Order-Types, Trees, and a Problem of Erdős and Hajnal

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§0. Introduction

We work in ZFC set theory throughout, and use the usual notation and conventions. In particular, an ordinal is the set of all its predecessors and a cardinal is an initial ordinal. The cardinality of a set $X$ is denoted by $|X|$. As usual, if $\sigma$, $\rho$ are order types, $\sigma \leq \rho$ denotes that there is an order preserving monomorphism $F: \sigma \to \rho$. (We use $\sigma$, $\rho$ to denote order types, all other lower case Greek letters denoting ordinals, with $\kappa$, $\lambda$, in particular, for cardinals). We assume some familiarity with the notion of constructibility, as in, say, [1]. In particular, we assume the reader knows the usual proof that $V = L \Rightarrow \text{GCH}$ (see [1].) We use $L_\alpha, \alpha \in \Omega$, to denote the levels in the constructible hierarchy.

Let $\dagger$ denote the following proposition: Whenever $\rho$ is an order type of cardinality $\omega_2$ such that $\omega_2 \leq \rho$ and $\omega^*_2 \leq \rho$ there is $\sigma \subset \rho$ of cardinality $\omega_1$ such that $\omega_1 \leq \sigma$ and $\omega^*_1 \leq \sigma$. Without assuming GCH, the status of $\dagger$ is of little interest, of course, since a set of reals of cardinality $\omega_2$ embeds neither $\omega_2$, $\omega^*_2$ nor $\omega_1$, $\omega^*_1$. In [2] (§ 7, Problem I), Erdős and Hajnal ask what happens to $\dagger$ if we do assume GCH. In this paper, we prove that if $V = L$, then $\Vdash \dagger$. The proof uses Jensen's answer to an earlier question of ours concerning subtrees of $\omega_1$-trees, and we end the paper with a brief discussion of this topic, and a remark concerning the consistency of $\dagger$. 
A tree is a poset $\mathcal{T} = (T, \preceq)$ such that for any $x \in T$, 
$\{y \in T | y \prec_T x\}$ is well ordered by $\preceq_T$. (For such $\mathcal{T}$, $x$, we call the order type of this set (under $\prec_T$) the height of $x$ in $\mathcal{T}$.) For any ordinal $\alpha$, we set $T_{\alpha} = \{x \in T | x$ has height $\alpha$ in $\mathcal{T}\}$, the $\alpha$'th level of $\mathcal{T}$. We write $\mathcal{T}|\alpha$ for $\langle \cup_{\beta<\alpha} T_{\beta}, \preceq_T \cap (\cup_{\beta<\alpha} T_{\beta}) \rangle$.

A branch of $\mathcal{T}$ is a maximal linearly ordered subset of $T$. If it has order type $\alpha$ it is an $\alpha$-branch.

Let $\theta$ be an ordinal, $\kappa$ a cardinal. $\mathcal{T}$ is a $(\theta, \kappa)$-tree iff $(\forall \alpha < \theta) (T_{\alpha} \neq \emptyset \land |T_{\alpha}| < \kappa) \land T_{\theta} = \emptyset$. $\mathcal{T}$ is normal iff whenever $\alpha < \beta < \theta$ and $x \in T_{\alpha}$ there are distinct $y, y' \in T_{\beta}$ with $x <_{T} y, y'$. A tree $\mathcal{T}$ is $\kappa$-Aronszajn iff it is a $(\kappa, \kappa)$-tree with no $\kappa$-branches. $\mathcal{T}$ is $\kappa$-Kurepa iff it is a $(\kappa, \kappa)$-tree with at least $\kappa^{+}$ $\kappa$-branches. In both cases, we do not bother to mention the $\kappa$ if $\kappa = \omega_{1}$.

(Solovay has proved that if $V = L$, there is a Kurepa tree. For our purposes, a somewhat stronger result is required.

Let us begin by proving three fundamental lemmas on constructibility.

Lemma 1

Assume $V = L$. If $M \ll L_{\omega_{1}}$, then $M = L_{\alpha}$ for some $\alpha \leq \omega_{1}$.

Proof: By absoluteness considerations, it suffices to show that $M$ is transitive. Let $x \in L_{\omega_{1}}$. Then $x$ is countable.

Now, $L_{\omega_{1}}$ has an $L_{\omega_{1}}$-definable well ordering, $<_L$. Let $J_{x}$ be the $<_L$-least bijection $J_{x}: \omega \leftrightarrow x$. Then $J_{x} \in L_{\omega_{1}}$ and is $L_{\omega_{1}}$-definable from $x$. Hence, $x \in M \iff J_{x} \in M$.

