

Selfpolar forms and their applications  
to the  $C^*$ -algebra theory

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Abstract:

A general notion of a selfpolar form is introduced and investigated. It turns out, that selfpolar forms are distinguished by a sort of maximum principle. As an application we prove, that the purification map is concave and upper semi-continuous.

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0. Introduction.

To explain detailly the content of the paper, we consider the following simple example. Let  $B(H)$  be the algebra of all bounded operators acting in a Hilbert space  $H$  and  $\eta$  be a normal state of  $B(H)$ . Then ( $A \in B(H)$ ) :

$$\eta(A) = \text{Tr}(A\rho) ,$$

where  $\rho$  is a positive, trace class operator acting in  $H$  . For any  $A, B \in B(H)$  we put

$$s_{\eta}(A, B) = \text{Tr}(A^* \rho^{\frac{1}{2}} A \rho^{\frac{1}{2}}) .$$

It turns out that the sesquilinear form  $s_{\eta}$  defined on  $B(H) \times B(H)$  introduced by the above formula exhibits many interesting properties. For example :  $s_{\eta}(I, B) = \eta(B)$  for any  $B \in B(H)$  and  $s_{\eta}(A, B) \geq 0$  for any  $A, B \in V$  . Here,  $V$  denotes the cone of all positive operators acting in  $H$ . A less trivial result says that  $s_{\eta}$  is the maximal sesquilinear positive form satisfying these two properties.

More exactly, let  $s_1$  be a positive sesquilinear form defined on  $B(H) \times B(H)$  such that  $s_1(I, B) = \eta(B)$  for any  $B \in B(H)$  and  $s_1(A, B) \geq 0$  for any  $A, B \in V$  . Then for any  $A \in B(H)$  we have

$$s_{\eta}(A, A) \geq s_1(A, A) .$$

We would like to stress that all the properties of  $s_{\eta}$  mentioned above are not related directly to the whole algebraic structure of  $B(H)$  . It turns out, that they can be derived in a more general setting involving a complex vector space  $M$

(instead of  $B(H)$ ), a cone  $V \subset M$ , a distinguished element  $1 \in V$  and a selfpolar form  $s$  defined on  $M \times M$  (instead of  $s_\eta$ ).

The notion of a selfpolar form (forme autopolaire) was introduced by A. Connes in a recent paper [1]. We would like to point out, that his definition is too narrow, at least for two reasons. At first, he considers only non-degenerate forms, at second, only normal positive functionals defined on  $M$  admit extension to selfpolar form in his sense (see [1], Proposition 1.2).

For these reasons the Connes notion of a selfpolar form works perfectly as far as the analysis of faithful normal states of von Neumann algebras is concerned, but it is not applicable for the investigation of (in general neither faithful nor normal) states of  $C^*$ -algebras.

In section 1 we introduce a new, more general notion of a selfpolar form. It turns out, that selfpolar forms are distinguished among other forms positive on  $V \times V$  by a sort of maximum principle (thm 1.1). As a simple conclusion we shall see that any selfpolar form is determined by its restriction to  $\{1\} \times M$  (thm.1.2). This theorem contains some older results: similar A. Connes theorem [1] and the uniqueness of the purification map proved in [7]. It should be noticed, that the present proof is much simpler than the previous ones.

In section 2, selfpolar forms on  $C^*$ -algebras are investigated. We discover a tight relation between these forms and  $j$ -positive exact states introduced in [6]. As an application we derive new, interesting properties of the purification map.

Remark:

Throughout the paper, all forms defined on  $M \times M$  (or on  $\mathcal{A} \times \mathcal{A}$  in section 2) are assumed to be sesquilinear and positive (for details see page 3).

