Relativistic Quantum Statistical Mechanics in two-dimensional Space - Time

by

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Abstract

We construct for a boson field in two-dimensional space-time with polynomial or exponential interactions and without cut-offs, the positive temperature state or the Gibbs state at temperature $^1/\beta$. We prove that at positive temperatures i.e. $\beta < \infty$, there is now phase transitions and the thermodynamic limit exists and is unique for all interactions. It turns out that the Schwinger functions for the Gibbs state at temperature $^1/\beta$ is after interchange of space and time equal to the Schwinger functions for the vacuum or temperature zero state for the field in a periodic box of length β, and the lowest eigenvalue for the energy of the field in a peroidic box is simply related to the pressure in the Gibbs state at temperature $^1/\beta$.

1. Introduction.

Although the study of the statistical mechanics for quantum systems has made good progress the last ten years [1], the progress has been best for the descrete systems or the lattice systems. The main dificulty in connections with the continuous systems has been that the group of time automorphisms α_t for the Schrödinger particles is non local. The consequence of this non locallity is that the infinite system of interacting Schrödinger particles do not agree well with the generally accepted picture of a quantum statistical mechanics described in terms of a local C^*-algebra or a C^*-algebra of local operators, on which the time acts as a group α_t of C^*-automorphisms. Hence we get a somewhat discouraging situation, that the only known realistic model of a statistical quentum mechanics, namely the system of interacting Schrödinger particles, does not conform to the highly developed abstract theory of quantum statistical mechanics.

For this very reason the question of studying relativistic particles in stead of Schrödinger particles comes up quite natural, since in any relativistic theory there should be an upper bound for the propagation speed and this would force the group of time automorphisms α_t to be local. And this is the motivation for this paper.

Interacting relativistic particles or interacting quantum fields is by now resonably well understood in the case of two space time dimensions. In the case of weak polynomial interaction and strong exponential interactions in two space time dimensions one also has a very clear picture of what happens with the vacuum in the infinite volume limit, or as we would like to say it here,

one has a very clear picture of the theromodynamic limit in the case of temperature zero. For the weak polynomial interactions this was done by Glimm and Spencer [2], and in the case of exponential interactions by Albeverio and Høegh-Krohn [3]. Hence good candidates for a quantum statistical mechanics of interacting relativistic particles are the polynomial and exponential interactions in two space-time dimensions.

In this paper I study the thermodynaimc limit of the positive temperature Gibbs state for the polynomial and exponential interactions in two space time dimensions.

The methode I use is strongly influenced by recent works by Nelson [4], and may be denoted as Markoff field approach. The Markoff field approach was also a main ingredience in [3] and played also a certain role in [2]. One of the advantages of the Markoff field approach is to make available for quantum fields the menthods of classical statistical mechanics, and this is the way it is used in [3], lending havily on the work of Guerra, Rosen and Simon [5] ; that introduces a framework which describes the Markoff fields as Ising ferromagnetic systems.

The way the Markoff field approach is used here is somewhat different. In this paper we use the Markoff field to transform the problem about the thermodynamic limit for the Gibbs state at temperature $1/\beta$ for the relativistic quantum statistical system into the problem of the uniqueness of the vaccum for the system in a periodic box of length β .

In fact it turns out that for any of the interactions we consider, namely the polynomial and the exponential interaction, the Markoff fields for the Gibbs state at temperature $1/\beta$ is the Markoff field on the cylinder $S_\beta \times R$, where S_β is a circle

of length β , that correspond to the Markoff field for the vacuum in the plane $R \times R$, and this last Markoff field is the limit of the first one as the temperature $1/\beta$ goes to zero.

Using this methods it is proved that the termodynamic limit for the Gibbs state exists for all positive temperatures $1/\beta$ and all interactions i.e. for strong exponential interactions as well as strong polynomial interactions.

We see that this is in strong contrast to the vacuum or temperature zero case for the polynomial interactions, where Glimm and Spencer were only able to prove the existence of the infinite volume limit for weak interactions, and from Dobrushin and Minlos [6] we know by now that this is best possible, in fact for any polynomial interaction in two space-time dimensions they get that the thermodynamic limit is not unique in the temperature zero case for strong enough interactions. The reason for this difference is the above mentioned fact that while for the temperature zero case we have a Markoff field in plane $R \times R$ so that the problem is two dimensional, we have for positive temperature a Markoff field on the cylinder $S_\beta \times R$ so that the problem is essentially one dimensional, and therefore in a sence much simpler.

The Gibbs state at positive temperature $1/\beta$ is of course not invariant under the Lorentz group since it is given in terms of the energy operator. There is however, a Lorentz invariant analogy of the Gibbs state at positive temperature $1/\beta$. But this Lorentz invariant Gibbs state is only to be found in a closed universe, the so called De Sitter universe, and it will lead too far to give the construction of the positive temperature state in the De Sitter universe in this paper. This will be delt with separately in a forthcomming paper.

2. The Gibbs-state for the harmonic oscillator.

Consider the self adjoint operator

$$H_o = -\tfrac{1}{2}\Delta + \tfrac{1}{2}(x, A^2 x) - \tfrac{1}{2}\mathrm{tr}A \tag{2.1}$$

on the Hilbert space $\mathcal{H} = L_2(R^N)$, where $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ and A is a real symmetric $N \times N$ matrix bounded below by a positive constant, $A \geq cI$, $c > 0$, $x \in R^N$ and $(,)$ is the natural inner product in R^N.

Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of A. It is well known that H_o has descrete spectrum consisting of the points of the form

$$\sum_{k=1}^{n} \lambda_{i_k} \tag{2.2}$$

and zero. Hence for a positive β, $e^{-\beta H_o}$ is of trace class and we get

$$\mathrm{tr}\, e^{-\beta H_o} = \sum_{n_1 \geq 0, \ldots, n_N \geq 0} e^{-\beta \sum_{i=1}^{N} n_i \lambda_i},$$

hence

$$\mathrm{tr}\, e^{-\beta H_o} = \prod_{i=1}^{N} (1 - e^{-\beta\lambda_i})^{-1} \tag{2.3}$$

Let $V(x) \geq -b$ be a real measurable function bounded below such that

$$H = H_o + V(x) \tag{2.4}$$

is essentially self adjoint. We say that H is the Hamiltonian for the anharmonic oscillator. From $V \geq -b$ we get $H \geq H_o - b$, which gives us that H has descrete spectrum and together with (2.2) it gives a lower bound for the eigenvalues of H, which is transformed into an upper bound for the eigenvalues of $e^{-\beta H}$.

Hence $e^{-\beta H}$ is of trace class. Therefore we may form the normal state ω_β on the von Neumann algebra $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$, given by

$$\omega_\beta(A) = (\mathrm{tr}\, e^{-\beta H})^{-1}\, \mathrm{tr}(Ae^{-\beta H}) \tag{2.5}$$

for $A \in B(\mathcal{H})$. ω_β is called the Gibbs-state for the anharmonic oscillator.

By the Feynmann-Kac formula we know that the kernel $e^{-\beta H}(x,y)$ of the operator $e^{-\beta H}$ is given by

$$e^{-\beta H}(x,y) = E^\beta_{(x,y)}[e^{-\int_0^\beta U(x(\tau))d\tau}] , \tag{2.6}$$

with $U(x) = \frac{1}{2}(x,A^2x) + V(x)$ and $E^\beta_{(x,y)}$ is the conditional expectation with respect to the Brownian motion in R_N given that $x(o) = x$ and $x(\beta) = y$. So that $E^\beta_{(o,o)}$ is the expectation with respect to the normal distribution indexed by the real Hilbert space h of continuous functions $x(\tau)$ from $[0,\beta]$ into R^N, such that $x(0) = x(\beta) = 0$ and the norm square

$$\int_0^\beta \left(\frac{dx(\tau)}{d\tau}, \frac{dx(\tau)}{d\tau}\right)d\tau \tag{2.7}$$

is finite.

Consider the Hilbert space $L_2([0,\beta];R^N)$ of L_2-integrable functions from $[0,\beta]$ into R^N, and let $k_{i,j}(s,t)$ be the kernel of the inverse operator of the self adjoint operator $-\frac{d^2}{d\tau^2}$ with boundary conditions $x(0) = x(\beta) = 0$ on $L_2([0,\beta];R^N)$. Then

$$k_{ij}(s,t) = k(s,t)\delta_{ij}$$

and

$$k(s,t) = \begin{cases} \frac{1}{\beta}s(\beta-t) & s \leq t \\ \frac{1}{\beta}(\beta-s)t & s \geq t \,. \end{cases} \tag{2.8}$$

The normal distribution indexed by h is the same as the Gaussian prosses with mean zero and covariance function $k_{ij}(s,t)$.

It is well known that the Brownian motion hence also the prosses above has support on the continuous functions from $[0,\beta]$ into R^N.

In terms of the measures introdused above $E^{\beta}_{(x,y)}$ is the expectation with respect to the measure obtained from the normal distribution endexed by h by a transformation on the continuous functions from $[0,\beta]$ into R^N given by

$$x(\tau) \rightarrow x + \frac{\tau}{\beta}(y-x) + x(\tau) \tag{2.9}$$

From (2.6) we now get that the kernel $e^{-\beta H}(x,y)$ is a continuous function of x and y. It is well known in that case that $\operatorname{tr} e^{-\beta H} = \int e^{-\beta H}(x,x)dx$, which together with (2.6) gives

$$\operatorname{tr} e^{-\beta H} = \int E^{\beta}_{(x,x)}[e^{-\int_0^{\beta} U(x(\tau))d\tau}]dx \,. \tag{2.10}$$

By (2.9) $E^{\beta}_{(x,x)}$ is the expectation with respect to the measure on the continuous periodic functions from $[0,\beta]$ into R^N obtained from the normal distribution indexed by h by the transformation $x(\tau) \rightarrow x(\tau) + x$.

Since $U(x) = \frac{1}{2}(x,A^2x) + V(x)$ we have that

$$\begin{aligned} &\int_{R^N} E^{\beta}_{(x,x)}[e^{-\int_0^{\beta} U(x(\tau))d\tau}]dx \\ &= \int_{R^N} E^{\beta}_{(x,x)}[e^{-\frac{1}{2}\int_0^{\beta}(x(\tau),A^2x(\tau))} \cdot e^{-\int_0^{\beta} V(x(\tau))d\tau}]dx \,. \end{aligned} \tag{2.11}$$

On the other hand we easily verify that for any real continuous function F defined on the space of continuous periodic functions from $[0,\beta]$ into R^N

$$\int_{R^N} E^\beta_{(x,x)}[e^{-\frac{1}{2}\int_0^\beta (x(\tau),A^2x(\tau))} F]dx = CE^\beta[F] \qquad (2.12)$$

where E^β is the expectation with respect to the normal distribution indexed by the real Hilbert space g of continuous periodic functions from $[0,\beta]$ into R^N , $x(0) = x(\beta)$, such that the norm square

$$\int_0^\beta [(\frac{dx(\tau)}{d\tau}, \frac{dx(\tau)}{d\tau}) + (x(\tau),A^2x(\tau))]d\tau \qquad (2.13)$$

is finite. C is some positive constant independent of F . By setting $V = 0$ in (2.11) we get that $C = \operatorname{tr} e^{-\beta H_o}$.