Thus $x \in M \iff J_{x}: \omega \in M \iff x \in M$, as required.
Lemma 2

Assume $V = L$. If $M \prec L_{\omega_2}$, then $M \cap L_{\omega_1} = L_\alpha$ for some $\alpha \leq \omega_1$.

Proof: $L_{\omega_1}$ is $L_{\omega_2}$-definable, so clearly $M \cap L_{\omega_1} \subseteq L_{\omega_1}$. By lemma 1, $M \cap L_{\omega_1} = L_\alpha$ for some $\alpha \leq \omega_1$.

Lemma 3

Let $\beta > \alpha > \omega$, $p \in L_\beta$. Suppose that $L_{\beta+1} \models "\alpha \text{ is a regular uncountable cardinal}"$ and that $L_\beta$ is the smallest $X \prec L_\beta$ such that $p \in X$ and $X \cap \alpha$ is transitive. Then there is an $L_{\beta+1}$-definable map of $\omega$ cofinally into $\alpha$.

Proof: Let us use the notation $X \prec_{\Sigma_n} L_\beta$ to mean that $X$ is an elementary substructure of $(L_\beta, \in)$ when we restrict our attention to the $\Sigma_n$ formulas of set theory only. For each $n \in \omega$, let $X_n$ be the smallest $X \prec_{\Sigma_n} L_\beta$ such that $p \in X$ and $X \cap \alpha$ is transitive. Thus $X_n$ is $L_\beta$-definable. But $L_{\beta+1} \models "\alpha \text{ is a regular uncountable cardinal}"$, so clearly $X_n \cap \alpha \in \alpha$. Let $\alpha_n = X_n \cap \alpha$. Since $\langle X_n | n < \omega \rangle$ is $L_{\beta+1}$-definable, so is $\langle \alpha_n | n < \omega \rangle$. But $\bigcup_{n<\omega} X_n = L_\beta$ by assumption on $L_\beta$, so $\sup_{n<\omega} \alpha_n = \alpha$, and we are done.

The following theorem was proved by Jensen in answer to an old question of ours.

Theorem 4 (Jensen)

Assume $V = L$. Then there is a Kurepa tree no subset of which is (under the inherited ordering) an Aronszajn tree.

Proof: Define a function $h: \omega_1 \to \omega_1$ by setting $h(\alpha) =$ the least $\gamma$ such that $L_{\gamma+1} \models "\alpha \text{ is countable}"$. We define an $(\omega_1, \omega_1)$-tree $T$ by induction on the levels. The elements of $T$ will be countable ordinals, and we have $\alpha \prec_T \beta \rightarrow \alpha < \beta$. 

Each $T \upharpoonright \alpha$ will be a normal $(\alpha, \omega_1)$-tree.

As we proceed, we simultaneously define a function $f: \omega_1 \to \omega_1$ by setting $f(\alpha) =$ the least $\gamma$ such that $\alpha, T \upharpoonright \alpha \in L \uparrow \gamma \subseteq L_{\omega_1}$ (by lemma 1).

Set $T_0 = \{0\}$. If $T_\alpha$ is defined, $T_{\alpha+1}$ is the result of appointing (in a minimal way) two new ordinals to succeed each member of $T_\alpha$. We are left with the definition of $T_\alpha$ when $\lim(\alpha)$ and $T \upharpoonright \alpha$ is defined, and is a normal $(\alpha, \omega_1)$-tree.

Let $S(\alpha)$ be the set of all $U \subseteq T \upharpoonright \alpha$ such that:

(i) $U \subseteq \bigcup \{h(\alpha) : \gamma < \alpha\}$;

(ii) $U$ is a normal $(\alpha, \omega_1)$-tree (under the inherited ordering);

(iii) $U$ is thin in $T \upharpoonright \alpha$ (i.e. for any $x \in U$ there is a $y \in T \upharpoonright \alpha - U$ such that $x < T y$);

(iv) $L_\alpha(\alpha) \models "U$ is an Aronszajn tree".

We let $T_\alpha$ consist of (minimally appointed ordinals as) one point extensions of each $\alpha$-branch $b$ of $T \upharpoonright \alpha$ such that $b \in L_f(\alpha)$ and $b \not\in U$ for any $U \in S(\alpha)$. By condition (iii) above, it is clear that (since $L_f(\alpha) \models "\alpha$ and $S(\alpha)$ are countable") $T \upharpoonright \alpha + 1$ is a normal $(\alpha + 1, \omega_1)$-tree.

Set $T = \bigcup_{\alpha < \omega_1} T \upharpoonright \alpha$, a normal $(\omega_1, \omega_1)$-tree.