1. Selfpolar forms and  
the maximum principle.

We shall investigate the following triple  $(M, V, 1)$ , where  $M$  is a complex vector space,  $V$  is a convex cone in  $M$  and  $1$  is an element of  $V$ . We assume, that  $V \cap (-V) = \{0\}$ , that any element of  $M$  can be written as  $a_1 - a_2 + i(a_3 - a_4)$ , where  $a_1, a_2, a_3, a_4 \in V$  and that for any  $a \in V$  one can find a real number  $t$  such that  $t \cdot 1 - a \in V$ .

Let  $M'$  denote the set of all linear functionals defined on  $M$ . For any  $\eta \in M'$  and  $a \in M$ , the value of  $\eta$  at the point  $a$  will be denoted by  $\langle \eta, a \rangle$ . We introduce vector space structure in  $M'$  such that  $\langle \eta, a \rangle$  becomes antilinear with respect to  $\eta$ .

The set of all functionals, which are positive on  $V$  will be denoted by  $V'$ . Clearly,  $V'$  is a convex cone in  $M'$  and  $V' \cap (-V') = \{0\}$ .

Cones  $V$  and  $V'$  introduce partially ordering relations compatible with the vector space structures of  $M$  and  $M'$  respectively. For example we write  $a \leq b$  iff  $b - a \in V$ . We shall use the following "interval notation":

$$[a, b] = \{c \in M : a \leq c \leq b\}$$

In the similar way one defines interval  $[\eta, \xi]$  for  $\eta, \xi \in M'$ .

Let us notice that the linear span of  $[0, 1]$  coincides with  $M$ . This fact follows immediately from our conditions imposed on triple  $(M, V, 1)$ .

Let  $s$  be a sesquilinear (i.e. linear with respect to the second and antilinear with respect to the first argument) positive

(i.e.  $s(a,a) \geq 0$  for any  $a \in M$ ) form defined on  $M \times M$ . Any form of that kind defines a linear mapping

$$s^* : M \rightarrow M'$$

such that ( $a, b \in M$ ) :

$$\langle s^*(a), b \rangle = s(a, b)$$

Now, we are ready to introduce our basic notions:

Definition 1.

Form  $s$  is said to be  $V$ -positive iff  $s(a,b) \geq 0$  for all  $a, b \in V$  (one can easily check that this condition is equivalent to inclusion  $s^*(V) \subset V'$ ).

Definition 2.

Form  $s$  is said to be selfpolar iff the weak closure of  $s^*([0,1])$  coincides with  $[0, s^*(1)]$  :

$$\overline{s^*([0,1])} = [0, s^*(1)] \quad (1.1)$$

Remark 1.

Weak topology in  $M'$  is, by definition, the weakest topology such that functionals  $\eta \rightarrow \langle \eta, a \rangle$  are continuous for all  $a \in M$  (so called  $\sigma(M', M)$  - topology).

Remark 2.

It follows immediately from definition 2 that  $s^*(V) \subset V'$ . So selfpolar forms are  $V$ -positive.

Remark 3.

Let  $s$  be a sesquilinear, positive, non-degenerate form defined on  $M \times M$ . Assume, that  $s^*$  maps  $V$  onto a face of  $V'$ .

It means that  $s$  is selfpolar in the sense of A. Connes [1].

Then

$$s^*([0,1]) = [0, s^*(1)] \quad (1.2)$$

Indeed relation  $s^*(V) \subset V'$  means, that  $s^*$  preserves relation  $\leq$  and we get immediately  $s^*([0,1]) \subset [0, s^*(1)]$ .

On the other hand, remembering that  $s^*(V)$  is a face of  $V'$ , one can easily show that

$$[0, s^*(1)] \subset s^*(V) \cap s^*(1-V)$$

It means that any element  $\eta \in [0, s^*(1)]$  can be written in the following two ways:

$$\eta = s^*(a) = s^*(1-b) ,$$

where  $a, b \in V$ . Then  $a = 1-b$  ( $s$  is non-degenerate, so  $s^*$  is an injection) and  $a \in [0,1]$ . Therefore  $[0, s^*(1)] \subset s^*([0,1])$  and equation (1.2) is proven. Since  $[0, s^*(1)]$  is weakly closed, (1.1) is the more satisfied.

