Kurepa's Hypothesis and the Continuum
by
Keith J. Devlin
(Oslo)

Abstract

Silver [5] proved that \( \text{Con}(\text{ZFC} + "\text{there is an inaccessible cardinal"}) \) implies \( \text{Con}(\text{ZFC} + \text{CH} + "\text{there are no Kurepa trees"}) \). In order to obtain this result, he generically collapses an inaccessible cardinal to \( w_2 \). Hence \( \text{CH} \) necessarily holds in his final model. In this paper we sketch Silver's proof, and then show how it can be modified to obtain a model in which there are no Kurepa trees and the continuum is anything we wish.

Introduction

We work in \( \text{ZFC} \) and use the usual notation and conventions. For details concerning the forcing theory we require, see Jech [3] or Shoenfield [4]. A tree is a poset \( T = \langle T, \leq_T \rangle \) such that \( \hat{x} = \{y \in T \mid y <_T x\} \) is well-ordered by \( <_T \) for any \( x \in T \). The order-type of \( \hat{x} \) is the height of \( x \) in \( T \), \( \text{ht}(x) \). The \( \alpha \)'th level of \( T \) is the set \( T_\alpha = \{x \in T \mid \text{ht}(x) = \alpha\} \). \( T \) is an \( w_1 \)-tree iff:

(i) \((\forall \alpha < w_1)(T_\alpha \neq \emptyset) \& (T_{w_1} = \emptyset)\);
(ii) \((\forall \alpha < \beta < w_1)(\forall x \in T_\alpha)(\exists y_1, y_2 \in T_\beta)(x <_T y_1, y_2 \& y_1 \neq y_2)\);
(iii) \((\forall \alpha < w_1)(\forall x, y \in T_\alpha) (\lim(\alpha) \rightarrow [x = y \leftrightarrow \hat{x} = \hat{y}])\);
(iv) \((\forall \alpha < w_1)(|T_\alpha| \leq \omega) \& |T_0| = 1 \). For further details of \( w_1 \)-trees, see Jech [2].
If $T$ is an $\omega_1$-tree, a branch of $T$ is a maximal totally ordered subset of $T$. A branch $b$ of $T$ is cofinal if $(\forall \alpha < \omega_1)(T_\alpha \cap b \neq \emptyset)$. $T$ is Kurepa if it has at least $\omega_2$ cofinal branches. If $V = L$, then there is a Kurepa tree. This result is due to Solovay. For a proof, see Devlin [1] or Jech [2]. More generally, if $V = L[A]$, where $A \subseteq \omega_1$, then there is a Kurepa tree, from which it follows that if there are no Kurepa trees, then $\omega_2$ is inaccessible in $L$. (All of this is still due to Solovay, and is proved in [1] and [2].) Hence, in order to establish $\text{Con}(\text{ZFC} + K)$, where $K$ denotes the statement "there are no Kurepa trees", one must at least assume $\text{Con}(\text{ZFC} + I)$, where $I$ denotes the statement "there is an inaccessible cardinal".

Now, if $M$ is any cardinal absolute extension of $L$, and if $T$ is a Kurepa tree in $L$, then $T$ will clearly be a Kurepa tree in $M$. Hence, if $\kappa$ is any cardinal of cofinality greater than $\omega$, we can, by standard arguments, find a generic extension of $L$, with the same cardinals as $L$, such that, in the extension, there is a Kurepa tree and $2^\omega = \kappa$. Johnsbråten has pointed out that the consistency of $K + 2^\omega = \kappa$ (for such $\kappa$) is not so easily obtained. Now, Silver [5] has shown that $\text{Con}(\text{ZFC} + I) \rightarrow \text{Con}(\text{ZFC} + 2^\omega = \omega_1 + K)$. (And by Solovay's result above, the hypothesis here is as weak as possible). However, the method Silver employs necessarily makes $2^\omega = \omega_1$ hold, so as it stands the only hope to obtain $K + 2^\omega = \kappa$ would seem to be to take Silver's model and blow-up the continuum generically to $\kappa$. In fact this procedure does work (i.e. $K$ is preserved), but the proof that it does is fairly delicate, as opposed to the corresponding argument for $\neg K$. Since we shall need all of the tricks
employed by Silver in his proof of $\text{Con}(\text{ZFC} + K)$, we may as well commence by describing his argument.