We prove that $T$ is Kurepa. Suppose not. Then we may let $\langle c_\alpha \mid \alpha < \omega_1 \rangle$ be the $\langle \cdot \rangle$-least enumeration of all the $\omega_1$-branches of $T$. $T$ is clearly $L_{\omega_2}$-definable, and hence so is $\langle c_\alpha \mid \alpha < \omega_1 \rangle$. Using this fact, we define an $\omega_1$-branch $b$ of $T$ such that $b \neq c_\alpha$ for all $\alpha < \omega_1$, giving the required result.

Define a chain of submodels $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_\gamma \subseteq \bigcup L_{\omega_2}$ ($\gamma < \omega_1$) as follows:
$X_0$ = the smallest $X < L_{\omega_2}$;

$X_{\nu+1}$ = the smallest $X < L_{\omega_2}$ such that $X_\nu \cup \{X_\nu\} \subseteq X$;

$X_\gamma = \bigcup_{\nu < \gamma} X_\nu$ if $\lim(\gamma)$.

For each $\nu < \omega_1$, let $\pi_\nu : X_\nu \sim L_\psi(\nu)$. By lemma 2, $\pi_\nu(\omega_1) = \omega_1 \cap X_\nu = \alpha_\nu$, say. (Note that $\langle \alpha_\nu | \nu < \omega_1 \rangle$ is a continuous sequence). Hence $\pi_\nu(\mathcal{T}) = \mathcal{T} \cap \alpha_\nu$ and $\pi_\nu(\langle c_\alpha | \alpha < \omega_1 \rangle) = \langle c_\alpha \cap \alpha_\nu | \alpha_\nu \rangle$.

Also, $L_f(\alpha_\nu) = "\alpha_\nu is countable"$ whereas $L_\psi(\nu) = "\alpha_\nu = \omega_1", so $\psi(\nu) < f(\alpha_\nu)$ for each $\nu < \omega_1$. It follows that $\langle c_\alpha \cap \alpha_\nu | \alpha < \alpha_\nu \rangle \in L_f(\alpha_\nu)$.

We define, by induction, a sequence $b_0, \ldots, b_\nu, \ldots (\nu < \omega_1)$ such that, for each $\nu < \omega_1$, $b_\nu$ is an $\alpha_\nu$-branch of $\mathcal{T} \cap \alpha_\nu$, with an extension on level $\alpha_\nu$ of $\mathcal{T}$, such that $\alpha < \alpha_\nu \rightarrow b_\nu \neq c_\alpha \cap \alpha_\nu$.

We shall then just set $b = \bigcup_{\nu < \omega_1} b_\nu$ and be done, of course.

To define $b_0$, observe that $\langle c_\alpha \cap \alpha_0 | \alpha < \alpha_0 \rangle, S(\alpha_0), \alpha_0,$ $\mathcal{T} \cap \alpha_0 \in L_f(\alpha_0)$ and $L_f(\alpha_0) = "\alpha_0, S(\alpha_0), \mathcal{T} \cap \alpha_0 are countable"$. Hence, working inside $L_f(\alpha_0)$, we may let $b_0$ be the $<_L$-least $\alpha_0$-branch of $\mathcal{T} \cap \alpha_0$ such that $b_0 \notin U$ for any $U \in S(\alpha_0)$ and $\alpha < \alpha_0 \rightarrow b_0 \neq c_\alpha \cap \alpha_0$. Then, since $S(\alpha_0) \subseteq L_f(\alpha_0)$, $b_0$ has an extension on $\mathcal{T} \cap \alpha_0$.

If $b_0, \ldots, b_\nu$ are already (suitably) defined, $b_{\nu+1}$ is defined similarly (to contain the $\mathcal{T} \cap \gamma$-extension of $b_\nu$, of course).

Finally, suppose $\lim(\nu)$ and that $b_0, b_1, \ldots, b_\xi, \ldots (\xi < \nu)$ are suitably defined. Let $b_\nu = \bigcup_{\xi < \nu} b_\xi$. Since $\langle \alpha_\nu | \nu < \omega_1 \rangle$ is continuous, $b_\nu$ is an $\alpha_\nu$-branch of $\mathcal{T} \cap \alpha_\nu$ and $\alpha < \alpha_\nu \rightarrow b_\nu \neq c_\alpha \cap \alpha_\nu$.

We must show that $b_\nu$ extends on $\mathcal{T} \cap \alpha_\nu$. We show first that $b_\nu \in L_f(\alpha_\nu)$.