Summarizing, we have shown, that any form selfpolar in the sense of A. Connes is selfpolar in the sense of our definition.

Our main results shows that there is a kind of "maximum principle" for selfpolar forms:

Theorem 1.1.

Let  $s$  be a selfpolar form and  $s_1$  be a  $V$ -positive form, both defined on  $M \times M$ . Assume that  $s_1^*(1) \leq s^*(1)$ . Then

$$s_1(a,a) \leq s(a,a)$$

for all  $a \in M$ .

Before the proof, let us notice some conclusions. At first we get immediately the uniqueness theorem:

Theorem 1.2.

Let  $s$  and  $s_1$  be selfpolar forms defined on  $M \times M$ . Assume that  $s_1^*(1) = s^*(1)$ . Then  $s = s_1$ .

We say that  $(M, V, 1)$  admits a complete system of selfpolar forms iff for any  $\eta \in V'$  there exists a selfpolar form  $s_\eta$  such that  $s_\eta^*(1) = \eta$ . If this is a case, then one can consider mapping  $\eta \rightarrow s_\eta$  (it is well defined in virtue of Thm 1.2).

Theorem 1.3.

Assume that  $(M, V, 1)$  admits a complete system of selfpolar forms. Then for any  $a \in M$  function  $\eta \rightarrow s_\eta(a, a)$  is concave and upper semicontinuous.

Proof:

Let  $\eta_1, \eta_2 \in V'$  and  $\eta = \lambda\eta_1 + (1-\lambda)\eta_2$  be a convex combination of  $\eta_1$  and  $\eta_2$ . We put  $s_1 = \lambda s_{\eta_1} + (1-\lambda)s_{\eta_2}$ . Then  $s_1$  is a  $V$ -positive form and  $s_1^*(1) = \eta = s_\eta^*(1)$ . In virtue of Thm 1.1 we have ( $a \in M$ ):

$$\lambda s_{\eta_1}(a, a) + (1-\lambda)s_{\eta_2}(a, a) \leq s_\eta(a, a).$$

To prove the semicontinuity let us consider a net  $(\eta_\alpha)$  of elements of  $V'$  converging weakly to  $\eta \in V'$ . Assume that  $\lim S_{\eta_\alpha}(a, a)$  exists. We have to show that

$$\lim s_{\eta_\alpha}(a, a) \leq s_\eta(a, a) \tag{1.3}$$

For any  $b, c \in [0, 1]$  we have  $0 \leq s_{\eta_\alpha}(b, c) \leq s_{\eta_\alpha}(1, 1) = \eta_\alpha(1)$ . Therefore one can assume that  $s_{\eta_\alpha}(b, c)$  are converging for all  $b, c \in [0, 1]$  (if this is not the case, one replace  $(\eta_\alpha)$  by a suitably chosen subnet). Then  $s_1 = \lim s_{\eta_\alpha}$  is a  $V$ -positive form and (1.3) follows immediately from Thm 1.1.

To prove Thm 1.1 we need the following lemmas.

Lemma 1.

Let  $a \in M$ . Assume that  $s$  is a  $V$ -positive form defined on  $M \times M$  and that  $\langle \eta, a \rangle = 0$  for all  $\eta \in [0, s^*(1)]$ . Then  $s^*(a) = 0$ .

Conversely if  $s$  is a selfpolar form defined on  $M \times M$  and  $s^*(a) = 0$ , then  $\langle \eta, a \rangle = 0$  for all  $\eta \in [0, s^*(1)]$ .

Proof:

Assume, that  $\langle \eta, a \rangle = 0$  for all  $\eta \in [0, s^*(1)]$ . Then  $\langle s^*(b), a \rangle = 0$  for all  $b \in [0, 1]$  (note that  $s^*(b) \in [0, s^*(1)]$ ) and the following formula

$$\langle s^*(a), b \rangle = s(a, b) = \overline{s(b, a)} = \overline{\langle s^*(b), a \rangle} \quad (1.4)$$

shows that  $\langle s^*(a), b \rangle = 0$ . Since the linear span of  $[0, 1]$  coincides with  $M$ , we have  $s^*(a) = 0$ .