**Silver's Model.**

We shall use $M$ to denote an arbitrary countable transitive model (c.t.m.) of ZFC throughout. By *poset*, we mean, as usual in forcing, a poset $P$, with a maximum element $\top$, such that every $p \in P$ has at least two incompatible extensions in $P$, where $p, q \in P$ are *compatible*, written $p \sim q$, if there is $r \in P$ such that $r \leq p, q$. We say $P$ satisfies the $\kappa$ chain condition ($\kappa$-c.c.), for $\kappa$ an uncountable cardinal, if there is no pairwise incompatible subset of $P$ of cardinality $\kappa$.

$P$ is $\sigma$-closed if whenever $\langle p_\alpha \mid \alpha < \lambda < \omega_1 \rangle$ is a decreasing sequence from $P$ there is $p \in P$ such that $p \leq p_\alpha$ for all $\alpha < \lambda$. The following lemmas are standard. (See Shoenfield [4] for example.)

**Lemma 1** (Cohen; Solovay)

Let $P$ be a poset in $M$, $\kappa$ an uncountable regular cardinal in $M$. Let $G$ be $M$-generic for $P$.

(i) If $M \models " P satisfies the $\kappa$-c.c." then $\lambda \geq \kappa$ is a cardinal in $M[G]$ iff $\lambda$ is a cardinal in $M$.

(ii) If $M \models " P is $\sigma$-closed", then for all $\lambda < \omega_1$, $(M^\lambda)^M = (M^\lambda)[M[G]]$, so in particular, $\omega_1^M = \omega_1^M[G]$ and $\mathcal{P}^M(\omega) = \mathcal{P}^M[G](\omega)$. 
Lemma 2 (Lévy)
Let \( \kappa \) be an inaccessible cardinal in \( M \), \( P \) a poset in \( M \) such that \( M \models "|P| < \kappa" \). If \( G \) is \( M \)-generic for \( P \), then \( \kappa \) is still inaccessible in \( M[G] \).

Lemma 3 (Solovay)
Let \( P_1, P_2 \) be posets in \( M \). If \( G_1 \) is \( M \)-generic for \( P_1 \) and \( G_2 \) is \( M[G_1] \)-generic for \( P_2 \), then \( G_1 \) is \( M[G_2] \)-generic for \( P_1 \), \( G_2 \) is \( M \)-generic for \( P_2 \), \( G_1 \times G_2 \) is \( M \)-generic for \( P_1 \times P_2 \), and \( M[G_1][G_2] = M[G_2][G_1] = M[G_1 \times G_2] \), where \( P_1 \times P_2 \) is the cartesian product of \( P_1 \) and \( P_2 \) with the partial ordering \( \langle p_1, p_2 \rangle \preceq \langle q_1, q_2 \rangle \iff p_1 \preceq_1 q_1 & p_2 \preceq_2 q_2 \). Conversely, if \( G \) is \( M \)-generic for \( P_1 \times P_2 \), then \( G_1 = \{ p \mid \langle p, \pi \rangle \in G \} \) is \( M \)-generic for \( P_1 \), \( G_2 = \{ q \mid \langle \eta, q \rangle \in G \} \) is \( M[G_1] \)-generic for \( P_2 \), and \( G = G_1 \times G_2 \).

Let \( \kappa \) be an uncountable cardinal. The poset \( P(\kappa) \) is defined as follows. An element \( p \) of \( P(\kappa) \) is a countable function such that \( \text{dom}(p) \subseteq \omega_1 \times \kappa \) and \( \text{ran}(p) \subseteq \kappa \), and if \( \langle \alpha, \delta \rangle \in \text{dom}(p) \), then \( p(\alpha, \delta) \in \delta \). The ordering on \( P(\kappa) \) is defined by \( p \preceq q \iff p \supseteq q \). If \( P = P(\kappa) \) and \( \lambda < \kappa \), we set \( P_\lambda = \{ p \upharpoonright (\omega_1 \times \lambda) \mid p \in P \} \), \( P^\lambda = \{ p - p \upharpoonright (\omega_1 \times \lambda) \mid p \in P \} \), and regard \( P_\lambda, P^\lambda \) as posets in the obvious manner. Clearly, \( P \cong P_\lambda \times P^\lambda \), by a canonical isomorphism.