We have thus proved the following formula

$$\operatorname{tr} e^{-\beta H} = \operatorname{tr} e^{-\beta H_o} E^\beta[e^{-\int_0^\beta V(x(\tau))d\tau}] \qquad (2.14)$$

where E^β is the expectation with respect to the normal distribution indexed by the real Hilbertspace g of periodic functions from $[0,\beta]$ into R^N with norm square given by (2.13). Now $\operatorname{tr} e^{-\beta H_o}$ is given by (2.3), and since $1-e^{-\beta\lambda_i}$ are the eigenvalues of the real symmetric matrix $1-e^{-\beta A}$ we get from (2.3) that

$$\operatorname{tr} e^{-\beta H_o} = |1-e^{-\beta A}|^{-1} \qquad (2.15)$$

where $|1-e^{-\beta A}|$ is the determinant of the matrix $1-e^{-\beta A}$. Hence (2.14) may be written

$$\operatorname{tr} e^{-\beta H} = |1-e^{-\beta A}|^{-1} E^\beta[e^{-\int_0^\beta V(x(\tau))d\tau}] . \qquad (2.16)$$

Let now $F_i \in B(\mathcal{H})$ $i = 0,\dots,n$ be multiplication operators by bounded continuous functions $F_i(x)$, $i = 0,\dots,n$, and let $0 = s_0 \leq s_1 \dots \leq s_{n-1} \leq s_n = \beta$.

Consider the operator

$$F_0 e^{-s_1 H} F_1 e^{-(s_2-s_1)H} \cdots F_n e^{-(s_n-s_{n-1})H} . \tag{2.17}$$

From (2.6) we have that the kernel of $F_i e^{-(s_{i+1}-s_i)H}$ is given by

$$F_i(x) e^{-(s_{i+1}-s_i)H}(x,y) = E_{(x,y)}^{s_{i+1}-s_i}[F_i(x(0))\, e^{-\int_0^{s_{i+1}-s_i} U(x(\tau))d\tau}]. \tag{2.18}$$

Since the Brownian motion is a homogeneous prosess, (1.18) may be written in the form

$$E_{(x,y)}^{[s_i,s_{i+1}]}[F_i(x(s_i))\, e^{-\int_{s_i}^{s_{i+1}} U(x(\tau))d\tau}] \tag{2.19}$$

where $E_{(x,y)}^{[s_i,s_{i+1}]}$ is the conditional expectation with respect to the Brownian motion given that $x(s_i) = x$ and $x(s_{i+1}) = y$. Utilizing now the Markovian properties of the Brownian motion we get that the kernel of the operator $F_0 e^{-s_1 H} F_1 e^{-(s_2-s_1)H}$ is given by

$$\int dz\, E_{(x,z)}^{[o,s_1]}[F_0(x(0))\, e^{-\int_0^{s_1} U(x(\tau))d\tau}] \cdot [E_{(z,y)}^{[s_1,s_2]}[F_1(x(s_1))\, e^{-\int_{s_1}^{s_2} U(x(\tau))d\tau}] \tag{2.20}$$

$$= E_{(x,y)}^{[o,s_2]}[F_0(x(0))F_1(x(s_1))\, e^{-\int_0^{s_2} U(x(\tau))d\tau}]$$

By induction we get the kernel of the operator (2.17) is given by

$$E_{(x,y)}^{\beta}[\prod_{i=o}^{n-1} F_i(x(s_i))\, e^{-\int_0^{\beta} U(x(\tau))d\tau}] \tag{2.21}$$

By computing the trace of this kernel in the same way as we computed the trace (2.16) of the kernel (2.6), we prove the following theorem.

Theorem 2.1:

Let $F_i \in B(\mathcal{H})$ $i = 0,\dots,n$ be multiplication operators by bounded continuous functions $F_i(x)$, $i = 0,\dots,n$, let $0 = s_o \leq s_1 \dots \leq s_n = \beta$, and let H be the Hamiltonian for the anharmonic oscillator (2.4) then

$$\operatorname{tr}(F_o e^{-s_1 H} F_1 e^{-(s_2-s_1)H} \dots F_{n-1} e^{-(\beta-s_{n-1})H})$$

$$= |1-e^{-\beta A}|^{-1} E^{\beta}[e^{-\int_0^{\beta} V(x(\tau))d\tau} \prod_{i=o}^{n-1} F_i(x(s_i))] \tag{2.22}$$

where $|1-e^{-\beta A}|$ is the determinant of the matrix $1-e^{-\beta A}$ and E^{β} is the expectation with respect to the normal distrubution indexed by the real Hilbert space g of continuous periodic functions from $[0,\beta]$ into R_N , $x(0) = x(\beta)$, with norm square equal to

$$\int_0^{\beta} [(\frac{dx(\tau)}{d\tau}, \frac{dx(\tau)}{d\tau}) + (x(\tau), A^2 x(\tau))] d\tau \ .$$

By a direct calculation one easily verifies the following remark.

Remark: The expectation E^{β} in the theorem above is the expectation with respect to the homogeneous Gaussian prosess on a circle of length β with values in R^N given by the covariance matrix $E^{\beta}(x_i(s)x_j(t))$ equal to the matrix

$$\frac{1}{\beta}A^2 + \frac{2}{\beta}\sum_{n=1}^{\infty}(\frac{4\pi^2}{\beta^2}n^2 + A^2)^{-1}\cos\frac{2\pi}{\beta}n(s-t) \ . \tag{2.23}$$

Summing up this series we get a more explicit expression for the

covariance matrix

$$E^{\beta}(x_i(0)x_j(t)) = (2A(e^{\beta A}-1))^{-1}[e^{(\beta-t)A}+e^{tA}] \tag{2.24}$$

for $0 \leq t \leq \beta$.

Let α_t be the C*-automorphism of $B(\mathcal{H})$ defined by

$$\alpha_t(B) = e^{-itH} B \, e^{itH} \tag{2.25}$$

then

$$\begin{aligned} &\operatorname{tr}(B\alpha_t(C)e^{-\beta H}) \\ &= \operatorname{tr}(Be^{-itH} C \, e^{-(\beta-it)H}) \\ &= \operatorname{tr}(Ce^{-(\beta-it)H} B \, e^{-itH}) \end{aligned} \tag{2.26}$$

is analytic in t in the strip $\beta < \operatorname{Im} t < 0$, with boundary values at real t equal to $\operatorname{tr}(B\alpha_t(C)e^{-\beta H})$ and at $t - i\beta$ equal to $\operatorname{tr}(C\alpha_{-t}(B)e^{-\beta H})$.

Consider now an operator of the form (2.17).

$$\operatorname{tr}(F_o e^{-s_1 H} F_1 e^{-(s_2-s_1)H} \cdots F_{n-1} e^{-(\beta-s_{n-1})H}) \tag{2.27}$$

is obviously analytic in the domain $0 < \operatorname{Re} s_1 < \operatorname{Re} s_2 \ldots < \operatorname{Re} s_{n-1} < \beta$ with boundary values at $\operatorname{Re} s_i = 0$, $i = 0,\ldots,n$ which are continuous and uniformly bounded and given by

$$\operatorname{tr}(F_o \alpha_{t_1}(F_1)\alpha_{t_2}(F_2) \cdots \alpha_{t_{n-1}}(F_{n-1})e^{-\beta H}) \tag{2.28}$$

for $s_k = it_k$, $k = 1,\ldots,n-1$.

The continuity of (2.28) follows from the strong continuity of e^{itH} .

<u>Lemma 2.1</u>: Let $t_i \in R$ and F_i be bounded continuous functions on R^N , then $B(\mathcal{H})$ is the smallest strongly closed linear space of operators that contains all operators of the form

$\alpha_{t_1}(F_1)\cdot\alpha_{t_2}(F_2)\cdots\alpha_{t_n}(F_n)$.

Proof: Since the smallest strongly closed linear space containing the operators above is obviously a strongly closed C^*-algebra of operators, it is enough to prove that if $B \in B(\mathcal{H})$ commute with $\alpha_t(F)$ for all t and all continuous functions F then $B = \lambda I$. Therefore assume $[B,\alpha_t(F)] = 0$ for all t and F . Then $[\alpha_t(B),F] = 0$ for all F and t , hence $\alpha_t(B)$ is a multiplication operator by an L_∞- function for all t . Hence for any real L_∞- function W

$$e^{isW}\alpha_t(B)e^{-isW} = \alpha_t(B) \tag{2.29}$$

so that

$$(e^{i\frac{t}{n}W}e^{-i\frac{t}{n}H})^n\, B(e^{i\frac{t}{n}H}e^{-i\frac{t}{n}W})^n = \alpha_t(B) \tag{2.30}$$

By the Trotter-Kato product formula

$$\text{strong}\lim_{n\to\infty}(e^{i\frac{t}{n}H}e^{-i\frac{t}{n}W})^n = e^{it(H-W)} \tag{2.31}$$

and therefore by (2.30)

$$\alpha_t(B) = e^{-it(H-W)}Be^{it(H-W)} \tag{2.32}$$

By letting $W(x)$ increase to $U(x)$, we get that $(1+H-W)^{-1}$ increase to $(1-\frac{1}{2}\Delta)^{-1}$ so that $(1+H-W)^{-1}$ converge strongly to $(1-\frac{1}{2}\Delta)^{-1}$ and so by the semigroup theorem $e^{it(H-W)}$ converge strongly to $e^{-i\frac{t}{2}\Delta}$. Hence by (2.32) we get that

$$\alpha_t(B) = e^{i\frac{t}{2}\Delta}Be^{-i\frac{t}{2}\Delta} .$$

Since $\alpha_t(B)$ is a multiplication operator for all t we have that B is a multiplication operator. But it is easy to see that if B is not equal to λI , then $e^{i\frac{t}{2}\Delta}Be^{-i\frac{t}{2}\Delta}$ is not a

multiplication operator. This proves the lemma.