Assume now that  $s$  is selfpolar and that  $s^*(a) = 0$ . Then  $\langle s^*(b), a \rangle = 0$  (formula (1.4)) for all  $b \in [0, 1]$  and (1.1) shows that  $\langle \eta, a \rangle = 0$  for all  $\eta \in [0, s^*(1)]$ .

Lemma 2.

Let  $H$  be a Hilbert space,  $A$  be a positive operator acting in  $H$  and  $\Delta$  be a subset of the domain of  $A$ . Assume that  $\Delta$  is bounded, total (i.e. closed linear span of  $\Delta$  coincides with  $H$ ) and that  $A\Delta \subset \bar{\Delta}$ . Then  $A \leq I$  (it means that  $(x|Ax) \leq (x|x)$  for any  $x$  belonging to the domain of  $A$ ).

Proof:

Let  $\tilde{A}$  be a positive selfadjoint extension of  $A$  (see for



example [4]) and let  $dE$  be spectral measure associated with  $\tilde{A}$  :

$$\tilde{A} = \int_0^{\infty} \lambda dE(\lambda)$$

For any  $t \geq 0$  we put  $E_t = \int_0^{\infty} dE(\lambda)$ . Let  $t > 1$  and  $0 < \epsilon < 1 - \frac{1}{t}$ . Pick up a vector  $x \in \Delta$  such that

$$\|E_t x\| \geq (1 - \epsilon) \sup_{y \in \Delta} \|E_t y\|$$

Then

$$\|E_t Ax\| = \|\tilde{A} E_t x\| \geq t \|E_t x\| \geq t(1 - \epsilon) \sup_{y \in \Delta} \|E_t y\| .$$

On the other hand  $Ax \in \bar{\Delta}$ ,  $E_t Ax \in E_t \bar{\Delta} \subset \overline{E_t \Delta}$ .

Therefore

$$\|E_t Ax\| \leq \sup_{y \in \Delta} \|E_t y\|$$

Comparing the last two inequalities we have

$$\sup_{y \in \Delta} \|E_t y\| \geq t(1 - \epsilon) \sup_{y \in \Delta} \|E_t y\| .$$

Since  $t(1 - \epsilon) > 1$  we get  $E_t y = 0$  for all  $y \in \Delta$ . It shows that  $E_t = 0$  ( $\Delta$  is assumed to be total in  $H$ ). This way we have proven that spectral measure  $dE(\cdot)$  vanishes on  $]1, \infty[$ . It means that  $\tilde{A} \leq I$  and relation  $A \leq I$  follows immediately.

Proof of the theorem 1.1.

By using standard procedure (see [4] page 31) one can find a Hilbert space  $H$  and a linear mapping  $i$  of  $M$  onto a dense subset of  $H$  such that ( $a, b \in M$ ) :

$$s(a, b) = (i(a)|i(b)) .$$

The adjoint mapping  $i' : H \rightarrow M'$  is introduced by the

formula  $(x \in H, a \in M)$  :

$$\langle i'(x), a \rangle = (x | i(a)) .$$

Note that  $s^* = i'i$  . One can easily check that  $i([0,1])$  is bounded and total in  $H$  .

Let  $s_1$  be a  $V$ -positive form defined on  $M \times M$  such that  $s_1^*(1) \leq s^*(1)$  . We shall prove that there exists an operator  $A$  acting in  $H$  defined on  $D_A = i(M)$  such that  $s_1^* = i'Ai$  .

To this end it is sufficient to show that

$$1^{\circ} \quad i(a) = 0 \Rightarrow s_1^*(a) = 0$$

$$2^{\circ} \quad s_1^*(M) \subset i'(M)$$

$$3^{\circ} \quad i' \text{ is an injection.}$$

Statement  $1^{\circ}$  follows from lemma 1, statement  $3^{\circ}$  is implied by (1.5) and relation  $\overline{i(M)} = H$  . We are going to prove  $2^{\circ}$  .