Lemma 4 (Lévy)
Let \( \kappa \) be an inaccessible cardinal in \( M_1 \), and set \( P = [P(\kappa)]^M \). Then, \( M \models "P is \sigma\text{-closed and satisfies the } \kappa\text{-c.c."} \). If \( G \) is \( M \)-generic for \( P \), then \( \omega_1^M = \omega_1^{M[G]} \) and \( \kappa = \omega_2^{M[G]} \). Furthermore, if \( \lambda < \kappa \) is an uncountable regular cardinal in \( M \), then \( M[G \cap P_\lambda] \models "P^\lambda \text{ is } \sigma\text{-closed and satisfies } \kappa\text{-c.c."} \).
Proof: See Jech [3] or Silver [5]. For the last part, notice that as $P_\lambda$ is $\sigma$-closed in $M$, $M[G \cap P_\lambda]$ has no new countable sequences from $P_\lambda$, whence $P_\lambda$ is still $\sigma$-closed in $M[G \cap P_\lambda]$. Also, as we clearly have $P^\lambda \subseteq [P(x)]^{M[G \cap P_\lambda]}$, lemma 2 will ensure that $P^\lambda$ has the $\lambda$-c.c. in $M[G \cap P_\lambda]$. □

For later use, we shall give the proof of the next lemma in full.

Lemma 5 (Silver)

Let $P$ be a poset in $M$ such that $M \models "P$ is $\sigma$-closed". Let $\mathcal{T}$ be an $\omega_1$-tree in $M$. Let $G$ be $M$-generic for $P$. If $b$ is a cofinal branch of $\mathcal{T}$ in $M[G]$, then in fact $b \in M$.

Proof: We may assume $\mathcal{T} = \langle \omega_1, \leq_T \rangle$. Suppose that, in fact $b \notin M$. Working in $M$, we define sequences $\langle p_s \mid s \in 2^{\omega_1} \rangle$, $\langle x_s \mid s \in 2^{\omega_1} \rangle$ so that $p_s \in P$; $t \subseteq s \rightarrow p_s \leq p_t$; $x_s \in T$; $t \subseteq s \rightarrow x_t <_T x_s$; $|s| = |t| \rightarrow \text{ht}(x_s) = \text{ht}(x_t)$; and $x_s \langle 0 \rangle \neq x_s \langle 1 \rangle$. The definition is by induction on $|s|$. Pick $p_\emptyset \in P$ so that $p_\emptyset \models "b$ is a cofinal branch of $\mathcal{T}$ & $b \notin \dot{M}"$. Let $x_\emptyset$ be the minimal element of $\mathcal{T}$. Suppose $p_s$, $x_s$ are defined for all $s \in 2^{n}$, and that $p_s \models "x_s \notin b$", where $p_s \leq p_\emptyset$ in particular. Since $p_\emptyset \models "b \notin \dot{M}"$, we can clearly find $p_{s \langle 0 \rangle}$, $p_{s \langle 1 \rangle} \leq p_s$ (each $s \in 2^{n}$) and points $x_{s \langle 0 \rangle}$, $x_{s \langle 1 \rangle} >_T x_s$ such that $\text{ht}(x_{s \langle 0 \rangle}) = \text{ht}(x_{s \langle 1 \rangle})$ and $x_{s \langle 0 \rangle} \neq x_{s \langle 1 \rangle}$, for which $p_{s \langle 1 \rangle} \models "x_{s \langle i \rangle} \in \dot{b}"$, $i = 0, 1$. Furthermore, we may clearly do this in such a way that for any $s, t \in 2^{n+1}$, $\text{ht}(x_s) = \text{ht}(x_t)$. Since $P$ is $\sigma$-closed, for each $f \in 2^{\omega_1}$ we may pick $p_f \in P$ such that $p_f \leq p_f ^{\restriction n}$ for all $n < \omega$. Also, as $|2^{\omega_1}| = \omega$, we may pick $\alpha < \omega_1$
such that $\text{ht}(x_s) < \alpha$ for all $s \in 2^\omega$. Since $p_f \leq p_\emptyset$ (each $f \in 2^\omega$), we can find $p_f' \leq p_f$ such that for some $x_f \in T_\alpha$, $p_f' \vdash \exists \check{x_f} \in \check{b}$. But, clearly, $p_f' \vdash \check{x_f} \in T_e$ for all $n < \omega$, so by our construction, $f \not= g \rightarrow x_f \not= x_g$.