Using that ω_β is a normal state we get the following theorem

<u>Theorem 2.2</u> Let B and C be in $B(\mathcal{H})$, then

$$\omega_\beta(B\alpha_t(C)) = \omega_\beta(\alpha_{-t}(B)\cdot C)$$

is analytic in the strip $-\beta < \mathrm{Im}t < 0$, and continuous and uniformly bounded in $\beta \leq \mathrm{Im}t \leq 0$. The boundary values satisfy the KMS condition

$$\omega_\beta(B\alpha_{t-i\beta}(C)) = \omega_\beta(C\alpha_{-t}(B))$$

for real t.

Moreover, any operator B in $B(\mathcal{H})$ may be approximated strongly by linear combinations of operators of the form $\alpha_{t_1}(F_1)\alpha_{t_2}(F_2)\ldots$ $\ldots\alpha_{t_n}(F_n)$, where $F_1,\ldots,F_n$ are multiplication operators by continuous functions $F_1(x),\ldots,F_n(x)$, hence $\omega_\beta(B)$ will also be approximated by the same linear combinations of $\omega_\beta(\alpha_{t_1}(F_1)\alpha_{t_2}(F_2)\ldots\alpha_{t_n}(F_n))$. Furthermore $\omega_\beta(F_o\alpha_{t_1}(F_1)\ldots$ $\ldots\alpha_{t_n}(F_n))$ is analytic in $0 > \mathrm{Im}t_1 > \ldots > \mathrm{Im}t_n > -\beta$ and its value for $t_k = -is_k$ $k = 1,\ldots,n$ with $0 = s_o \leq s_1 \leq \ldots \leq s_n \leq \beta$ is given by

$$\omega_\beta(F_o\alpha_{-is_1}(F_1)\ldots\alpha_{-is_n}(F_n))$$

$$= \left(E^\beta\left[e^{-\int_o^\beta V(x(\tau))d\tau}\right]\right)^{-1} E^\beta\left[\prod_{i=o}^{n} F_i(x(s_i))e^{-\int_o^\beta V(x(\tau))d\tau}\right]$$

where E^β is the expectation given in theorem 2.1.

3. The Gibbs-state for the free scalar quantum field.

Let $\Lambda \subset R^n$ be a bounded domain in R^n with a regular boundary $\partial\Lambda$. Let $A_\Lambda^2 = -\Delta + m^2$ where Δ is the Laplace operator in Λ with some self adjoint boundary conditions on $\partial\Lambda$. If the constant functions satisfies the boundary conditions we shall assume that $m > 0$ if not only that $m \geq 0$, so that in any case, A_Λ^2 is a self adjoint operator on the real Hilbert space $L_2^R(\Lambda)$ and

$$A_\Lambda \geq c\,I \quad , \quad c > 0 \; . \tag{3.1}$$

It is well known that A_Λ has descrete spectrum and that $e^{-\beta A_\Lambda}$ is of trace class for all $\beta > 0$, so that the Fredholm determinant $|1-e^{-\beta A_\Lambda}|$ exists, and by (3.1) it is different from zero.

Let h_Λ be the real Hilbert space $D(A_\Lambda) \subset L_2^R(\Lambda)$ with norm square equal to

$$2(x, A_\Lambda x) \tag{3.2}$$

for $x \in D(A_\Lambda)$ where $(\ ,\ )$ is the inner product in $L_2^R(\Lambda)$. h_Λ depends of course also on the boundary conditions on $\partial\Lambda$. Let now $L_2(h_\Lambda)$ be the complex Hilbert space of L_2 integrable functions with respect to the normal distribution indexed by the real Hilbert space h_Λ.

The Hamiltonian $H_0(\Lambda)$ for the free scalar field in Λ with mass m and the given boundary conditions is a self adjoint operator on $L_2(h_\Lambda)$ which is denoted by

$$H_0(\Lambda) = -\tfrac{1}{2}\Delta_\Lambda + \tfrac{1}{2}(x, A_\Lambda^2 x) - \tfrac{1}{2}\mathrm{tr}A_\Lambda \tag{3.3}$$

where Δ_Λ denotes the Laplace operator on $L_2^R(\Lambda)$ and $(\ ,\ )$ is the inner product on $L_2^R(\Lambda)$. (3.3) is not a definition of $H_0(\Lambda)$

but just a convenient notation. We shall now give the proper definition of $H_o(\Lambda)$.

Let $\{e_k\}_{k=1}^{\infty}$ be the complete orthonormal base in $L_2^R(\Lambda)$ of eigenfunctions for A_Λ

$$A_\Lambda e_k = \lambda_k e_k \ . \tag{3.4}$$

The probability space for the normal distribution dn_{h_Λ} indexed by the real Hilbert space h_Λ is then in a natural way identified with infinite product of the probability spaces for the one dimensional normal distributions

$$dn_{\lambda_k} = (\frac{\lambda_k}{\pi})^{\frac{1}{2}} e^{-\lambda_k x_k^2} dx_k \ , \tag{3.5}$$

so that

$$dn_{h_\lambda} = \bigtimes_{k=1}^{\infty} dn_{\lambda_k} \ . \tag{3.6}$$

Hence $L_2(h_\Lambda)$ may be identified with the infinite tensor product

$$L_2(h_\Lambda) = \bigotimes_{k=1}^{\infty} L_2(dn_{\lambda_k}) \tag{3.7}$$

relative to the vectors $f_k \in L_2(dn_{\lambda_k})$ given by $f_k(x_k) = 1$.
Now $L_2(dn_{\lambda_k})$ may be identified with $L_2(R)$ by the identification

$$g(x_k) \quad \Longleftrightarrow \quad (\frac{\lambda_k}{\pi})^{1/4} e^{-\frac{1}{2}\lambda_k x_k^2} g(x_k) \tag{3.8}$$

for $g \in L_2(dn_{\lambda_k})$. Therefore $L_2(h_\Lambda)$ may be identified with the infinite tensor product

$$L_2(h_\Lambda) = \bigotimes_{k=1}^{\infty} L_2(R) \tag{3.9}$$

relative to the vectors $g_k \in L_2(R)$ given by

$$g_k(x_k) = (\frac{\lambda_k}{\pi})^{1/4} e^{-\frac{1}{2}\lambda_k x_k^2} \ . \tag{3.10}$$

Let now H_{λ_i} be the Hamiltonian for a one dimensional harmonic oscillator given by

$$H_{\lambda_k} = -\tfrac{1}{2}\frac{d^2}{dx_i^2} + \frac{\lambda_k^2}{2}x_k^2 - \frac{\lambda_k}{2} \tag{3.11}$$

as a self adjoint operator on the k-th component in the tensor product (3.9). e^{itH_k} is then a strongly continuous unitary group on the k-th component which leaves the vector g_k invariant. It is then well known that the infinite tensor product $\bigotimes_{k=1}^{\infty} e^{itH_k}$ exists and forms a strongly continuous unitary group on the infinite tensor product (3.9). We now define $H_o(\Lambda)$ as the self adjoint infinitesimal generator of this unitary group on $L_2(h_\Lambda)$.

<u>Definition</u>:

$$e^{itH_o(\Lambda)} = \bigotimes_{k=1}^{\infty} e^{itH_k} \tag{3.12}$$

relative to the tensor decomposition (3.9).

From this definition we get immediately that $e^{-\beta H_o(\Lambda)}$ is of trace class for $\beta > 0$ and that

$$\operatorname{tr} e^{-\beta H_o(\Lambda)} = |1-e^{-\beta A_\Lambda}|^{-1} . \tag{3.13}$$

We now define the Gibbs-state for the free scalar field of mass m in Λ with the given boundary conditions by

$$\omega_\beta^o(\Lambda)(B) = (\operatorname{tr} e^{-\beta H_o(\Lambda)})^{-1}\operatorname{tr}(B\,e^{-\beta H_o(\Lambda)}) \tag{3.14}$$

for any B in the C*-algebra $B(L_2(h_\Lambda))$.

Let F be a bounded continuous function on R^N . From (3.9) we get the following tensor decomposition

$$L_2(h_\Lambda) = L_2(R^N) \otimes [\bigotimes_{k=N+1}^{\infty} L_2(R)] \tag{3.15}$$

where the infinite tensor product here is also relative to the vektors (3.10). F may then be identified with an element $F \otimes 1$ of $B(L_2(h_\Lambda))$ in accordance with the tensor decomposition (3.15). We shall denote this element in $B(L_2(h_\Lambda))$ also by F. By $L_\infty(h_\Lambda)$ we shall understand the maximal abelian algebra in $B(L_2(h))$ containing all bounded continuous functions F on R^N for all values of N. It is obvious that $L_\infty(h_\Lambda)$ is the space of L_∞-functions on the probability space assosiated with the normal distribution indexed by h_Λ.

Let $H_o^N(\Lambda)$ be the infinitesimal generator of the unitary group on $L_2(R^N)$ given by

$$e^{itH_o^N(\Lambda)} = \bigotimes_{k=1}^{N} e^{itH_k}, \tag{3.16}$$

and let $F_o,\dots,F_{n-1}$ be bounded continuous functions on R^N and $0 = s_o \leq \dots \leq s_n = \beta$. It follows then immediately from the definition (3.12) of $H_o(\Lambda)$ that if we consider $F_o,\dots,F_{n-1}$ as elements in $L_\infty(h_\Lambda)$ then

$$\begin{aligned}&\operatorname{tr}(F_o e^{-s_1 H_o(\Lambda)} F_1 e^{-(s_2-s_1)H_o(\Lambda)} \dots F_{n-1} e^{-(\beta-s_{n-1})H_o(\Lambda)}) \\ &= \prod_{k=N+1}^{\infty} (1-e^{-\beta\lambda_k})^{-1} \operatorname{tr}_N(F_o e^{-s_1 H_o^N(\Lambda)} \dots F_{n-1} e^{-(\beta-s_{n-1})H_o^N(\Lambda)})\end{aligned} \tag{3.17}$$

where tr_N is the trace in $L_2(R^N)$. By theorem 2.1

$$\begin{aligned}&\operatorname{tr}_N(F_o e^{-s_1 H_o^N(\Lambda)} \dots F_{n-1} e^{-(\beta-s_{n-1})H_o^N(\Lambda)}) \\ &= \prod_{k=1}^{N} (1-e^{-\beta\lambda_k})^{-1} E_N^\beta[\prod_{i=o}^{n-1} F(x(s_i))]\end{aligned} \tag{3.18}$$

where E_N^β is the expectation with respect to the normal distribution indexed by the real Hilbert space g_N of continuous func-

tions from the circle S_β of length β into R^N with norm square equal to

$$\sum_{k=1}^{N} \int_0^\beta [(\frac{dx_k}{d\tau})^2 + \lambda_k^2 (x_k(\tau))^2] d\tau \; . \tag{3.19}$$