Let  $a \in [0,1]$  . Then  $s_1^*(a) \in [0, s_1^*(1)] \subset [0, s^*(1)]$  . In virtue of (1.1) one can find a net  $(a_{\alpha})$  of elements of  $[0,1]$  such that  $s_1^*(a) = \lim s^*(a_{\alpha})$  . Since all  $i(a_{\alpha})$  belong to bounded convex set  $i([0,1])$  , one can assume that the net  $(i(a_{\alpha}))$  is weakly converging to a point  $x \in \overline{i([0,1])}$  . Then  $s_1^*(a) = i'(x)$  . Indeed for any  $b \in M$  we have

$$\begin{aligned} \langle s_1^*(a), b \rangle &= \lim \langle s^*(a_{\alpha}), b \rangle = \lim (i(a_{\alpha}) | i(b)) \\ &= (x | i(b)) = \langle i'(x), b \rangle . \end{aligned}$$

This way we have shown that

$$s_1^*([0,1]) \subset i'(\overline{i([0,1])}) \tag{1.6}$$

and statement  $2^{\circ}$  follows immediately.

For any  $a \in M$  we have

$$\begin{aligned} (Ai(a)|i(a)) &= \langle i' Ai(a), a \rangle \\ &= \langle s_1^*(a), a \rangle = s_1(a, a) \geq 0 . \end{aligned}$$

It shows that  $A$  is positive. Moreover by using (1.6) we have  $Ai([0,1]) \subset \overline{i([0,1])}$ . According to lemma 2 :  $A \leq I$ .

Let  $a \in M$ . Then

$$s_1(a, a) = (Ai(a)|i(a)) \leq (i(a)|i(a)) = s(a, a) .$$

This ends the proof.

2. Applications to the  $C^*$ -algebra theory.

In this section we investigate selfpolar forms associated with triple  $(\mathcal{A}, V, 1)$ , where  $\mathcal{A}$  is a  $C^*$ -algebra,  $V$  is the cone containing all positive elements of  $\mathcal{A}$  and  $1$  is the unity of  $\mathcal{A}$ . It can be easily checked that  $(\mathcal{A}, V, 1)$  satisfies all requirements mentioned in section 1.

We adopt the notation used in [6]. In particular for any  $C^*$ -algebra  $\mathcal{A}$ ,  $\mathcal{A}^0$  denotes the opposite  $C^*$ -algebra. The connection between  $\mathcal{A}$  and  $\mathcal{A}^0$  is given by an antilinear, multiplicative,  $*$ -invariant 1-1 mapping

$$\mathcal{A} \ni a \rightarrow \bar{a} \in \mathcal{A}^0 .$$

Tensor product  $\mathcal{A}^0 \otimes \mathcal{A}$ , after a suitable completion, becomes a  $C^*$ -algebra. It will be denoted by  $\tilde{\mathcal{A}}$ .

Let us recall that a state  $\tilde{\omega}$  of  $\tilde{\mathcal{A}}$  is said to be  $j$ -positive iff  $\tilde{\omega}(\bar{a} \otimes a) \geq 0$  for all  $a \in \mathcal{A}$ . State  $\tilde{\omega}$  is said to be exact iff the representation  $\pi_{\tilde{\omega}}$  of  $\tilde{\mathcal{A}}$  defined (GNS-construction) by  $\tilde{\omega}$  obeys the following property:  $\{\pi_{\tilde{\omega}}(\bar{a} \otimes 1) : a \in \mathcal{A}\}$  is weakly dense in the von Neumann algebra of all operators commuting with  $\pi_{\tilde{\omega}}(\bar{1} \otimes a)$  for any  $a \in \mathcal{A}$ :

$$\overline{\pi_{\tilde{\omega}}(\mathcal{A}^0 \otimes 1)} = \pi_{\tilde{\omega}}(\bar{1} \otimes \mathcal{A})' .$$

Let us note the following result ([6] thm 1.1).