(There are just two remarks called for here. Firstly, since $\check{x} \in M$, if $p_f' \vdash \check{x_f} \in T_e$ then in fact $x_f < T_e x_f \in n$. Secondly, if $f \not= g$ then for some $n < \omega$, $f \upharpoonright n \not= g \upharpoonright n$.) Thus $\{x_f | f \in 2^\omega\}$ is an uncountable subset of $T_\alpha$, which is absurd. \(\square\)

**Theorem 6 (Silver)**

Let $\kappa$ be an inaccessible cardinal in $M$. Let $P = [P(\kappa)]^M$.

Let $G$ be $M$-generic for $P$. Then $M[G] \models 2^\omega = \omega_1 + K$.

**Proof:** By lemmas 4 and 1, $M[G] \models 2^\omega = \omega_1$ and $\omega_2^{M[G]} = \kappa$.

Also, $\omega_1^{M[G]} = \omega_1^M$, so the notion of an "$\omega_1$-tree" is absolute here. Let $\check{T}$ be an $\omega_1$-tree in $M[G]$. We may assume $\check{T} = \langle w_1, \leq_T \rangle$. By the truth lemma, we can find an uncountable regular cardinal $\lambda < \kappa$ of $M$ such that $\check{T} \in M[G \cap P_\lambda]$. By lemma 2, $\check{T}$ has fewer than $\kappa$ cofinal branches in $M[G \cap P_\lambda]$. But by lemma 4, $P^\lambda$ is $\sigma$-closed in $M[G \cap P_\lambda]$, and by lemma 3, $G \cap P^\lambda$ is $M[G \cap P_\lambda]$-generic for $P^\lambda$, so by lemma 5, $\check{T}$ has no cofinal branches in $M[G \cap P_\lambda][G \cap P^\lambda]$ other than those in $M[G \cap P_\lambda]$. Again by lemma 3, $M[G \cap P_\lambda][G \cap P^\lambda] = M[G]$, so we see that $\check{T}$ has fewer than $\kappa$ cofinal branches in $M[G]$.

Q.E.D.
We shall require the following well-known result, proved in Jech [3].

**Lemma 7 (Marczewski)**

Let \( \lambda \) be a limit ordinal, \( \text{cf}(\lambda) = \omega_1 \). Let \( J \) be a collection of \( \omega_1 \) finite subsets of \( \lambda \). There is a finite subset \( X \) of \( \lambda \) and an uncountable subfamily \( J' \) of \( J \) such that \( Y, Z \in J' \Rightarrow Y \cap Z \subseteq X \).

Let \( \kappa \) be an ordinal. The poset \( C(\kappa) \) is defined as follows. An element of \( C(\kappa) \) is a finite function \( p \) such that \( \text{dom}(p) \subseteq \kappa \) and \( \text{ran}(p) \subseteq 2 \). The partial ordering on \( C(\kappa) \) is defined by \( p \leq q \Leftrightarrow p \supseteq q \). Thus, if \( \kappa \) is an uncountable regular cardinal in \( M \), \( [C(\kappa)]^M \) is the usual poset for adding \( \kappa \) Cohen generic subsets of \( \omega \) to \( M \). Note that in this case, \( [C(\kappa)]^M = C(\kappa) \), both of these being defined by the same, absolute formula of set theory.

It is well known that if \( \kappa \) is an uncountable regular cardinal in \( M \) and \( G \) is \( M \)-generic for \( C = [C(\kappa)]^M \), then \( M \) and \( M[G] \) have the same cardinals, by virtue of the fact that \( M = "C \text{satisfies the countable chain condition}" \), and \( M[G] \models 2^\omega \geq \kappa \). For our purposes, however, it will be useful to regard the procedure of forcing with \( C \) over \( M \) here as an iteration of length \( \kappa \).

Accordingly, we make the following definitions.

Let \( U \) be the poset consisting of all maps \( p \) such that \( \text{dom}(p) = n \) for some \( n \in \omega \) and \( \text{ran}(p) \subseteq 2 \), ordered by \( p \leq q \Leftrightarrow p \supseteq q \). Thus \( U \in M \) and \( U \) is the usual poset for adding one Cohen generic subset of \( \omega \) to \( M \).
Let $\kappa \in \text{On}$. Set $C^*(\kappa) = \{ \varphi \mid \varphi : \kappa \to U \}$ for some finite set $X \subseteq \kappa$, $\varphi(\alpha) \neq \emptyset \iff \alpha \in X$ (we call $X$ the support of $\varphi$, $\text{supp}(\varphi)$), and partially order $C^*(\kappa)$ by $\varphi \leq \psi \iff 
exists \alpha \in \kappa (\varphi(\alpha) \supseteq \psi(\alpha))$. It is easily seen that forcing with $C^*(\kappa)$ is equivalent to forcing with $C(\kappa)$. In fact, the complete boolean algebra associated with both of these posets is the Borel algebra on $2^\kappa$ factored by the ideal of all meager Borel subsets of $2^\kappa$, where $2^\kappa$ is given the product topology for the discrete topology on 2. Note also that the definition of $C^*(\kappa)$ is, like $C(\kappa)$, absolute for transitive models of ZFC containing $\kappa$. The point of all of this is that forcing with $C^*(\kappa)$ can be regarded as a process of forcing with $U \times \kappa$ times, successively, using lemma 3.