Let $g_\beta(\Lambda)$ be the real Hilbert space of functions from $S_\beta \times \Lambda$ into R such that the norm square

$$\int_0^\beta \int_\Lambda [(\frac{\partial \xi}{\partial t})^2 + \sum_{i=1}^{n} (\frac{\partial \xi}{\partial x_i})^2 + m^2 \xi^2] dx dt \tag{3.20}$$

is finite

and such that for all t , $0 \leq t \leq \beta$, $\xi(x,t)$ satisfies the self adjouint boundary conditions given by A^2 . If we consider $\xi(x,t)$ as a function $\eta(t)$ from S_β into $L_2^R(\Lambda)$, then (3.20) takes the form

$$\int_0^\beta [(\frac{d\eta}{d\tau}, \frac{d\eta}{d\tau}) + (\eta(\tau), A_\Lambda^2 \eta(\tau))] d\tau \tag{3.21}$$

From (3.19) it then follows that g_N is a closed subspace of $g_\beta(\Lambda)$ generated by all functions $\eta(t)$ such that $\eta(t)$ is in the subspace of $L_2^R(\Lambda)$ generated by the N first eigenvectors $e_1, \dots, e_N$ of A , for all t , $0 \leq t \leq \beta$. This together with (3.17) and (3.18) gives then that

$$\begin{aligned} &\operatorname{tr}(F_o e^{-s_1 H_o(\Lambda)} F_1 e^{-(s_2-s_1)H_o(\Lambda)} \dots F_{n-1} e^{-(\beta-s_{n-1})H_o(\Lambda)}) \\ &= |1-e^{-\beta A_\Lambda}|^{-1} \; E_\Lambda^\beta [\prod_{i=o}^{n-1} F_i(\eta(s_i))] \; , \end{aligned} \tag{3.22}$$

where E_Λ^β is the expectation with respect to the normal distribution indexed by the real Hilbert space $g_\beta(\Lambda)$.

Since the bounded continuous functions on R^N are obviously weakly dense in $L_\infty(h_\Lambda)$, we may extend (3.22) to arbitrary

$F_o, \ldots, F_{n-1}$ in $L_\infty(h_\Lambda)$. Utilizing the remark following theorem 2.1 we may also compute the covariance for E_Λ^β . We have thus the following theorem.

<u>Theorem 3.1</u>: Let $F_o, \ldots, F_{m-1}$ be in $L_\infty(h_\Lambda)$, and $0 = s_o \leq s_1 \leq \ldots \leq s_m = \beta$, then

$$tr(F_o e^{-s_1 H_o(\Lambda)} F_1 e^{-(s_2 - s_1)H_o(\Lambda)} \ldots F_{m-1} e^{-(\beta - s_{m-1})H_o(\Lambda)})$$

$$= |1 - e^{-\beta A_\Lambda}|^{-1} E_\Lambda^\beta [\prod_{i=o}^{m-1} F_i(\eta(s_i))]$$

where E^β is the expectation with respect to the normal distribution indexed by the real Hilbert space $g_\beta(\Lambda)$, of functions from the circle S_β of length β into $D(A_\Lambda)$ with norm square equal to

$$\int_o^\beta [(\frac{d\eta}{d\tau}, \frac{d\eta}{d\tau}) + (\eta(\tau), A_\Lambda^2 \eta(\tau))] d\tau ,$$

where $(\, , \,)$ is the inner product in $L_2^R(\Lambda)$. E^β may also be characterized as the Gaussian distribution with mean zero and covariance which is invariant on S_β and given by

$$E_\Lambda^\beta [(\varphi, \eta(0))(\psi, \eta(t))] = (\varphi, (2A(1 - e^{-\beta A_\Lambda})^{-1}(e^{-tA_\Lambda} + e^{-(\beta - t)A_\Lambda})\psi)$$

for $0 \leq t \leq \beta$,

where φ and ψ are in $L_2(\Lambda)$ and $(\, ,)$ is the inner product in $L_2(\Lambda)$.

If we consider the elements in $g_\beta(\Lambda)$as functions $\xi(x,t)$ from $S_\beta \times \Lambda$ into R satisfying the proper boundary conditions on $\partial\Lambda$, then E_Λ^β may be characterized as the expectation with respect to the generalized Gaussian prosess on $S_\beta \times \Lambda$ with covariance function given by

$$E_\Lambda^\beta(\xi(x,s)\xi(y,t)) = G_\Lambda^\beta(x,y,s-t)$$

where $G_\Lambda^\beta(x,y,s-t)$ is the Greensfunction for the self adjoint operator $-\frac{\partial^2}{\partial t^2}-\Delta+m^2$ on $S_\beta \times \Lambda$ with the corresponding self adjoint boundary conditions on $\partial\Lambda$.

We define now the C*-automorphism $\alpha_t^o(\Lambda)$ on $B(L_2(h_\Lambda))$ by

$$\alpha_t^o(\Lambda)(B) = e^{-itH_o(\Lambda)} B e^{itH_o(\Lambda)} . \tag{3.23}$$

Then for B and C in $B(L_2(h_\Lambda))$ we have that

$$\omega_\beta^o(\Lambda)(B\cdot\alpha_t^o(\Lambda)(C)) = \omega_\beta^o(\Lambda)(\alpha_{-t}^o(\Lambda)(B)\cdot C) . \tag{3.24}$$

Moreover (3.24) is an analytic function of t in the strip $-\beta < \mathrm{Im}t < 0$, which is continuous and uniformly bounded in $-\beta \leq \mathrm{Im}t \leq 0$, and the boundary values satisfies the KMS condition

$$\omega_\beta^o(\Lambda)(B\cdot\alpha_{t-i\beta}^o(\Lambda)(C)) = \omega_\beta^o(\Lambda)(C\alpha_{-t}^o(\Lambda)(B)) . \tag{3.25}$$

Futher more, if $F_o,\dots,F_n$ are in $L_\infty(h_\Lambda)$ then

$$\omega_\beta^o(\Lambda)(F_o\alpha_{t_1}^o(\Lambda)(F_1)\dots\ \alpha_{t_m}^o(\Lambda)(F_m))$$

is analytic in $0 > \mathrm{Im}t_1 > \dots > \mathrm{Im}t_m > -\beta$ and continuous and uniformly bounded in $0 \geq \mathrm{Im}t_1 \geq \dots \geq \mathrm{Im}t_m \geq -\beta$, and its value for $t_k = -is_k$, $k = 1,\dots,m$ with $0 = s_o \leq s_1 \dots \leq s_m = \beta$ is given by

$$\omega_\beta^o(\Lambda)(F_o\alpha_{-is_1}^o(\Lambda)(F_1)\dots\ \alpha_{-is_m}^o(\Lambda)(F_m)) = E_\Lambda^\beta[\prod_{k=o}^m F_k(\eta(s_k))] \tag{3.26}$$

Let $\mathcal{O} \subset R^n$ be a bounded open set in R^n and let $\varphi \in C_o^\infty(\mathcal{O})$ and real. It is easily seen that the normal distribution indexed by h_Λ is quasi-invariant under the transformation $\eta \to \eta+\varphi$, if $\mathcal{O} \subset \Lambda$. Hence this transformation induces a unitary transformation $U(\varphi)$ on $L_2(h_\Lambda)$. Let $V(\varphi)$ be the unitary transformation of multiplication by the $L_\infty(h_\Lambda)$ function $e^{i(\varphi,\eta)}$

where $(\, , \,)$ is the inner product in $L_2(\Lambda)$. $\mathcal{A}_0(\mathcal{O})$ is then the smallest norm closed algebra in $B(L_2(h_\Lambda))$ containing $U(\varphi)$ and $V(\varphi)$ for all real $\varphi \in C_0^\infty(\mathcal{O})$. Since $\mathcal{A}_0(\mathcal{O})$ is a C*-algebra which is fatefully represented in each $B(L_2(h_\Lambda))$ for all $\Lambda \supset \mathcal{O}$, $\mathcal{A}_0(\mathcal{O})$ will not depend on the particular Λ as soon as $\Lambda \supset \mathcal{O}$.

By (3.2) the normal distribution indexed by h_Λ , may be characterized as the generalized Gaussian prosess with mean zero and covarians function $G_\Lambda(x,y)$, where $G_\Lambda(x,y)$ is the Greens function for the selfadjoint operator A_Λ . Let now $\mathcal{O}$ be contained in the interior of Λ_1 and Λ_2 . Since $G_{\Lambda_1}(x,y) - G_{\Lambda_2}(x,y)$ is a smooth function for x and y in $\mathcal{O}$, it follows that the conditional expectations of the normal distributions indexed by h_{Λ_1} and h_{Λ_2} with respect to the σ-algebra generated by functions of the form (φ, η) with $\varphi \in C_0^\infty(\mathcal{O})$ are equivalent measures. From this it immediately follows that $\mathcal{A}_0(\mathcal{O})$ has equivalent representations in $B(L_2(h_{\Lambda_1}))$ and $B(L_2(h_{\Lambda_2}))$, so that the strong closure $\bar{\mathcal{A}}(\mathcal{O})$ of $\mathcal{A}_0(\mathcal{O})$ in $B(L_2(h_\Lambda))$ is independent of Λ as soon as $\mathcal{O}$ is contained in the interior of Λ . We have obviously that $\bar{\mathcal{A}}(\mathcal{O}_1) \subseteq \bar{\mathcal{A}}(\mathcal{O}_2)$ if $\mathcal{O}_1 \subset \mathcal{O}_2$. Let $\mathcal{A}$ be the norm closure of $\cup \{ \bar{\mathcal{A}}(\mathcal{O}) \mid \mathcal{O} \subset R^n \}$.

Let now $B \in \bar{\mathcal{A}}(\mathcal{O})$. It is then well known that $\alpha_t^0(\Lambda)(B) \in \bar{\mathcal{A}}(\mathcal{O}_t)$, where $\mathcal{O}_t$ is the open set of points with distance smaller than t from $\mathcal{O}$, and that $\alpha_t^0(\Lambda)(B)$ is independent of Λ as soon as $\mathcal{O}_t$ is contained in the interior of Λ . We shall denote this independent value by $\alpha_t^0(B)$. α_t^0 is then a C*-isomorphism from $\bar{\mathcal{A}}(\mathcal{O})$ into $\bar{\mathcal{A}}(\mathcal{O}_t)$ for any $\mathcal{O}$, hence it extends to a C*-automorphism of $\bar{\mathcal{A}}$.

Now let $\mathcal{F}(\mathcal{O})$ be all functions in $L_\infty(h_\Lambda)$ of the form $f((\eta,\varphi_1),(\eta,\varphi_2),\ldots,(\eta,\varphi_N))$, where $f(x_1,\ldots,x_N)$ is bounded continuous function on R^N and $\varphi_1,\ldots,\varphi_N$ is in $C_o^\infty(\mathcal{O})$. We denote by $\mathcal{A}_o$ the smallest norm closed C*-algebra in $\bar{\mathcal{A}}$ which contains all operators of the form $\alpha_t^o(F)$, for $F \in \mathcal{F}(\mathcal{O})$ for some $\mathcal{O}$. $\mathcal{A}_o$ is then obviously invariant under α_t^o , and we shall say that $\mathcal{A}_o$ is the local algebra for the free field.