Clearly, $b_\nu$ is definable from $\mathcal{T} \cap \alpha_\nu$, $\langle \alpha_\xi | \xi < \nu \rangle$, $\langle S(\alpha_\xi) | \xi < \nu \rangle$, $\langle c_\alpha \cap \alpha_\nu | \alpha < \alpha_\nu \rangle$, so it reduces to proving that $\langle \alpha_\xi | \xi < \nu \rangle \in L_f(\alpha_\nu)$.

And since $\alpha_\xi = \omega_1 L_\psi(\xi)$ for each $\xi$, this will be so providing
\[ \langle L_\psi(\xi) \mid \xi < \nu \rangle \in L_f(\alpha_\nu) . \] Now, \( \psi(\nu) < f(\alpha_\nu) \), so \( L_\psi(\nu) \in L_f(\alpha_\nu) \).

Thus, working inside \( L_f(\alpha_\nu) \), we may define a chain
\[ Y_0 \prec Y_1 \prec \ldots \prec Y_\xi \prec \ldots \prec L_\psi(\nu) \] (\( \xi < \nu \)) exactly as \( \langle X_\xi \mid \xi < \omega_1 \rangle \) was defined from \( L_{\omega_2} \). But look, in defining \( \langle X_\xi \mid \xi < \nu \rangle \), we could equally well have used \( X_\nu \) in place of \( L_{\omega_2} \) (since \( \xi < \nu \rightarrow X_\xi \prec X_\nu \prec L_{\omega_2} \)). Then, since \( X_\nu \cong L_\psi(\nu) \), an easy induction argument shows that \( X_\xi \cong Y_\xi \) for all \( \xi < \nu \). Hence \( Y_\xi \cong L_\psi(\xi) \), \( \xi < \nu \). But \( \langle Y_\xi \mid \xi < \nu \rangle \in L_f(\alpha_\nu) \), so \( \langle L_\psi(\xi) \mid \xi < \nu \rangle \in L_f(\alpha_\nu) \), as required.

Thus, \( b_\nu \) will be proved to extend on level \( \alpha_\nu \) if we can show that for any \( U \in S(\alpha_\nu) \), \( b_\nu \notin U \). To this end, note first that our above argument did more than prove that
\[ b_\nu \in L_f(\alpha_\nu) \]. What we actually proved was that \( b_\nu \) is \( L_\psi(\nu)+1 \)-definable (from elements of \( L_\psi(\nu)+1 \) ). Now, \( \alpha_\nu \) is uncountable in \( L_\psi(\nu)+1 \) but countable in \( L_{h(\alpha_\nu)+1} \).
(For the former, note that \( \pi^{-1}_\nu : L_\psi(\nu) \not\leq L_{\omega_2} \) and \( \pi^{-1}(\alpha_\nu) = \omega_1 \)). Hence \( \psi(\nu) < h(\alpha_\nu) \).

**Case A:** \( \psi(\nu) + 1 < h(\alpha_\nu) \).

Then \( b_\nu \in L_h(\alpha_\nu) \). So as \( L_h(\alpha_\nu) \vdash "U \ is \ Aronszajn" \) for any \( U \in S(\alpha_\nu) \), we cannot have \( b_\nu \in U \) for any such \( U \).

**Case B:** \( \psi(\nu) + 1 = h(\alpha_\nu) \).

Thus \( S(\alpha_\nu) \subset L_\psi(\nu) \). Let \( U \in S(\alpha_\nu) \). For \( \iota < \tau \leq \nu \), set \( \pi_{\iota \tau} = \pi_{\tau \cap(\iota+1)} \). Then \( \langle L_\psi(\nu) , \langle \pi_{\tau \nu} \rangle_{\tau < \nu} \rangle \) is (isomorphic to) the direct limit of the elementary system \( \langle (L_\psi(\tau))_{\tau < \nu} , \langle \pi_{\tau \nu} \rangle_{\tau < \nu} \rangle \). So for some \( \tau < \nu \) and some \( U' \in L_\psi(\tau) \), \( U = \pi_{\tau \nu}(U') \). Let \( \tau \) be the least such. It is then easily observed that \( L_\psi(\tau) \) is the smallest \( X \not\leq L_\psi(\tau) \) such that \( U' \in L_\psi(\tau) \) and \( X \cap \alpha_\tau \) is transitive. Also, \( L_\psi(\tau) + 1 \models "\alpha_\tau \ is \ a \ regular \ uncountable \ cardinal" \). (By applying \( \pi^{-1}_\tau \).)
Thus by lemma 3, it follows that \( \alpha_\tau \) is countable in \( L_\psi(\tau) + 2 \). Hence \( \psi(\tau) + 1 = h(\alpha_\tau) \). But look, there is no \( L_\psi(\nu) \)-definable \( \alpha_\nu \)-branch of \( U \). (By applying \( \pi_\nu^{-1} \)). Hence there can be no \( L_\psi(\tau) \)-definable \( \alpha_\tau \)-branch of \( U' \). Thus \( L_h(\alpha_\tau) |= "U' is Aronszajn" \). It follows readily that \( U' \in S(\alpha_\tau) \). Hence \( b_\tau \not\in U' \).