Lemma 8.

Let $\kappa$ be an uncountable cardinal in $M$, $\text{cf}^M(\kappa) > \omega$. Let $C = [C(x)]^M$. If $G$ is $M$-generic for $C$, then $M[G] = 2^\omega \geq \kappa$, $M$ and $M[G]$ have the same cardinals and cofinality function, and if $M = 2^\omega \leq \kappa$, then $M[G] = 2^\omega = \kappa$. Furthermore, if $T = (\omega_1^M, \leq_T)$ is an $\omega_1$-tree in $M$, and $b$ is a cofinal branch of $T$ in $M[G]$, then $b \in M$.

Proof: The last part of the lemma is the only non-standard part. Let $C^* = [C^*(\kappa)]^M$. We may assume, by virtue of our above remarks, that $G$ is $M$-generic for $C^*$ rather than $C$. Let $\mathcal{T} = (\omega_1^M, \leq_T)$ be an $\omega_1$-tree in $M$. We may assume that $\nu <_T \tau \vdash \nu < \tau$. Note that as $\omega_1^M[G] = \omega_1^M$, $\mathcal{T}$ is still an $\omega_1$-tree in $M[G]$.

If $\gamma < \kappa$, then clearly $C^*(\gamma) = \{ \varphi \upharpoonright \gamma \mid \varphi \in C^* \}$. Set
$G_\gamma = \{ \varphi \mid \gamma \mid \varphi \in G \}$. By lemma 3, $G_\gamma$ is $M$-generic for $C^*(\gamma)$ and $M[G]$ is a generic extension of $M_\gamma = M[G_\gamma]$. Clearly, $M_\kappa = M[G]$, so it suffices to prove, by induction on $\gamma < \kappa$, that if $b$ is a cofinal branch of $\bar{\nu}$ in $M_\gamma$, then $b \in M$.

For $\gamma = 0$ there is nothing to prove. Suppose the result holds for $\gamma < \kappa$. If $H = \{ \varphi(\gamma) \mid \varphi \in G \}$, then by lemma 3, $H$ is $M_\gamma$-generic for $U$ and $M_{\gamma+1} = M_\gamma[H]$. Let $b$ be a cofinal branch of $\bar{\nu}$ in $M_{\gamma+1}$. It suffices, by virtue of the induction hypothesis, to show that $b \in M_\gamma$. This will be so if, whenever $p \in U$ and $p \Vdash "b$ is a cofinal branch of $\bar{\nu}"$, there is $q < p$ such that $q \Vdash "b \in \bar{\nu}"$. We work in $M_\gamma$. Let such a $p$ be given. For each $q < p$, let $a(q)$ be the supremum of all ordinals $\xi < w_1$ such that $q \Vdash "\bar{\nu} \in b"$ for some $\bar{\nu}$ on level $\xi$ of $\bar{\nu}$. Set $a = \sup\{a(q) \mid q < p\}$. By the truth lemma for forcing with $U$ over $M_\gamma$, $a = w_1$. Hence, as $|U| = w$, $a(q) = w_1$ for some $q < p$. Set $b' = \{ \bar{\nu} \in T \mid q \Vdash "\bar{\nu} \in b" \}$. Then $b' \in M_\gamma$, and clearly $q \Vdash "b = b'"$, so we are done. Finally, suppose $\gamma < \kappa$, $\lim(\gamma)$, and the result holds for all $\delta < \gamma$. There are three cases to consider.

Case 1 $\cf^M(\gamma) = w$.

