s^C, \theta_s^D)||} \tag{A.3}$$

Then, the dynamics of $F_t^{C,D}$ under (A.3) is

$$dF_t^{C,D} = ||(\theta_t^C, \theta_t^D)|| dW_t^{C,D} \tag{A.4}$$

Define a new Brownian motion

$$\bar{W}_t^{C,D} := \int_0^t \frac{\sum_{i=1}^n \frac{\partial \phi}{\partial c_i}\beta_i(t, C_t^i)dW_s^{C,i} + \frac{\partial \phi}{\partial d}b(t, D_t)dW_s^D}{\xi_s} \tag{A.5}$$

with

$$\xi_s^2 := \sum_{i=1}^n (\frac{\partial \phi}{\partial c_i}\beta_i(t, C_t^i))^2 + (\frac{\partial \phi}{\partial d}b(t, D_t))^2$$

Then

$$\frac{\partial \phi}{\partial c_i}\beta_i(t, C_t^i)dW_s^{C,i} + \frac{\partial \phi}{\partial d}b(t, D_t)dW_s^D = \xi_t d\bar{W}_t^{C,D} \tag{A.6}$$

Now the quadratic covariation between $W^{C,D}$ and $\bar{W}^{C,D}$(i.e. $d < W^{C,D}, \bar{W}^{C,D} >_t = \rho_t dt$) is

$$\rho_t = \frac{\sum_{i=1}^n \theta_t^{C,i}\frac{\partial \phi}{\partial c_i}\beta_i(t, C_t^i) + \theta_t^D \frac{\partial \phi}{\partial d}b(t, D_t)}{||(\theta_t^C, \theta_t^D)||\xi_t}$$

Define a new Brownian motion

$$W_t^{\perp} = \int_0^t \frac{d\bar{W}^{C,D} - \rho_s dW_s^{C,D}}{\sqrt{1-\rho_s^2}}$$

independent of $W^{C,D}$ such that

$$\bar{W}_t^{C,D} = \int_0^t \rho_s dW_s^{C,D} + \int_0^t \sqrt{1-\rho_t^2}dW_s^{\perp} \tag{A.7}$$

Now (A.6) can be written as:

$$\frac{\partial \phi}{\partial c_i}\beta_i(t, C_t^i)dW_s^{C,i} + \frac{\partial \phi}{\partial d}b(t, D_t)dW_s^D = \xi_t(\rho_t dW_t^{C,D} + \sqrt{1-\rho_t^2}dW_t^{\perp}) \tag{A.8}$$

By replacing $dW_t^{C,D} = \frac{dF_t^e - \theta_t^s d\hat{W}_t}{||(\theta_t^C, \theta_t^D)||}$ into (A.8), we get that

$$\frac{\partial \phi}{\partial c_i}\beta_i(t, C_t^i)dW_s^{C,i} + \frac{\partial \phi}{\partial d}b(t, D_t)dW_s^D = \xi_t(\frac{\rho_t}{||(\theta_t^C, \theta_t^D)||}(dF_t^e - \theta_t^s d\hat{W}_t) + \sqrt{1-\rho_t^2}dW_t^{\perp}) \tag{A.9}$$

and by the dynamics of F_t^e we have

$$\theta_t^{S,i}d\hat{W}_t^i = h_i F^i(t, C_t, D_t)dF_t^i \tag{A.10}$$

Now, moving back to (A.2), by replacing, we get

$$V_t^H = \int_0^t \underbrace{\frac{\xi_s\rho_s}{||(\theta_s^C, \theta_s^D)||}}_{\xi_t^e} dF_s^e + \int_0^t \sum_{i=1}^n \underbrace{(\frac{\partial \phi}{\partial y_i} - \frac{\xi_s\rho_s}{||(\theta_s^C, \theta_s^D)||}h_i G_i(s, C_s, D_s))}_{\xi_t^i} dF_s^i + \int_0^t \underbrace{\xi_s\sqrt{1-\rho_s^2}}_{L_t^H} dW_s^{\perp} \tag{A.11}$$

A.3 The optional stopping theorem

Theorem 13. *Let $(X_t)_{t\geq 0}$ be a càdlàg adapted integrable process. Then the following are equivalent:*
(a). X is a martingale
(b). $X^T = (X_{t\wedge T})_{t\geq 0}$ is a martingale, for all bounded stopping times T
(c). For all bounded stopping times S, T, $E[X_T|\mathcal{F}_S] = X_{S\wedge T}$
(d). $E[X_T] = E[X_0]$, for all bounded stopping times T

Proof. (a)$\Longrightarrow$ ...(d) hold directly by definition of martingale. (d)$\Longrightarrow$(a) Let $s < t$ and $u > t$. Consider $A \in \mathcal{F}_s$ and take a random times T and S defined by T= $\begin{cases} \text{t, if A occurs} \\ \text{u, otherwise} \end{cases}$ and

S= $\begin{cases} \text{s, if A occurs} \\ \text{u, otherwise} \end{cases}$ By (d),

$$E[X_T] = E[X_0] = E[X_s] \tag{A.12}$$

Also, we can write

$$E[X_T] = E[X_t\mathbf{1}_A] + E[X_u\mathbf{1}_{A^c}]$$
$$E[X_S] = E[X_s\mathbf{1}_A] + E[X_u\mathbf{1}_{A^c}]$$