But \( U' = U \cap \alpha_\tau \) and \( b_\tau = b_\nu \cap \alpha_\tau \). Hence \( b_\nu \not\in U \).

This completes the proof that \( T \) is Kurepa.

Now suppose that \( T \) has an Aronszajn subtree, \( U \). It is easily seen that there is no loss of generality in assuming that \( U \) is normal and thin in \( T \). We may also assume that \( U \) is the \( \prec_L \)-least such. Hence \( U \) is \( L_{\omega_2} \)-definable. So \( U \in X_\alpha \). Clearly, \( U' = \pi_\alpha(U) = U \cap T|\alpha_\alpha \). As \( U \) is Aronszajn, there is no \( L_{\omega_2} \)-definable \( \omega_1 \)-branch of \( U \). Thus \( L_\psi(\alpha_0) + 1 |= "U' is Aronszajn" \).

But, using lemma 3, it is immediately clear that \( \psi(\alpha_0) + 1 = h(\alpha_\alpha) \).

Hence we see that \( U' \in S(\alpha_\alpha) \). But by construction, no \( \alpha_\alpha \)-branch of \( U' \) ever extended on \( T_{\alpha_\alpha} \), so how ever can \( U \supset U' \) be cofinal in \( \omega_1 \)? This contradiction completes the proof.

The above theorem, together with our next result, shows that \( V = L \rightarrow \exists \bar{\alpha} \).

**Theorem 5**

Assume GCH. Suppose that there is a Kurepa tree \( T \), no subset of which is an Aronszajn tree. Then \( V \).

**Proof:** Without loss of generality, we may assume that \( T \subset \omega_\alpha \omega_1^{\omega_1} \) and that \( s \leq_T t \) iff \( s \) is an initial segment of \( t \) (written \( s \in_{\text{inl}} t \)).

Let \( \rho \) be the set of all \( \omega_1 \)-branches of \( T \), ordered lexicographically. Thus \( |\rho| = \omega_2 \). We show that \( \omega_2, \omega_2 \not\leq \rho \).
Let \( D = \{ s \in \cup_{\alpha < \omega_1} 2^\alpha : (\exists b \in \rho)(s \text{ inl } b) \} \). Then \( |D| = \omega_1 \) by GCH.

Suppose \( \omega_2 \leq \rho \), and let \( \langle b_\nu | \nu < \omega_2 \rangle \) be an \( \omega_2 \)-sequence from \( \rho \). Let \( A = \{ \nu \in \omega_2 | \lim(\nu) \} \). By induction on \( \nu \in A \), pick \( s_\nu \in D \) such that \( \tau \in A \cap \nu \rightarrow s_\tau \neq s_\nu \) (By demanding that \( s_\nu \text{ inl } b_\nu \) but \( \not\exists s_\nu \text{ inl } b_{\nu+1} \) for each \( \nu \in A \)). Then \( \{ s_\nu | \nu \in A \} \) is a set of \( \omega_2 \) distinct members of \( D \), which is absurd. Similarly if \( \omega_2^* \leq \rho \).

Now let \( \sigma \subset \rho \), \( |\sigma| = \omega_1 \). We show that \( \omega_1 \leq \sigma \) or \( \omega_2^* \leq \sigma \). Suppose \( \rho = \langle \rho, \_ \rangle \).

Let \( U = \{ s \in \cup_{\alpha < \omega_1} 2^\alpha : (\exists b \in \sigma)(s \text{ inl } b) \} \). Thus \( U \subset T \).

**Case 1:** For some \( b \in \sigma \) it is the case that whenever \( s \text{ inl } b \) there is \( b' \in \sigma \) such that \( s \text{ inl } b' \) and \( b' \neq b \). By induction, pick a strictly \( \prec \) -increasing sequence \( \langle s_\nu | \nu < \omega_1 \rangle \) of initial sections of \( b \), and a pairwise distinct sequence \( \langle b_\nu | \nu < \omega_1 \rangle \) of members of \( \sigma - \{ b \} \) such that \( s_\nu \text{ inl } b_\nu \).