Let $b$ be a cofinal branch of $\bar{\nu}$ in $M_\gamma$. In $M$, let $\langle \gamma_n \mid n < w \rangle$ be cofinal in $\gamma$. Work in $M_\gamma$. By the truth lemma for forcing with $C^*(\gamma)$ over $M$, for each $\bar{\nu} \in b$ we can find $p_\bar{\nu} \in G_\gamma$ such that $p_\bar{\nu} \Vdash "\bar{\nu} \in b"$. Let $X_\bar{\nu} = \sup p_\bar{\nu})$. Since each $X_\bar{\nu}$ is finite, and $\cf(w_1) > w$, we can find an uncountable set $b' \subseteq b$ such that $\bar{\nu} \in b' - X_\bar{\nu} \subseteq \gamma_n$ for some fixed $n < w$. But clearly,
Let \( b \) be a cofinal branch of \( \mathcal{L} \) in \( M_\gamma \). Suppose, by way of contradiction, that \( b \notin M_\delta \), also. Work in \( M_\gamma \). For each \( \nu \in b \), pick \( p_\nu \in G_\gamma \) such that \( p_\nu \models \nu \in \check{b} \) and let \( X_\nu = \text{supp}(p_\nu) \). If \( \text{sup}\{\text{max}(X_\nu) | \nu \in b\} < \gamma \), then arguing as in case 1 we see that \( b \in M_\delta \) for \( \delta = \text{sup}\{\text{max}(X_\nu) | \nu \in b\} \), and we are done.

Hence we may assume \( \text{sup}\{\text{max}(X_\nu) | \nu \in b\} = \gamma \). It follows, by lemma 6, that we can find an uncountable set \( b' \subseteq b \) and a finite set \( X \subseteq \gamma \) such that \( \nu, \tau \in b' \) and \( \nu < \tau \) implies \( X_\nu \cap X_\tau = X \) and such that \( \nu \in b' \) implies \( X_\nu \neq X \). Since \( |U| = \omega \), we can find an uncountable set \( b'' \subseteq b' \) such that \( \nu, \tau \in b'' \) implies \( p_\nu \upharpoonright X = p_\tau \upharpoonright X = p \), say. From now on \( \models \) refers to the forcing relation for \( C^*(\gamma) \) over \( M \).

Claim. There is \( q \in C^*(\gamma) \), \( \text{supp}(q) \cap X = \emptyset \), and \( \nu < \omega_1 \) such that \( \nu \notin b \) but \( p \cup q \models \nu \in \check{b} \), where \( p \cup q \in C^*(\gamma) \) is defined from \( p \) and \( q \) in the obvious manner.

Suppose the claim is false. In \( M \), set \( d = \{ \nu \in T | (\exists q \in C^*(\gamma)) [\text{supp}(q) \cap X = \emptyset \& p \cup q \models \nu \in \check{b}] \} \). Since the claim fails, \( d \subseteq b \). But for each \( \nu \in b'' \), if \( q = p_\nu \upharpoonright (X_\nu - X) \), then \( \text{supp}(q) \cap X = \emptyset \) and \( p \cup q = p_\nu \) and \( p_\nu \models \nu \in \check{b} \), so \( b \subseteq d \).

Hence \( b = d \in M \), a contradiction. This proves the claim.

Pick \( q \in C^*(\gamma) \) as in the claim and let \( \nu < \omega_1 \) be such that \( \nu \notin b / p \cup q \models \nu \in \check{b} \). Pick \( \tau \in b'' \), \( \tau > \nu \), such that \( X_\tau \cap \text{supp}(q) = \emptyset \). (This is clearly possible). Clearly, \( p_\tau \cup q = p \cup q \cup [p_\tau \upharpoonright (X_\tau - X)] \in C^*(\gamma) \).
look, 

But if $p \uparrow q \leq p \uparrow \tau$, so $p \uparrow q \vDash " \check{\tau} \in b "$, and $p \uparrow q \leq p \cup q$, so $p \uparrow q \vDash " \check{\nu} \in b "$. Hence, as $\nu < \tau$, $p \uparrow q \vDash " \check{\nu} \in _\tau \check{\tau} "$, which means $\nu < _\tau \tau$, of course. Thus, as $\tau \in b$, $\nu \in b$, a contradiction.

**Case 3** \[ \text{cf}^M(\gamma) > w^M_1 \] 

This case is trivial by the lemma for forcing with $C^*(\gamma)$ over $M$.

The lemma is proved. \[ \square \]

The following is an analogue of lemma 5.

**Lemma 9**

Let $C, P$ be posets in $M$ such that $M \models " C$ satisfies c.c.c. and $P$ is $\sigma$-closed". Let $G$ be $M$-generic for $C \times P$. (Thus $w^M_1 = w^M_1[G]$.) Let $G_C = \{ p \in C \mid \langle p, 1 \rangle \in G \}$, $G_P = \{ q \in P \mid \langle 1, q \rangle \in G \}$.