$\rightarrow$  Let  $\omega$  be a state of  $\mathcal{A}$ . Then there exists a  $j$ -positive exact state  $\tilde{\omega}$  of  $\tilde{\mathcal{A}}$  such that  $\omega(a) = \tilde{\omega}(\bar{1} \otimes a)$  for any  $a \in \mathcal{A}$ .

The proof presented in [6] have used the following two additional assumptions: that  $\omega$  is a factor state and that the represen-

tation of  $\mathcal{U}$  defined by  $\omega$  acts is a separable Hilbert space. However the first assumption has been used only to show that  $\tilde{\omega}$  is a pure state; the second can be eliminated by using the theory of standard forms of von Neuman algebras developed in [3].

The main results of the section are contained in the following theorem.

Theorem 2.1.

There exists 1-1 correspondence between the set of all normalized ( $s$  is normalized iff  $s(1,1) = 1$ ) selfpolar forms  $s$  defined on  $\mathcal{U} \times \mathcal{U}$  and the set of all  $j$ -positive, exact states  $\tilde{\omega}$  of  $\tilde{\mathcal{U}}$ . The correspondence is established by the formula ( $a, b \in \mathcal{U}$ ) :

$$s(a,b) = \tilde{\omega}(\bar{a} \otimes b) \quad (2.1)$$

Proof:

In the first part we shall show that the sesquilinear form  $s$  introduced by (2.1) is selfpolar for any given  $j$ -positive exact state  $\tilde{\omega}$ . Obviously,  $s$  is positive. Moreover for any  $a, b \in V$ ,  $\bar{a} \otimes b$  is a positive element of  $\tilde{\mathcal{U}}$  and therefore

$$s(a,b) = \tilde{\omega}(\bar{a} \otimes b) \geq 0 \quad (2.2)$$

It means that  $s$  is  $V$ -positive and relation  $s^*([0,1]) \subset [0, s^*(1)]$  follows immediately.

Let  $\eta \in [0, s^*(1)]$ . To end the first part of the proof we have to find a net  $(a_\alpha)$  of elements of  $[0,1]$  such that  $s^*(a_\alpha)$  is weakly convergent to  $\eta$ .

Let  $\pi$  be the representation of  $\tilde{\mathcal{U}}$  induced by  $\tilde{\omega}$ ,  $H$  be the carrier Hilbert space of  $\pi$  and  $x$  be the corresponding cyclic vector. Then

$$\tilde{\omega}(\tilde{a}) = (x | \pi(\tilde{a})x)$$

for any  $\tilde{a} \in \tilde{\mathcal{U}}$ . In particular setting  $\tilde{a} = \bar{1} \otimes b$  we get

$$\begin{aligned} \langle s^*(1), b \rangle &= s(1, b) = \tilde{\omega}(\bar{1} \otimes b) \\ &= (x | \pi(\bar{1} \otimes b)x) . \end{aligned}$$

Let  $\mathcal{A}$  be the von Neumann algebra generated by  $\pi(\bar{1} \otimes \mathcal{U})$  and let  $\mathcal{A}'$  denote the commutant of  $\mathcal{A}$ . Starting from the last formula one can show (see [2] page 35) that any functional  $\eta \in [0, s^*(1)]$  is given by the formula ( $b \in \mathcal{U}$ ):

$$\langle \eta, b \rangle = (x | A \pi(\bar{1} \otimes b)x) ,$$

where  $A$  is an operator belonging to  $\mathcal{A}'$  such that  $0 \leq A \leq I$ .