Then either \( \{ b_\nu | \nu < \omega_1 \text{ & } b_\nu \rightarrow b \} \) or else \( \{ b_\nu | \nu < \omega_1 \text{ & } b \rightarrow b_\nu \} \) has cardinality \( \omega_1 \). But \( \langle b_\nu | \nu < \omega_1 \text{ & } b_\nu \rightarrow b \rangle \) is a \( \rightarrow \) -increasing sequence from \( \sigma \) and \( \langle b_\nu | \nu < \omega_1 \text{ & } b \rightarrow b_\nu \rangle \) is a \( \rightarrow \) -decreasing sequence from \( \sigma \).

**Case 2:** Otherwise.

Then, for each \( b \in \sigma \) there is \( s_b \in U \) such that \( s_b \text{ inl } b \) and for all \( b' \in \sigma \), \( s_b \text{ inl } b' \rightarrow b' = b \).

Let \( U' = \{ s | (\exists b \in \sigma)(s \text{ inl } s_b) \} \). Then \( U' \subset U \). We know that \( U' \) cannot be an Aronszajn tree. And yet \( U' \) is an \( (\omega_1, \omega_1) \)-tree. Hence \( U' \) has an \( \omega_1 \)-branch, \( d \). For \( b_1, b_2 \in \sigma \), \( s_{b_1} \text{ and } s_{b_2} \) must be \( \prec \) -incomparable. Hence for each \( s \text{ inl } d \) there must be a \( b \in \sigma \) such that \( s \text{ inl } s_b \).
and \( \text{lsb inl d} \). So, for each \( s \text{ inl d} \) there is \( b \in \sigma \) such that \( \text{lsb inl b} \) and \( b \neq d \). So, as in Case 1, \( \omega_1 \leq \sigma \) or \( \omega_1^* \leq \sigma \).

§3. Subtrees of \( \omega_1 \)-trees.

We saw above that it is consistent that there is a Kurepa tree with no Aronszajn subtree. Since the existence of Kurepa trees is not provable in \( \text{ZFC} \) (see [4]), we could not hope to eliminate the use of \( V = L \) in establishing that result. However, in \( \text{ZFC} \), it is possible to construct a normal \( (\omega_1, \omega_1) \)-tree with no Aronszajn subtree. In fact, we have:

**Theorem 6.**

there is a normal \( (\omega_1, \omega_1) \)-tree \( T \) such that:

(i) \( T \) has no Aronszajn subtree.

(ii) if \( T' \) is any normal \( (\omega_1, \omega_1) \)-tree then either \( T' \) has an Aronszajn subtree on else \( T \leq T' \).

**Proof:** Let \( T = \{ s \in 2^{\omega_1} \mid |\{ \alpha \in \omega_1 \mid s(\alpha) = 1 \}| < \omega \} \), and make \( T \) into a tree by setting \( s \leq t \) iff \( s \subseteq t \). Clearly, \( T \) is a normal \( (\omega_1, \omega_1) \)-tree such that every point in \( T \) lies on an \( \omega_1 \)-branch of \( T \).

(i) Suppose \( U \subseteq T \), \( U \) a normal \( (\omega_1, \omega_1) \) tree. We show that \( U \) is not an Aronszajn tree. It is easily seen that we may assume that \( U \) is an initial segment of \( T \). (If the initialisation of \( U \) in \( T \) has an \( \omega_1 \)-branch, so must \( U \) itself !)

Set \( C = \{ \alpha \in \omega_1 \mid \lim (\alpha) \} \). For each \( \alpha \in C \), let \( s_\alpha \in U \) be arbitrary. For each \( \alpha \in C \), define \( f(\alpha) = \) the largest \( \beta < \alpha \) such that \( s_\alpha(\beta) = 1 \), or else \( f(\alpha) = 0 \) if no such \( \beta \) exists. Then \( f: C \rightarrow \omega_1 \) is regressive, so by a well known
theorem of Fodor (see [1], Chapter 3, for example) we can find stationary set $X \subseteq C$ such that $f''X = \{\alpha_0\}$ for some fixed $\alpha_0$. It follows immediately that there must be an uncountable set $Y \subseteq X$ such that $\alpha, \beta \in Y \& \alpha < \beta \rightarrow s_\alpha \leq_T s_\beta$. Hence $\{|s_\alpha| \alpha \in Y\}$ determines an $\omega_1$-branch of $U$.