Proof: Notice that as $P$ is not necessarily $\sigma$-closed in the sense of $M[G_C]$, we cannot argue exactly as in lemma 5. However, with a little extra work, we can carry through an argument parallel to that of lemma 5. We shall assume that $\bar{T} = \langle w_1, \leq_T \rangle$, as before. In $M[G_C]$, for each $\alpha < w_1$, let $D_\alpha = \{ q \in P \mid q \vDash " \check{x} \in b \cap \check{T}_\alpha " \text{ for some } x \in T \}$, where $\vDash_P$ denotes $P$-forcing over $M[G_C]$.

Clearly each $D_\alpha$ is a dense open subset of $P$.

Suppose that $b \notin M[G_C]$. To cut down on notation, let us suppose that, in fact $\emptyset \vDash_P " \check{b} \in b \cap \check{T}_\alpha "$. (In the general case, we simply work beneath some $q_0$ in $P$, of course.) Pick $p^* \in G_C$ so
that $p^* \Vdash C "[\check{i}]$ is an $\omega_1$-tree with domain $\check{\omega}_1$ and $[\emptyset \vdash \check{i} " b$ is a cofinal branch of $\check{z}$ not in $\check{M}[\check{G}_C"]$ and $[<\check{D}_\alpha | \alpha < \check{\omega}_1]$ is a sequence of dense open subsets of $\check{F}$.

Claim. Let $\alpha < \omega_1$, $q \in P$. There is $q' \leq_P q$ and $x \in T$ such that $\langle p^*, q' \rangle \models_{C \times P} x \in \check{b} \cap \check{T}_\check{z} "$. By induction, we define in $M$ a sequence $\langle \langle p_\nu, q_\nu \rangle | \nu < \delta \rangle$, some $\delta < \omega_1$, so that $\nu < \delta - p_\nu \leq_C p^*$ and $q_\nu \leq_P q$, $\nu < \tau < \delta - p_\nu \Vdash p_\tau$ and $q_\tau \leq_P q_\nu$, and $p_\nu \Vdash " q_\nu \in \check{D}_\check{z} "$. The ordinal $\delta$ will be determined by the failure of the definition. Since $C$ satisfies c.c.c., the incompatibility condition on the $p_\nu$'s will ensure that $\delta < \omega_1$; and in fact, the definition will stop precisely when $\{p_\nu | \nu < \delta \}$ is a maximal pairwise incompatible subset of $\{p \in C | p \leq_C p^* \}$. Let $\langle p_0, q_0 \rangle$ be chosen so that $p_0 \leq_C p^*$, $q_0 \leq_P q$, and $p_0 \Vdash " q_0 \in \check{D}_\check{z} ".$ This clearly causes no problems. Suppose $\langle \langle p_\tau, q_\tau \rangle | \tau < \nu \rangle$ is defined. Thus $\nu < \omega_1$, so we can find $q_\nu' \in P$, $q_\nu' \leq_P q_\nu$ for all $\tau < \nu$, by the $\sigma$-closed nature of $P$. (Remember that we are working in $M$ here!) Pick (if possible) $p_\nu \in C$, $p_\nu \leq_C p^*$, and $q_\nu \in P$, $q_\nu \leq_P q_\nu'$, so that $\tau < \nu - p_\tau \Vdash p_\nu$ and $p_\nu \Vdash " q_\nu \in \check{D}_\check{z} "$. Clearly, if we can find $p_\nu$ such that $p_\nu \leq_C p^*$ and $\tau < \nu - p_\tau \Vdash p_\nu$, then the choice of $q_\nu$ causes no trouble. That completes the definition. Some $P$ is $\sigma$-closed, let $q' \in P$ be such that $q' \leq_P q_\nu$ for all $\nu < \delta$. Then $q'$ is as required. It suffices to show that $p^* \Vdash " q' \in \check{D}_\check{z} ", and for this it is enough to show that $\{p \in C | p \Vdash " q' \in \check{D}_\check{z} " \}$ is dense below $p^*$ in $C$. So let $p \leq_C p^*$. Then for some $\alpha < \delta$, $p \sim p_\alpha$. Pick $p' \leq q P$, $p_\alpha$. Thus $p' \Vdash " q_\alpha \in \check{D}_\check{z} "$. But $q' \leq_P q_\alpha$. Hence $p' \Vdash " q' \in \check{D}_\check{z} ", as required. The claim is proved.
Using the claim, we can now argue as in lemma 5. By induction, pick sequences \( \langle q_s | s \in 2^\omega \rangle \), \( \langle x_s | s \in 2^\omega \rangle \) so that, in particular, \( \langle p^*, q_s \rangle \models \forall C \exists x_s \in \check{b} \) and so that \( \langle q_s | s \in 2^\omega \rangle \) decreases along branches in \( 2^\omega \), etc. Since \( \langle p^*, \emptyset \rangle \models \forall C \exists \check{b} \in \check{M}[G_C] \), this follows from the claim just as it followed in lemma 5. Since all of this is done in \( M \), where \( P \) is \( \sigma \)-closed, we obtain a contradiction exactly as before. The lemma is proved.\( \square \)