And by introducing (A.12) into these two, we get that $E[X_t\mathbf{1}_A] = E[X_s\mathbf{A}], s < t, \forall A$. Then, this is the same as $E[X_t|\mathcal{F}_s] = X_s$. And this, plus the assumption of adaptability and integrability of X renders X a martingale. ☐

A.4 The Lévy-Itô decomposition

Proposition 7. *Let (X_t)_t be a Lévy process on $\mathbf{R}^d$ and Π its Lévy measure. Let the following hold:*
(i) Π is a Radon measure on $\mathbf{R}^d\backslash\{0\}$ and

$$\int_{|x|\leq 1} |x|^2\Pi(dx) < \infty \qquad \int_{|x|\geq 1} \Pi(dx) < \infty$$

(ii) The Jump measure J_X defined as $J_X(B) = \#\{(t, X_t - X_{t-}) \in B\}, \forall B \in \mathcal{B}(\mathbf{R}^d)$ is a Poisson random measure on $[0,\infty]x\mathbf{R}^d$ with intensity measure $\Pi(dx)dt$
(iii) $\exists\mu \in \mathbf{R}^d$ and a d-dim Brownian motion $(B_t)_t$with covariance matrix $\Sigma \in \mathcal{M}_d(\mathbf{R})$ such that

$$X_t = \mu t + B_t + X_t^l + \lim_{\varepsilon\to 0} \tilde{X}_t^\varepsilon$$

where

$$X_t^l = \int_{|x|\geq 1, s\in[0,t]} xJ_X(ds\times dx)$$

$$\tilde{X}_t^\varepsilon = \int_{\varepsilon\leq|x|<1, s\in[0,t]} x\{J_X(ds\times dx) - \Pi(dx)ds\} = \int_{\varepsilon\leq|x|<1, s\in[0,t]} x\tilde{J}_X(ds\times dx)$$

Then (μ, Σ, Π) are the ***Lévy characteristics****.*

Appendix B

BSDE approach

B.1 General results about BSDEs

We refer to El Karui et al. (1997) [14] for definitions and results on BSDEs. The next paragraphs are under the assumptions made in Section 3.2. Consider $(B_t)_{0\leq t\leq T}$ a Brownian motion wrt the filtration $\mathcal{F}$.

Definition 30. A BSDE associated to a coefficient $f(t,\omega,x,y):\Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ and a terminal value $\xi\in\mathcal{L}^2_T$ which is $\mathcal{F}_T$-measurable has the form:

$$\begin{aligned} -dX_t &= f(t,\omega,X_t,Y_t)dt - Y_t dB_t \\ X_T &= \xi \end{aligned} \tag{B.1}$$

Definition 31. A solution to the BSDE (B.1) is a pair of square integrable $\mathcal{F}$-adapted processes (X,Y) satisfying:

$$X_t = \xi + \int_t^T f(s,\omega,X_s,Y_s)ds - \int_t^T Y_s dB_s, \quad 0\leq t\leq T \tag{B.2}$$

Pardoux and Peng (1990)[27] proved that the solution (B.2) exists if under some assumptions.

Theorem 14. *There exists a unique solution (X,Y) to the BSDE (B.1) if the following conditions are given:*
(i). $f(.,\omega,0,0)\in\mathcal{H}^2_T$
(ii). f is uniformly Lipschitz in (x,y): $\exists C\geq 0$ st $\forall(x,y),(x\prime,y\prime)\in\mathbf{R}^2$ and $\forall t\in[0,T]$

$$|f(t,\omega,x,y)-f(t,\omega,x\prime,y\prime)|\leq C(|x-x\prime|+|y-y\prime|)$$

B.2 Orthogonality conditions

By FS decomposition, we need to impose that L^H is orthogonal to $F(T^\star),F^e(T^\star)$. Let us write again, for clarity, their $\mathbf{P}$-dynamics:

$$L^H_t = \int_0^t \alpha_s\cdot dW_s + \int_0^t \beta_s\cdot dW^C_s + \int_0^t \gamma_s dW^D_s \tag{B.3}$$

with $\alpha = (\alpha^i)_{i=1,n}, \;\; \beta = (\beta^k)_{k=1,n}$

$$dF^i_t(T^\star) = e^{r(T^\star - t)}\frac{1}{h_i}\sum_{j\le i}(\mu_j Y^j_t dt + \sigma_j Y^j_t dW^j_t) \tag{B.4}$$

$$dF^e_t(T^\star) = \theta^{drift}_t dt + \theta^S_t \cdot dW_t + \theta^D_t dW^D_t + \theta^C_t \cdot dW^C_t \tag{B.5}$$

By Remark 1, we have that our processes are orthogonal if $dL^H dF(T^\star) = 0$ and $dL^H dF^e(T^\star) = 0$. Plugging this in our dynamics and operating, we get that:

$$\begin{aligned} \alpha^i_t e^{r(T^\star - t)}\frac{1}{h_i}\sum_{j\le i} Y^j_t \sigma_j &= 0 \qquad i = 1..n \\ \theta^S_t \cdot \alpha_t + \theta^C_t \cdot \beta_t + \theta^D_t \gamma_t &= 0 \end{aligned} \tag{B.6}$$

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