We know that  $\pi(\bar{\mathcal{U}} \otimes 1)$  is weakly dense in  $\mathcal{A}'$  ( $\tilde{\omega}$  is exact!). In virtue of Kaplansky theorem there exists a net  $(c_\alpha)$  of elements of  $\mathcal{U}$  such that  $\pi(\bar{c}_\alpha \otimes 1)$  is weakly convergent to  $A$  and  $0 \leq \pi(\bar{c}_\alpha \otimes 1) \leq I$  for all  $\alpha$ .

Let  $f$  be the real function on  $\mathbb{R}$  such that  $f(t) = 0$  for  $t \leq 0$ ;  $f(t) = t$  for  $t \in [0, 1]$  and  $f(t) = 1$  for  $t \geq 1$ . Then  $f(B) = B$  for any operator  $B$  such that  $0 \leq B \leq I$ . Therefore  $\pi(\bar{c}_\alpha \otimes 1)$  remains unchanged if one replace  $c_\alpha$  by  $a_\alpha = f(\frac{c_\alpha + c_\alpha^*}{2})$ . On the other hand, obviously  $a_\alpha \in [0, 1]$ . Now we have ( $b \in \mathcal{U}$ ):

$$\begin{aligned} \langle \eta, b \rangle &= (x | A \pi(\bar{1} \otimes b)x) \\ &= \lim (x | \pi(\bar{a}_\alpha \otimes 1) \pi(\bar{1} \otimes b)x) \\ &= \lim (x | \pi(\bar{a}_\alpha \otimes b)x) = \lim \tilde{\omega}(\bar{a}_\alpha \otimes b) \\ &= \lim s(a_\alpha, b) = \lim \langle s^*(a_\alpha), b \rangle \end{aligned}$$

This ends the first part of the proof.

Now, let  $s$  be a selfpolar normalized form defined on  $\mathcal{A} \times \mathcal{B}$ . Then  $\omega = s^*(1)$  is a state of  $\mathcal{B}$  and one can find a  $j$ -positive exact state  $\tilde{\omega}$  of  $\tilde{\mathcal{A}}$  such that  $\omega(b) = \tilde{\omega}(\bar{1} \otimes b)$ . According to the first part of the proof mapping  $s_1 : (a, b) \rightarrow \tilde{\omega}(\bar{a} \otimes b)$  is a selfpolar form. We have to show, that  $s = s_1$ . But this follows immediately from Thm 1.2 (note that  $s_1^*(1) = \omega = s^*(1)$ ). The theorem is proven.

Let us note that the argument leading to (2.2) works for any (not necessarily exact)  $j$ -positive state  $\tilde{\omega}$ . Therefore, for the  $C^*$ -algebra case, theorems 1.1 and 1.2 admit the following nice reformulation:

Theorem 2.2.

Let  $\tilde{\omega}$  and  $\tilde{\eta}$  be  $j$ -positive states of  $\tilde{\mathcal{A}}$ . Assume that  $\tilde{\omega}$  is exact and  $\tilde{\omega}(\bar{1} \otimes a) = \tilde{\eta}(\bar{1} \otimes a)$  for any  $a \in \mathcal{A}$ . Then  $\tilde{\eta}(\bar{a} \otimes a) \leq \tilde{\omega}(\bar{a} \otimes a)$  for any  $a \in \mathcal{A}$ . If in addition  $\tilde{\eta}$  is exact, then  $\tilde{\omega} = \tilde{\eta}$ .

Remark:

The last statement of the theorem had been already proven in much stronger version [7].

Let us also note the following nice result implied directly by Thm 1.3:

Theorem 2.3.

Let  $\omega \rightarrow \tilde{\omega}$  be the purification map (i.e.  $\tilde{\omega}$  is the only  $j$ -positive exact state of  $\tilde{\mathcal{A}}$  such that  $\tilde{\omega}(\bar{1} \otimes a) = \omega(a)$  for any  $a \in \mathcal{A}$ ; the terminology is taken from [5]). Then for any  $a \in \mathcal{A}$  function

$$\omega \rightarrow \tilde{\omega}(\bar{a} \otimes a)$$

is concave and upper semicontinuous.

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