Let $F_o,\ldots,F_m$ be in $\mathcal{F}(\mathcal{O})$ for some $\mathcal{O}$. We shall then show that

$$\omega_\beta^o(\Lambda)(F_o\alpha_{t_1}^o(F_1)\ldots\ \alpha_{t_m}^o(F_m)) \tag{3.27}$$

converge as Λ tends to R^n in such a way that it finally contains all bounded sets, independently of the boundary conditions on $\partial\Lambda$. To see this, choose any $T > 0$. Then for $|t_k| \leq T$, $k = 1,\ldots,m$, since Λ finally contains any bounded set, we have that from a certain point on $\mathcal{O}_T$ is contained in the interior of Λ , but then (3.27) is equal to

$$\omega_\beta^o(\Lambda)(F_o\alpha_{t_1}^o(\Lambda)(F_1)\ldots\ \alpha_{t_m}^o(F_m)) \tag{3.28}$$

Now (3.28) is analytic in $0 > \mathrm{Im}t_1 > \ldots > \mathrm{Im}t_m > -\beta$ and uniformly bounded and continuous in $0 \geq \mathrm{Im}t_1 \geq \ldots \geq \mathrm{Im}t_m \geq -\beta$. The value of (3.28) at the imaginary points $t_k = -is_k$, $k = 1,\ldots,m$ and $0 = s_o \leq s_1 \ldots \leq s_m = \beta$ is by (3.26) given by

$$\omega_\beta^o(\Lambda)(F_o\alpha_{-is_1}^o(\Lambda))(F_1)\ldots\ \alpha_{is_m}^o(\Lambda)(F_m)) = E_\Lambda^\beta[\prod_{k=o}^m F_k^{s_k}] , \tag{3.29}$$

where $F_k^{s_k}$ is the translated by an amount s^k around the circle S_β of the functions F_k in $\mathcal{F}(\mathcal{O})$. Since F_k , $k = 0,\ldots,n$ are continuous bounded functions of the stochastic variables

$(\eta,\varphi_1),\ldots,(\eta,\varphi_n)$, we get that (3.29) converge if the corresponding correlation function converge since E_Λ^β is the expectation with respect to a Gaussian distribution. We shall now assume that the mass $m > 0$. By theorem 3.1 the correlation function for E_Λ^β is given by $G_\Lambda^\beta(x,y,s-t)$. That $G_\Lambda^\beta(x,y,s-t)$ converges as Λ tends to R^n in such a way that it finally contains all bounded sets follows from the fact that G_Λ^β is the Greenfunction for the self adjoint operator

$$-\frac{\partial^2}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}+m^2 \tag{3.30}$$

on $S_\beta \times \Lambda$, with some self dajoint boundary conditions on $\partial\Lambda$. So that as Λ tends to R^n in such a way as to finally containing any bounded set we get that $G_\Lambda^\beta(x,y,s-t)$ converge weakly to $G^\beta(x,y,s-t)$ which is the Greensfunction on $S_\beta \times R^n$ for the operator (3.30).

Since the local algebra for the free field $\mathcal{A}_o$ is the smallest norm closed C*-algebra containing $\alpha_t^o(F)$ for all t and $F \in \mathcal{F}(\mathcal{O})$ for some $\mathcal{O}$, we have that elements of the form $F_o\alpha_{t_1}^o(F_1)\ldots\alpha_{t_m}^o(F_m)$ is norm dense in $\mathcal{A}_o$. Hence we have proved the following theorem.

<u>Theorem 3.2</u> Let $\mathcal{A}_o$ be the local algebra for the free field, then α_t^o defines a group of C*-automorphism of $\mathcal{A}_o$. There is a state ω_β^o on $\mathcal{A}_o$ which is invariant under α_t^o i.e.

$$\omega_\beta^o(B\cdot\alpha_t^o(C)) = \omega_\beta^o(\alpha_{-t}^o(B)\cdot C) ,$$

such that $\omega_\beta^o(B\alpha_t^o(C))$ is analytic in the strip $-\beta < \mathrm{Im}t < 0$ and uniformly bounded and continuous in $-\beta \leq \mathrm{Im}t \leq 0$, and satisfies the KMS conditions on the boundary

$$\omega_\beta^o(B\cdot\alpha_{t-i\beta}^o(C)) = \omega_\beta^o(C\cdot\alpha_{-t}^o(B))$$

for real t.

Moreover, if $F_o,\dots,F_m$ is in the subalgebra of $\mathcal{A}_o$ generated by the fields at time zero then $\omega_\beta^o(F_o\alpha_{t_1}^o(F_1)\dots\alpha_{t_m}^o(F_m))$ is analytic in $0 > \mathrm{Im}t_1 > \dots > \mathrm{Im}t_m > -\beta$ and continuous and uniformly bounded in $0 \geq \mathrm{Im}t_1 \geq \dots \geq -\beta$, and its value at the imaginary points $t_k = -is_k$, $k = 1,\dots,m$ with $0 = s_o \leq s_1 \dots \leq s_m = \beta$ is given by

$$\omega_\beta^o(F_o\alpha_{-is_1}^o(F_1)\dots\alpha_{-is_m}^o(F_m)) = E^\beta[\prod_{k=o}^m F_k^{s_k}] ,$$

where E^β is the expectation with respect to the generaliized Gaussian process with mean zero and covariance function $G^\beta(x-y,s-t)$, which is the Greensfunction on $S_\beta \times R^n$ for the self adjoint operator

$$-\frac{\partial^2}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + m^2$$

on $L_2(S_\beta \times R^n)$,

and $F_k^{s_k}$ is the translated by the action of the circle group S_β on $S_\beta \times R^n$ of the function F_k by the amount s_k.

Further more, if $B \in \mathcal{A}_o$ is in $\bar{\mathcal{A}}(\mathcal{O})$ for some bounded $\mathcal{O}$ then

$$\omega_\beta^o(B) = \lim_\Lambda \omega_\beta^o(\Lambda)(B)$$

as Λ tends to R^n in the sense that Λ finally contains any fixed bounded set.

<u>Remark</u>: Utilizing the formula (2.24) we get that $G^\beta(x,t)$ is given by

$$G^\beta(x,s) = \frac{1}{(2\pi)^n}\int_{R^n} \hat{G}^\beta(p,s)e^{-ipx}dp \qquad (3.31)$$

where for $0 \leq s \leq \beta$

$$\hat{G}^{\beta}(p,s) = (2\omega(1-e^{-\beta\omega}))^{-1}(e^{-s\omega}+e^{-(\beta-s)\omega}) \tag{3.32}$$

with $\omega = \omega(p) = \sqrt{p^2+m^2}$.

If we introduce the annihilation creation operators and the free fields we have the relations

$$\varphi(x,t) = \frac{1}{\sqrt{2}}(2\pi)^{-\frac{n}{2}}\int[e^{i(px+\omega t)}a^*(p) + e^{-i(px+\omega t)}a(p)]\frac{dp}{\omega(p)^{\frac{1}{2}}} \tag{3.33}$$

where $\varphi(x,t)$ is the free field at time t .

The operator that counts the number of particles with momentum $p \in \Omega$ in a region $\Omega \subset R^n$ of momentumspace is given by

$$N(\Omega) = \int_{\Omega} a^*(p)a(p)dp \ . \tag{3.34}$$

Introducing now the function

$$\Delta^{\beta}(x,t) = G^{\beta}(x,-it) \tag{3.35}$$

so that

$$\hat{\Delta}^{\beta}(p,t) = (2\omega(1-e^{-\beta\omega}))^{-1}(e^{it\omega}+e^{-\beta\omega}e^{-it\omega}) \tag{3.36}$$

we get the following formula for computing expectations of products of fields

$$\omega_{\beta}^{o}(\varphi(x_1,t_1)\ldots\varphi(x_n,t_n)) = \begin{cases} \Sigma\ \Delta^{\beta}(x_{i_1}-x_{i_2},t_{i_1}-t_{i_2})\ldots\Delta^{\beta}(x_{i_{n-1}}-x_{i_n},t_{i_{n-1}}-t_{i_n}) \\ \qquad \text{for } n \text{ even} \\ 0 \quad \text{for } n \text{ odd} \end{cases} \tag{3.37}$$

where the summation runs over all partitions of $(1,\ldots,2k)$ with $2k = n$ into disjoint pairs $(i_1,i_2)(i_3,i_4)\ldots(i_{n-1},i_n)$.

If we define the pressure for the free field at temperature $\frac{1}{\beta}$ in the usual way by

$$p_\beta^o = \beta^{-1} \lim_{\Lambda \to R^n} |\Lambda|^{-1} \log(\mathrm{tr}(e^{-\beta H_o(\Lambda)})) \ , \qquad (3.38)$$

where $|\Lambda|$ is the volume of Λ ,

we get by using the formla

$$\mathrm{tr}(e^{-\beta H_o(\Lambda)}) = |1-e^{-\beta A_\Lambda}|^{-1} \qquad (3.39)$$

together with well known asymptotic formulas for the eigenvalues of the Laplaeien Δ in Λ as Λ increase to R^n , we get that the limit (3.39) always exists and is given by

$$p_\beta^o = -(2\pi)^{-n}\beta^{-1}\int_{R^n} \log(1-e^{-\beta\omega(p)})dp \ . \qquad (3.40)$$

If we take $\Lambda_l = [-\frac{l}{2},\frac{l}{2}]^n$ with periodic boundary conditions we have that A_{Λ_l} has the eigenvalues

$$(\sum_{i=1}^{n} (\frac{2\pi n_i}{l})^2+m^2)^{\frac{1}{2}} \qquad (3.41)$$

where $(n_1,\dots,n_n) \in \mathbb{Z}^n$. In this case we have the annihilation creation operators $a_l^*(p)$ and $a_l(p)$ with $p \in \frac{2\pi}{l}\cdot\mathbb{Z}^n$, for $H_o(\Lambda_l)$. The operator that counts the number of particles with momentum $p \in \Omega$ in a region $\Omega \subset R^n$ is now given by

$$N_l(\Omega) = \sum_{p\in\Omega} a_l^*(p)a_l(p) \qquad (3.42)$$

If we now compute the expected number of particles for the system in Λ_l we get

$$\omega_\beta^o(\Lambda_l)(N_l(\Omega)) = \sum_{p\in\Omega} \frac{e^{-\beta\omega(p)}}{1-e^{-\beta\omega(p)}} \qquad (3.43)$$