(ii) Let $\mathcal{T}$ be a normal $(\omega_1, \omega_1)$-tree with no Aronszajn subtree. By replacing $\mathcal{T}$ by a subtree if necessary, we assume $\mathcal{T}'$ is such that every point in $\mathcal{T}'$ has exactly two distinct immediate successors. But look, as $\mathcal{T}'$ has no Aronszajn subtree, every point of $\mathcal{T}'$ lies on an $\omega_1$-branch of $\mathcal{T}'$. It is now an easy matter to inductively (on the levels) embed $\mathcal{T}$ into $\mathcal{T}'$.

Let us return now to the property $\xi$, and show how the failure of $\xi$ is closely connected with the existence of Aronszajn subtrees of trees. Let $\Delta$ denote the following proposition, often referred to as Chang's Conjecture: If $\mathcal{L} = \langle w_2, w_1, \ldots \rangle$ is an arbitrary first-order structure with a countable language, there is $\mathcal{L} = \langle B, B \cap w_1, \ldots \rangle \subseteq \mathcal{A}$ such that $|B| = w_1$ and $|B \cap w_1| \leq w$.

Silver [3] has shown that $\text{Con}(\text{ZFC} = \text{"there is a Ramsey cardinal"}) \rightarrow \text{Can}(\text{ZFC} + \text{GCH} + \Delta)$. (And since $\Delta$ implies the existence of Solovay's $\mathcal{O}^\#$, the large cardinal assumption here probably cannot be weakened very much, if at all.)

**Theorem 7.**

Assume $\Delta + \text{GCH}$. If $\xi$ fails, then there is an $\omega_2$-Aronszajn tree with no Aronszajn subtree.

**Proof:** Let $\rho$ be an order type of cardinality $\omega_2$ such that $\omega_2, \omega_2^\prec \not\leq \rho$, but for any $\sigma \leq \rho$ of cardinality $\omega_1$, either $\omega_1 \leq \sigma$ or else $\omega_1^\prec \leq \sigma$. Assume $\rho = \langle \omega_2, \mathcal{Z} \rangle$ for definiteness.
Define, by induction on the levels, a tree, \( T \), as follows:

The elements of \( T \) are non-empty intervals of \( \rho \) and the ordering, \( \prec_T \), is \( \supset \). Set \( T_0 = \{ \rho \} \). If \( I \in T_\alpha \) and \( |I| = 1 \), \( I \) has no successors in \( T \). If \( I \in T_\alpha \) and \( |I| > 1 \), let \( \alpha_I \) be the least ordinal in \( I \) not maximal in \( I \) and let \( \{ \xi \in I \mid \xi \prec \alpha_I \} \) and \( \{ \xi \in I \mid \xi \succ \alpha_I \} \) be the successors of \( I \) in \( T_{\alpha + 1} \). Finally, if \( \lim (\delta) \) and \( T_\delta \) is defined, let \( T_\delta = \{ \cap b \mid b \) is a \( \delta \)-branch of \( T_\delta \) and \( \cap b \neq \emptyset \} \).

We linearly order each level \( T_\alpha \) by setting, for each \( I, J \in T_\alpha \), \( I <_\alpha J \) iff \((\forall \xi \in I)(\forall \zeta \in J)(\xi \prec \zeta)\).

Suppose that for some \( \alpha < \omega_2 \), \( |T_\alpha| = \omega_2 \).

Let \( \alpha \) be the least such. Then, clearly, \( \lim (\alpha) \).

Let \( f: T_\alpha + 1 \rightarrow T_\alpha \), and consider the structure

\[ \mathcal{L} = \langle T_\alpha + 1, T_\alpha, T_\alpha, f, <_\alpha, <^* \rangle \], where \( <^* \) is the lexicographic order on \( T_\alpha + 1 \) induced by the \( <_\beta \), \( \beta < \alpha \).

Now, \( |T_\alpha + 1| = \omega_2 \) and \( |T_\alpha| = \omega_1 \) (by choice of \( \alpha \)), so let

\[ \mathcal{L} = \langle X, X \cap T_\alpha, \ldots \rangle \prec \mathcal{L} \text{ with } |X| = \omega_1 \text{ and } |X \cap T_\alpha| = \omega \],

by assumption. Since

(i) \( \mathcal{L} \models (\forall x, y \in T_\alpha)(x <_\alpha y \rightarrow (\exists z \in T_\alpha)(x \leq^* z \leq^* y)) \)

it follows that

(ii) \( \mathcal{L} \models (\forall x, y \in T_\alpha \cap X)(x <_\alpha y \rightarrow (\exists z \in T_\alpha \cap X)(x \leq^* z \leq^* y)) \)

For each \( I \in T_\alpha \cap X \), let \( \theta(I) \in I \) be arbitrary.