**Theorem 10**

Let \( \kappa \) be an inaccessible cardinal in \( M \), and let \( \lambda \) be an arbitrary cardinal in \( M \) such that \( \lambda \geq \kappa \) and \( \text{cf}^M(\lambda) > \omega \).

Let \( P = [P(\kappa)]^M \), \( C = [C(\lambda)]^M \). Let \( G \) be \( M \)-generic for \( P \times C \).

Then \( \omega^M_1 = \omega^M_1[G], \lambda = \omega^M_2[G], \lambda \) and all other cardinals of \( M \) above \( \kappa \) are cardinals in \( M[G] \) (so if \( \lambda = \omega^M_{\kappa+\gamma} \) then \( \lambda = \omega^M_{2+\gamma} M[G] \)), \( \text{cf}^M[G](\lambda) > \omega \), \( M[G] \models "2^\omega = \lambda" \), and \( M[G] \models "K" \).

**Proof:** Let \( G_P, G_C \) be as above. Let \( T = \langle w_1, \leq T \rangle \) be an \( w_1 \)-tree in \( M[G] \). By the truth lemma, pick \( \gamma < \kappa \) an uncountable regular cardinal of \( M \) such that \( T \in M[G_P \cap P_\gamma][G_C] \).

Let \( N = M[G_P \cap P_\gamma] \). Notice that by lemma 4, \( P^N \) is \( \sigma \)-closed in the sense of \( M \). Also, by absoluteness, \( C = [C(\lambda)]^N \), so \( C \) satisfies c.c.c. in \( N \). Now, by lemma 3, \( G_C \) is \( N \)-generic for \( C \), so by the truth lemma for \( C \)-forcing over \( N \) we can find, in \( N \), a set \( X \subseteq \lambda \), \( |X| = \omega_1 \), such that \( T \in N[G_C \cap C_X] \), where \( C_X = \{ p | X | p \in C \} \).

Now, \( X \in N \), so in \( N \) there is a canonical isomorphism \( C \cong C_X \times C^X \), where \( C^X = \{ p - p \cap X | p \in C \} \). Thus, by lemma 3 (applied to \( N \), \( G_C \cap C_X \) is \( N \)-generic for \( C_X \), \( G_C \cap C^X \) is \( N[G_C \cap C_X] \)-generic for \( C_X \), and \( N[G_C \cap C_X][G_C \cap C^X] = N[G_C] \). By lemma 2, \( \kappa \) is inaccessible in \( N[G_C \cap C_X] = \)
$M[G_p \cap P_y][G_C \cap C_X]$. Hence $\mathcal{P}$ has fewer than $\kappa$ cofinal branches in $N[G_C \cap C_X]$. In $N[G_C \cap C_X]$, there is a canonical isomorphism $C^X \cong [C(\lambda)]^N[G_C \cap C_X]$. Hence, by lemma 8 applied to $N[G_C \cap C_X]$, $\mathcal{T}$ has no extra cofinal branches in $N[G_C] = N[G_C \cap C_X][G_C \cap C_X]$. But by lemma 3 again, $M[G] = M[G_p][G_C] = M[G_p \cap P_y][G_p \cap P_y][G_C] = N[G_C][G_p \cap P_y]$ and $G_p \cap P_y$ is $N[G_C]$-generic for $P_y$. So, applying lemma 9 to $N$ and the posets $C, P_y$, we see that $\mathcal{T}$ has no extra cofinal branches in $M[G]$. Hence $\mathcal{T}$ is not Kurepa in $M[G]$. Q.E.D.

References