We now define the density of particles with momentum in Ω by

$$\int_\Omega \rho_\beta^o(p)dp = \lim_{\Lambda_l\to R^n} |\Lambda_l|^{-1}\omega_\beta^o(\Lambda_l)(N_l(\Omega)) \qquad (3.44)$$

Then this limit exists and is given by

$$\int_{\Omega} \rho_{\beta}^{o}(p)dp = (2\pi)^{-n} \int_{\Omega} \frac{e^{-\beta\omega(p)}}{1-e^{-\beta\omega(p)}} dp \ . \tag{3.45}$$

So that then density of particles with momentum p exists and is given by

$$\rho_{\beta}^{o}(p) = (2\pi)^{-n} \frac{e^{-\beta\omega(p)}}{1-e^{-\beta\omega(p)}} , \tag{3.46}$$

and the particle density is given by

$$\rho_{\beta}^{o} = (2\pi)^{-n} \int_{R^n} \frac{e^{-\beta\omega(p)}}{1-e^{-\beta\omega(p)}} dp \ . \tag{3.47}$$

In correspondence with (3.4) and (3.4) we may introduce the partial pressure due to particles with momentum p by

$$p_{\beta}^{o}(p) = -(2\pi)^{-n} \beta^{-1} \log(1-e^{-\beta\omega(p)}) \ . \tag{3.48}$$

If we want to express the state ω_{β}^{o} in terms of annihilation creation operators

$$a^{\#}(h) = \int h(p) a^{\#}(p) dp \tag{3.49}$$

where $a^{\#}$ stands for a or a^*, and $h \in L_2(R^n)$.
Then we have the formula

$$\omega_{\beta}^{o}(e^{i(a^*(\bar{h})+a(h))}) = e^{-\frac{1}{2}\int |h(p)|^2 \frac{1+e^{-\beta\omega(p)}}{1-e^{-\beta\omega(p)}} dp} \ . \tag{3.50}$$

As a comparison we have that the corresponding quantity for a system of free Schrödinger particles at temperature β and activity z is given by

$$e^{-\frac{1}{2}\int |h(p)|^2 \frac{1+ze^{-\frac{\beta}{2m}p^2}}{1-ze^{-\frac{\beta}{2m}p^2}} dp} \tag{3.51}$$

To within the non relativistic approximation

$$\omega(p) \sim m + \frac{1}{2m} p^2$$

we see that ω_β^o is the Gibbs state of free Schrödinger particles of mass m at temperature β and activity $z = e^{-m\beta}$.

4. The Gibbs-state for the interacting scalar quantum field in two space-time dimensions.

In the case of two space-time dimensions or equivalentely one space dimension, the interacting scalar field is relatively well understood in the case of polynomial interactions ([2],[7], [8],[9] and [10]) and exponential interactions ([3] and [11]). In the case of positive mass $m > 0$, it was proved by Glimm-Spencer [2] that the thermodynamic limit for the temperature zero ($\beta = \infty$) state existed and is unique for weak polynomial interactions. More recently Nelson [12] have established the existence of the thermodynamic limit for strong polynomial interactions with Dirichlet boundary conditions. Nelson's method which depends strongly on the Dirichlet boundary conditions leads to the question of whether this limit is unique, and in fact Dobrushin and Minlos [6] have announced the result that there is a certain critical value for the interaction strength for any polynomial interaction above which the limit is not unique. For the strong exponential interaction the existence and uniqueness of the termodynamic limit for the temperature zero state was recently proved by Albererio and Høegh-Krohn [3], in the case of even interactions.

From what is said above we see that the thermodynamic behavior of the temperature zero state is quite complex and that by the result of Dobrushin and Mihlos there are phasetransitions of the temprature zero state for the polynomial interactions.

In contrast with this complex picture for the temperature zero state , we shall see that for the positive temperature ($\beta < \infty$)

state or the Gibbs state the thermodynamic limit always exists and is unique for the polynomial interactions as well as for the exponential interactions, without any restriction on the strength of the interaction, in the case of two space-time dimensions.

Now let

$$H_1 = H_o + \int_{-1}^{1} :V(\varphi(x)): dx \qquad (4.1)$$

where $\varphi(x)$ is the time zero free field of positive mass $m > 0$, in two space time dimensions, and $V(s)$ is either a polynomial which is bounded below

$$V(s) = P(s) \qquad (4.2)$$

or an exponential function i.e.

$$V(s) = \int e^{\alpha s} d\mu(\alpha) \qquad (4.3)$$

where $d\mu$ is a positive measure of compact support in the open intervall $(-\sqrt{2\pi}, \sqrt{2\pi})$.

H_o is the Hamiltonian for the free scalar field φ. H_1 is then the Hamiltonian for the corresponding interacting field with a space cut off interaction. For details concerning the definition of H_1 the reader should consult the references [7] and [10] for the case (4.2) and the references [11] for the case (4.3).

It is known that H_1 is essentially self adjoint on the intersection of the domains of H_o and V_1

$$V_1 = \int_{-1}^{1} :V(\varphi(x): dx \qquad (4.4)$$

and that H_1 is bounded below

$$H_1 \geq -b \qquad (4.5)$$

where b is some real number depending on V_1, so that $e^{-\beta H_1}$ is a bounded operator.

We will now construct the Gibbs state for the space cut-off interaction (4.1).

Let Λ be an interval containing the interval $[-1,1]$ in its interior, then we set

$$H_1(\Lambda) = H_o(\Lambda) + \int_{-1}^{1} : V(\varphi(x)) : dx \ . \tag{4.6}$$

By the same methods that proves that H_1 is essentially self adjoint and bounded below we get that $H_1(\Lambda)$ is essentially self adjoint and bounded below. Moreover, we also get that $H_1(\Lambda)$ has descrete spectrum and that $e^{-\beta H_1(\Lambda)}$ is of trace class. We shall start by computing the trace of $e^{-\beta H_1(\Lambda)}$. By the method of hypercontractivity [13] in the same way as for H_1, we have that $H_1(\Lambda)$ may be approximated by operators H_n such that

$$\|e^{-\beta H_1(\Lambda)} - e^{-\beta H_n}\| \to 0 \tag{4.7}$$

and H_n has the form

$$H_1(\Lambda) = H_o(\Lambda) + V_1^{(n)} \tag{4.8}$$

where $V_1^{(n)}$ is in $\mathcal{F}(\mathcal{O})$ for some $\mathcal{O} \subset \Lambda$. We shall prove below that

$$\mathrm{tr}\ e^{-\beta H_n} \to \mathrm{tr}\ e^{-\beta H_1(\Lambda)} \ . \tag{4.9}$$

Since $V_1^{(n)}$ is bounded we get by the Trotter-Kato product formula that

$$\underset{k \to \infty}{\text{strong lim}} \left[e^{-\frac{\beta}{k} H_o(\Lambda)} \ e^{-\frac{\beta}{k} V_1^{(n)}} \right]^k = e^{-\beta H_n} \ , \tag{4.10}$$

but this may obviously also be written in the form

$$\operatorname*{strong\ lim}_{k\to\infty}\left[e^{-\frac{\beta}{2k}H_o(\Lambda)}\,e^{-\frac{\beta}{k}V_1^{(n)}}\,e^{-\frac{\beta}{2k}H_o(\Lambda)}\right]^k = e^{-\beta H_n} . \tag{4.11}$$

Now let $V_1^{(n)} \geq -c$, where c of course depends on n . Then

$$\| e^{-\frac{\beta}{k}V_1^{(n)}} \| \leq e^{\frac{\beta}{k}c} , \tag{4.12}$$

hence

$$e^{-\frac{\beta}{2k}H_o(\Lambda)}\,e^{-\frac{\beta}{k}V_1^{(n)}}\,e^{-\frac{\beta}{2k}H_o(\Lambda)} \leq e^{\frac{\beta}{k}c}\,e^{-\frac{\beta}{k}H_o(\Lambda)} , \tag{4.13}$$

so that the i-th eigenvalue of

$$\left[e^{-\frac{\beta}{2k}H_o(\Lambda)}\,e^{-\frac{\beta}{k}V_1^{(n)}}\,e^{-\frac{\beta}{2k}H_o(\Lambda)}\right]^k \tag{4.14}$$

is smaller or equal to $e^{\beta c}$ times the i-th eigenvalue of $e^{-\beta H_o(\Lambda)}$. On the other hand we have by (4.11) that the i-th eigenvalue of (4.14) converge to the i-th eigenvalue of $e^{-\beta H_n}$. Hence by dominated convergence we get that the trace of (4.14) converge to the trace of $e^{-\beta H_n}$, since $e^{-\beta H_o(\Lambda)}$ is of trace class.

However the trace of (4.14) is by theorem 3.1 given by

$$|1-e^{-\beta A_\Lambda}|^{-1}\; E_\Lambda^\beta[\exp\{-\frac{\beta}{k}\sum_{j=1} V_1^{(n)}(\eta(\tfrac{\beta}{k}(j-\tfrac{1}{2})))\}] \tag{4.15}$$

Since the transformation $V_1^{(n)}(\eta(0)) \to V_1^{(n)}(\eta(s))$ is induced by the action of the circle group S_β on $S_\beta \times \Lambda$, and the generalized Gaussian process corresponding to E_Λ^β is homogeneous with respect to this action, we have that the transformation is given by a strongly continuous unitary group on L_2 of the corresponding process, and therefore $V_1^{(n)}(\eta(s))$ is a strongly continuous function of s in the L_2 space of the process. Hence we get the

strong L_2-convergence

$$\frac{\beta}{k}\sum_{j=1}^{k} V_1^{(n)}(\eta(\tfrac{\beta}{k}(j-\tfrac{1}{2}))) \to \int_0^\beta V_1^{(n)}(\eta(s))ds \tag{4.16}$$

as $k \to \infty$. By passing to an almost everywhere convergent subsequence, we get by dominated convergence the corresponding convergence of (4.15). Hence we have that

$$\operatorname{tr} e^{-\beta H_n} = |1-e^{-\beta A_\Lambda}|^{-1} E_\Lambda^\beta\left[e^{-\int_0^\beta V_1^{(n)}(\eta(s))ds}\right]. \tag{4.17}$$

Now the approximation of $V_1 = \int_{-1}^{1} :V(\varphi(x)):dx$ by functions $V_1^{(n)}$ in $\mathcal{F}(\mathcal{O})$ may be carried out in two steps. First we approximate $V_1^{[a,b]}$, where $V_1^{[a,b]}$ is equal to V_1 in those points where V_1 has values in the interval $[a,b]$ and $V_1^{[a,b]}$ is equal to zero elsewhere. Under this approximation we have that $V_1^{(n)}$ is uniformly bounded below so that $e^{-\beta H_n} \le e^{c\beta}\cdot e^{-\beta H_0(\Lambda)}$, and by hypercontractivity $e^{-\beta H_n}$ converge to $e^{-\beta(H_0(\Lambda)+V_1^{[a,b]})}$ in norm, so that $\operatorname{tr} e^{-\beta H_n}$ converge by dominated convergence. On the other hand the right hand side of (4.17) will also converge by dominated convergence since under this approximation $V_1^{(n)}$ is uniformly bounded below.