Then, for \( I, J \in T_\alpha \cap X \), \( I <_\alpha J \) iff \( \theta(I) \prec \theta(J) \).

Set \( \sigma = \{ \theta(I) \mid I \in T_\alpha \cap X \} \). Thus \( \sigma \subset \rho \). And since

\[ f: X \rightarrow T_\alpha \cap X, |\sigma| = \omega_1 \]. We show that \( \omega_1, \omega_1^* \not\prec \sigma \), a contradiction.

Suppose \( \omega_1 \leq \sigma \). (The case \( \omega_1^* \leq \sigma \) is similar.) It follows that

\( \omega_1 \leq \langle T_\alpha \cap X, <_\alpha \cap X^2 \rangle \), so let \( \langle I_\nu \mid \nu < \omega_1 \rangle \) be a \( <_\alpha \)-increasing sequence from \( T_\alpha \cap X \). By (ii), we can inductively pick mimbers \( I_\nu \) from \( T_\alpha \cap X \), for \( \nu = \nu_1 < \omega_1 \), so that \( \lim (\nu_1) \) & \( \lim (\nu_2) \) & \( \nu_1 < \nu_2 \rightarrow I_\nu_1 \leq^* I_\nu_1 <^* I_\nu_2 \). But \( |T_\alpha \cap X| = \omega \), so we have a
contradiction.

Hence, for all $\alpha < \omega_2$, $|T_\alpha| \leq \omega_1$.

Suppose $T$ has an $\omega_2$-branch $b = \langle I_\alpha | \alpha < \omega_2 \rangle$. For each $\alpha < \omega_2$ there is $J_{\alpha+1} \in T_{\alpha+1}$ such that $I_\alpha \not<_{T} J_{\alpha+1}$ and $J_{\alpha+1} \not= I_{\alpha+1}$. Let $\theta_{\alpha+1} \in J_{\alpha+1}$ for each $\alpha$. Either

$|\{ \theta_{\alpha+1} | \alpha < \omega_2 \& I_{\alpha+1} <_{\alpha+1} J_{\alpha+1} \}| = \omega_2$ or else

$|\{ \theta_{\alpha+1} | \alpha < \omega_2 \& J_{\alpha+1} <_{\alpha+1} I_{\alpha+1} \}| = \omega_2$. In the first case, the requisite $\theta_{\alpha+1}$'s form a $\prec$-decreasing chain of type $\omega_2$, in the second case it is $\prec$-increasing. So, in either event, we have a contradiction, since $\omega_2, \omega_2^* \not\preceq \rho$. Hence $T$ is an $\omega_2$-Aronszajn tree.

Clearly, any Aronszajn subtree of $T$ will likewise correspond to a subtype $\sigma \subseteq \rho$, $|\sigma| = \omega_1$, such that $\omega_1, \omega_1^* \not\preceq \sigma$. Thus $T$ cannot have an Aronszajn subtree, and we are done. \[ \square \]

Remark. By a simple generalisation of the proof of Theorem 6, one can construct, in ZFC, a normal $(\omega_2, \omega_2)$-tree with no Aronszajn (and no $\omega_2$-Aronszajn) subtree. We do not know if it is possible to construct, in ZFC $+ \text{GCH}$, an $\omega_2$-Aronszajn tree with no Aronszajn subtree. If such were possible, however, we would immediately have a proof of $\neg \Phi$ in ZFC $+ \text{GCH}$, since the lexicographic ordering of such a tree is easily seen to provide a counterexample to $\Phi$. In view of an last result, this would seem to be the only hope of establishing $\neg \Phi$ in ZFC $+ \text{GCH}$. However, by $\Delta$ itself, the more obvious sorts of $\omega_2$-Aronszajn trees which one can construct in ZFC $+ \text{GCH}$ do have Aronszajn subtrees, so this approach does not appear to be very hopeful. Much more likely, in our opinion, is that in Silver's model of $\Delta$ (or perhaps a slight modification of it), every $\omega_2$-Aronszajn tree does have an Aronszajn subtree, whence, by theorem 7, we have at once the consistency of ZFC $+ \text{GCH} + \Phi$. Unfortunately, a proof of this has so far eluded us.
Postscript

Since writing this paper, we obtained a proof of the result
Con (ZFC + "there is a Ramsey cardinal") \rightarrow Con (ZFC + 2^\omega = 2^{\omega_1} = \omega_2 + \check{\gamma})

The proof will appear elsewhere. We still do not know if \check{\gamma} is consistent with GCH.
References


4. J. Silver. The Independence of Kurepa's Conjecture.....