Then we remove a and b by first letting $a \to -\infty$ and then $b \to \infty$. In both cases we have that both sides of the equation

$$\operatorname{tr} e^{-\beta(H_0(\Lambda)+V_1^{[a,b]})} = |1-e^{-\beta A_\Lambda}|^{-1} E_\Lambda^\beta\left[e^{-\int_0^\beta V_1^{[a,b]}(\eta(s))ds}\right] \tag{4.18}$$

converges by monoton convergence. Hence we have proved the following formula

$$\operatorname{tr} e^{-\beta H_1(\Lambda)} = |1-e^{-\beta A_\Lambda}|^{-1} E_\Lambda^\beta\left[e^{-\int_0^\beta V_1(\eta(s))ds}\right]. \tag{4.19}$$

Recalling the form of V_l this may also be written

$$\operatorname{tr} e^{-\beta H_l(\Lambda)} = |1-e^{-\beta A_\Lambda}|^{-1} E_\Lambda^\beta \left[e^{-\int_0^\beta \int_{-e}^{l} :V(\xi(x,s)):dxds} \right] . \tag{4.20}$$

In the same way as we proved the formula (4.20) we prove the following

Lemma 4.1

Let $F_0,\dots,F_{n-1}$ be in $L_\infty(h_\Lambda)$ and $0 = s_0 \leq s_1 \leq \dots \leq s_n = \beta$ then

$$\operatorname{tr}(F_0 e^{-s_1 H_l(\Lambda)} F_1 e^{-(s_2-s_1)H_l(\Lambda)} \dots F_{n-1} e^{-(\beta-s_n)H_l(\Lambda)})$$

$$= |1-e^{-\beta A_\Lambda}|^{-1} E_\Lambda^\beta \left[\prod_{k=0}^{n-1} F_k^{s_k} e^{-\int_0^\beta \int_{-e}^{l} :V(\xi(x,s)):dxds} \right]$$

where $F_k^{s_k}$ is the translation of F_k by the amount s_k in the action induced by the circle group S_β on the generalized Gaussian process $\xi(x,s)$.

As in section 3 we now define for any $B \in B(L_2(h_\Lambda))$

$$\alpha_t^l(\Lambda)(B) = e^{-itH_l(\Lambda)} B e^{itH_l(\Lambda)} \tag{4.21}$$

and

$$\omega_\beta^l(\Lambda)(B) = (\operatorname{tr} e^{-\beta H_l(\Lambda)})^{-1} \operatorname{tr}(B e^{-\beta H_l(\Lambda)}) . \tag{4.22}$$

We then have that if $B \in \mathcal{A}(\mathcal{O})$ and $\mathcal{O}_t \subset \Lambda$, then $\alpha_t^l(\Lambda)$ is independent of Λ , and we denote this Λ independent value by $\underline{\alpha_t^l(B)}$. This then gives us a group of C*-automorphism α_t^l on $\mathcal{A}$. It is well known that if $\mathcal{O}_t \subset [-l,l]$ then $\alpha_t^l(B)$ is independent of l , and we shall denote this l independent value by $\underline{\alpha_t}(B)$, and again α_t gives us a group of C*-automorphism on $\mathcal{A}$.

Let now $\mathcal{A}$ be the smallest norm closed C*-algebra in $\bar{\mathcal{A}}$ containing $\alpha_t(F)$ for all real t and all $F \in \mathcal{F}(\mathcal{O})$ for any bounded $\mathcal{O}$ in R. Elements in $\mathcal{A}$ of the form

$$F_o \alpha_{t_1}(F_1) \ldots \alpha_{t_n}(F_n) \tag{4.23}$$

with $F_o, F_1, \ldots, F_n$ in $\mathcal{F}(\mathcal{O})$ then spends a dense linear set in $\mathcal{A}$. We shall see that

$$\omega_\beta^l(\Lambda)(F_o \alpha_{t_1}(F_1) \ldots \alpha_{t_n}(F_n)) \tag{4.24}$$

converges as Λ tends to R and l tends to ∞.
If $\mathcal{O}_{t_i} \subset [-l,l] \subset \Lambda \quad i = 1,\ldots,n$ then (4.23) is equal to

$$\omega_\beta^l(\Lambda)(F_o \alpha_{t_1}^l(\Lambda)(F_1) \ldots \alpha_{t_n}^l(\Lambda)(F_n)) . \tag{4.25}$$

By the definition (4.22) of $\omega_\beta^l(\Lambda)$ we have that (4.25) is analytic in $0 > \mathrm{Im}t_1 > \ldots > \mathrm{Im}t_n > -\beta$ and uniformly bounded and continuous in $0 \geq \mathrm{Im}t_1 \geq \ldots \geq \mathrm{Im}t_n \geq -\beta$. Moreover, its values at the imaginary points $t_k = -is_k \quad k = 1,\ldots,n$ with $0 = s_o \leq s_1 \ldots \leq s_n$ is by lemma 4.1 given by

$$\begin{aligned} &\omega_\beta^l(\Lambda)(F_o \alpha_{-is_1}^l(\Lambda)(F_1) \ldots \alpha_{-is_n}^l(\Lambda)(F_n)) \\ &= (E_\Lambda^\beta[e^{-\int_o^\beta \int_{-e}^l :V(\xi(x,s)):dxds}])^{-1} E_\Lambda^\beta[\prod_{k=o}^n F_k^{s_k} e^{-\int_o^\beta \int_e^l :V(\xi(x,s)):dxds}] \end{aligned} \tag{4.26}$$

To prove that (4.24) converges as first Λ tends to R and then l tends to ∞, it is therefore enough to prove that the same limits exists for the right hand side of (4.26). Since $F_k \in \mathcal{F}(\mathcal{O})$ $k = 0,\ldots,n$, it is therefore enough; if we want to prove that the limit exists as $\Lambda \to R$, to prove that E_Λ^β converge weakly as $\Lambda \to R$. But since E_Λ^β is the expectation with respect to the generalized Gaussian process with mean zero and covariance

function $G_\Lambda^\beta(x,y;s-t)$, the weak convergence of E_Λ^β follows from that of the covariance function. Hence we find that the limit of (4.26) as $\Lambda \to R$ exists and is given by

$$E_l^\beta\left[\prod_{k=o}^{n} F_k^{s_k}\right] \tag{4.27}$$

$$= \left(E^\beta\left[e^{-\int_o^\beta\int_{-l}^{l} :V(\xi(x,s)):dxds}\right]\right)^{-1} E^\beta\left[\prod_{k=o}^{n} F_k^{s_k} e^{-\int_o^\beta\int_{-l}^{l} :V(\xi(x,s)):dxds}\right]$$

where E^β is the expectation with respect to the generalized Gaussian process with mean zero and covariance given by the Greensfunction $G^\beta(x-y,s-t)$ for the self adjoint operator

$$-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2 \tag{4.28}$$

on $S_\beta \times R$. For $0 \leq t \leq \beta$ we have that $G^\beta(x,t)$ is given by

$$G^\beta(x,t) = \frac{1}{2\pi}\int_R e^{-ixp}\hat{G}^\beta(p,t)dp \tag{4.29}$$

where

$$\hat{G}^\beta(p,t) = [2\omega(p)(1-e^{-\beta\omega(p)})]^{-1}[e^{-t\omega(p)} + e^{-(\beta-t)\omega(p)}] \tag{4.30}$$

where $\omega(p) = \sqrt{p^2+m^2}$.

To prove that (4.27) converges as $l \to \infty$ it is again enough, since $F_k \in \mathcal{F}(\mathcal{O})$ for $k = 0,\ldots,n$, to prove that E_l^β converges weakly as $l \to \infty$. To do this we consider the Fourier transform of the generalized process given by E_l^β

$$E_l^\beta\left[e^{i\iint\psi(x,s)\xi(x,s)dxds}\right] \tag{4.31}$$

where ψ is a C^∞- function of compact support in $S_\beta \times R$. By (4.27) we have that (4.31) is given by

$$\left\{E^\beta\left[e^{-\int_o^\beta\int_{-1}^{1}:V(\xi(x,s)):dxds}\right]\right\}^{-1} \tag{4.32}$$
$$\cdot\ E^\beta\left[e^{i(\psi,\xi)}\, e^{-\int_o^\beta\int_{-1}^{1}:V(\xi(x,s)):dxds}\right].$$

Let ψ have support in $S_\beta \times [-a,a]$.
Consider now the Hamiltonian H_β on $L_2(h_\beta)$ where $h_\beta = h_{\Lambda_\beta}$ with $\Lambda_\beta = [0,\beta]$ with periodic boundary conditions and H_β is the corresponding periodic Hamiltonian

$$H_\beta = H_o^\beta + \int_o^\beta :V(\varphi(x)):dx \tag{4.33}$$

where $H_o^\beta = H_o(\Lambda_\beta)$, $\Lambda_\beta = [0,\beta]$ with periodic boundary conditions. It is well known both in the polynomial and exponential case that H_β has a simple lowest eigenvalue with a normalized eigenvector which we denote Ω_β .

By letting in lemma 4.1 β tend to infinity and taking Λ in lemma 4.1 fixed equal to the Λ_β above, we obtain easily the formulas

$$E^\beta\left[e^{-\int_o^\beta\int_{-1}^{1}:V(\xi(x,s)):dxds}\right] = (\Omega_\beta^o, e^{-21H_\beta}\Omega_\beta^o) \tag{4.34}$$

and

$$E^\beta\left[e^{i(\psi,\xi)}\, e^{-\int_o^\beta\int_{-e}^{1}:V(\xi(x,s)):dxds}\right] \tag{4.35}$$
$$= (e^{-(1-a)H_\beta}\Omega_\beta^o,\ W_{[-a,a]}(if)e^{-(1-a)H_\beta}\Omega_\beta^o) ,$$

where Ω_β^o is the normalized eigenvector of H_o^β corresponding to it simple lowest eigenvalue and $W_{[s,t]}(if)$ is the unique bounded operator satisfying the strong differential equation

$$\frac{d}{dt} W_{[s,t]}(if) = W_{[s,t]}(if)\left[-H_\beta + i\int_0^\beta f(x,t)\varphi_\beta(x)dx\right] \tag{4.36}$$

where $\varphi_\beta(x)$ is the free field corresponding to H_0^β, with the initial condition that

$$W_{[s,s]}(if) = 1 \tag{4.37}$$

and

$$f(x,t) = \psi(t,x) \ . \tag{4.38}$$

The analog of (4.34) and (4.35) in the temperature zero case is well known and used for instance in [14] where they are called the Nelson symmetries.

Now that the limit of (4.31) and hence of (4.27) exists as $l \to \infty$ follows simply from the fact that H_β has a simple lowest eigenvalue. Hence we have

<u>Lemma 4.2</u>

Let $E_l^\beta(e^{i(\psi,\xi)}) = \left\{E^\beta\left[e^{-\int_0^\beta\int_{-l}^{l} :V(\xi(x,s)):dxds}\right]\right\}^{-1}$

$$\cdot E^\beta\left[e^{i(\psi,\xi)} e^{-\int_0^\beta\int_{-l}^{l} :V(\xi(x,s)):dxds}\right] .$$

Then the limit $E_\infty^\beta(e^{i(\psi,\xi)})$ as $l \to \infty$ exists and is given by

$$E_\infty^\beta(e^{i(\psi,\xi)}) = (\Omega_\beta, W_{[-a,a]}(if)\Omega_\beta)$$

where $f(x,t) = \psi(t,x)$. Moreover, the measure induced by E_∞^β is locally equivalent to the generalized Gaussian process given by E^β, i.e. restricted to the subalgebras generated by (ψ,ξ) for ψ with support in a fixed interval $S_\beta \times [-a,a]$ they are equivalent. The measure given by E_∞^β is strongly mixing with respect to space translations i.e.

$$\lim_{x \to \infty} E_\infty^\beta\left(e^{i(\psi_1,\xi)}\, e^{i(\psi_2^x,\xi)}\right) = E_\infty^\beta\left(e^{i(\psi_1,\xi)}\right)\cdot E_\infty^\beta\left(e^{i(\psi_2,\xi)}\right)$$

where $\psi^x(t,y) = \psi(t,y-x)$.

Proof: We have already proved everything up to the moreover part. The moreover part follows immediately from the formula for E_∞^β and the strongly mixing follows from the same formula together with the fact that Ω_β belongs to a simple lowest eigenvalue of H_β .

Theorem 4.1

Let $\mathcal{A}$ be the local algebra for the interacting field, i.e. the smallest norm closed C*-algebra in $\bar{\mathcal{A}}$ containing $\alpha_t(F)$ for all real t and all $F \in \mathcal{F}(\mathcal{O})$ for any bounded $\mathcal{O}$ in R . There exists then a state ω_β on $\mathcal{A}$ such that ω_β is invariant under α_t i.e.

$$\omega_\beta(B\alpha_t(C)) = \omega_\beta(\alpha_{-t}(B)\cdot C) ,$$

for any B and C in $\mathcal{A}$. $\omega_\beta(B\cdot\alpha_t(C))$ is analytic in the strip $-\beta < \mathrm{Im}t < 0$ and uniformly bounded and continuous in $-\beta \le \mathrm{Im}t \le 0$, and satisfies the KMS conditions on the boundary

$$\omega_\beta(B\cdot\alpha_{t-i\beta}(C)) = \omega_\beta(C\cdot\alpha_{-t}(B))$$

for real t . ω_β is invariant under space translations

$$\omega_\beta(B_x) = \omega_\beta(B)$$

and have the cluster property

$$\lim_{x \to \infty} \omega_\beta(B_x\cdot C) = \omega_\beta(B)\cdot\omega_\beta(C) .$$

ω_β is locally Fock, i.e. if we restrict ω_β to the subalgebra

generated by $\alpha_t(F)$ for t in a fixed interval $[-a,a]$ and $F \in \mathcal{F}(\mathcal{O})$. for a fixed bounded $\mathcal{O}$, then on this subalgebra ω_β induces the free Fock representation.

Moreover, if $F_o,\ldots,F_n$ is in $\mathcal{F}(\mathcal{O})$ for some bounded $\mathcal{O}$ then $\omega_\beta(F_o\alpha_{t_1}(F_1)\ldots\alpha_{t_n}(F_n))$ is analytic in $0 > \mathrm{Im}t_1 > \ldots > \mathrm{Im}t_n > -\beta$ and continuous and uniformly bounded in $0 \geq \mathrm{Im}t_1 \geq \ldots \geq \mathrm{Im}t_n \geq -\beta$, and its value at the imaginary points $t_k = -is_k$, $k = 1,\ldots,n$ with $0 = s_o \leq s_1 \leq \ldots \leq s_n = \beta$ is given by

$$\omega_\beta(F_o\alpha_{-is_1}(F_1)\ldots\alpha_{-is_n}(F_n)) = E_\infty^\beta[\prod_{k=o}^{n} F_k^{s_k}] ,$$

where E_∞^β is the expectation with respect to the generalized homogeneous process on $S_\beta \times R$ given in lemma 4.2, and $F_k^{s_k}$ is the translated of the function F_k by the amount s_k in the action of the circle group S_β on $S_\beta \times R$.

Furthermore, if B is in the subalgebra generated by $\alpha_t(F)$ for t in a fixed interval $[-a,a]$ and $F \in \mathcal{F}(\mathcal{O})$ for a fixed bounded $\mathcal{O}$ then

$$\omega_\beta(B) = \lim_{l\to\infty}\ \lim_{\Lambda\to R}\ \omega_\beta^l(\Lambda)(B) .$$

Proof: Linear combinations of elements of the form $F_o\alpha_{t_1}(F_1)\ldots$ $\ldots\alpha_{t_n}(F_n)$ are norm dense in $\mathcal{A}$. We may show in the same way as for the temperature zero case that $\omega_\beta^l(\Lambda)$ is locally Fock uniformly in l and Λ. Since for l and Λ big enough $\alpha_{t_k}(F_k) = \alpha_{t_k}^l(\Lambda)(F_k)$ and $\alpha_t^l(\Lambda)$ is strongly continuous in Fock space we therefore get that

$$\omega_\beta^l(\Lambda)(F_o\alpha_{t_1}(F_1)\ \ldots\ \alpha_{t_n}(F_n)) \tag{4.39}$$

when we restrict t_k, $k = 1,\ldots,n$ to a bounded interval is continuous in t_k, $k = 1,\ldots,n$ uniformly with respect to l and Λ. By passing to subsequences we therefore get as first $\Lambda \to R$ and then $l \to \infty$ through subsequences that (4.39) has a limit

$$\omega_\beta(F_o\alpha_{t_1}(F_1)\ \ldots\ \alpha_{t_n}(F_n)) \tag{4.40}$$

which is continuous in t_k, $k = 1,\ldots,n$. On the other hand we have already proved that (4.39) is analytic in $0 > \mathrm{Im}t_1 > \ldots > \mathrm{Im}t_n > -\beta$ and that at the imaginary points

$$\omega^l_\beta(\Lambda)(F_o\alpha_{-is_1}(F_1)\ \ldots\ \alpha_{-is_n}(F_n)) \tag{4.41}$$

with $0 \leq s_1 \leq \ldots \leq s_n \leq \beta$, (4.41) converges as first $\Lambda \to R$ and then $l \to \infty$.

If we denote the limit by $\omega_\beta(F_o\alpha_{-is_1}(F_1)\ \ldots\ \alpha_{-is_n}(F_n))$ we get by lemma 4.2 that

$$\omega_\beta(F_o\alpha_{-is_1}(F_1)\ldots\ \alpha_{-is_n}(F)) = E^\beta_\infty\Big(\prod_{k=o}^{n} F_k^{s_k}\Big)\ . \tag{4.42}$$

(4.42) being a limit of functions which are uniformly bounded and analytic in $0 > \mathrm{Im}t_1 > \ldots > \mathrm{Im}t_n > -\beta$ must itself be analytic and bounded in the same domain. Since (4.40) is the limit of boundary values of these functions, it must itself be the boundary value of (4.42). But this proves that (4.40) must be independent of the subsequences chosen, so that (4.39) converges as first $\Lambda \to R$ and then $l \to \infty$ to a limit (4.40) which is continuous in t_k, $k = 1,\ldots,n$. Hence (4.40) is the boundary value of a function which is analytic in $0 > \mathrm{Im}t_1 > \ldots > \mathrm{Im}t_n > -\beta$ and uniformly bounded and continuous in $0 \geq \mathrm{Im}t_1 \geq \ldots \geq \mathrm{Im}t_n > -\beta$ and its value at the imaginary points is given by (4.42).

Now ω_β as limit of states is again a state and extends by con-

tinuity to all of $\mathcal{A}$. The invariance under α_t follows from the corresponding invariance for $\omega_\beta^1(\Lambda)$ and similarily the KMS condition. The translation invariance under space translations follows from (4.42) and the homogenity of the generalized process given by E_∞^β. This homogenity follows from lemma 4.2. The cluster property follows from the fact that E_∞^β is strongly mixing with respect to space translations. This proves the theorem.

Remark: If we now define the pressure $p_\beta(V)$ at the temperature $\frac{1}{\beta}$ for the interacting field by

$$p_\beta(V) = \beta^{-1} \lim_{l\to\infty} |\Lambda_l|^{-1} \log(\mathrm{tr}(e^{-\beta H_l(\Lambda_l)})) \tag{4.43}$$

with $\Lambda_l = [-l,l]$, we get by lemma 4.1 and lemma 4.2 and its proof that

$$p_\beta(V) = p_\beta^o - \beta^{-1} e_\beta(V) \tag{4.44}$$

where V describes the interaction so that

$$H_l(\Lambda_l) = H_o(\Lambda_l) + \int_{-l}^{l} :V(\varphi(x)):dx$$

and p_β^o is the pressure for the free field given by (3.38) and (3.41), and $e_\beta(V)$ is the lowest eigenvalue of the periodic Hamiltonian

$$H_\beta = H_o^\beta + \int_0^\beta :V(\varphi(x)):dx \tag{4.45}$$

where $H_o^\beta = H_o([0,\beta])$ with periodic boundary conditions.

We see that theorem 4.1 gives a certain duality between the Gibbs state at temperature $1/\beta$ for the infinite volume interaction and the corresponding vacuum or zero temperature state for the interaction in a periodic box of length β. We shall denote this duality by the duality principle for the relativistic Gibbs state. This duality principle may also be expressed in terms of the Wightman functions or if we want also in terms of the Schwinger functions for the interaction.

Let $\varphi(x,t)$ be the interacting field at time t, i.e.

$$\varphi(x,t) = \alpha_t(\varphi(x)) \tag{4.46}$$

where $\varphi(x)$ is the field at time zero, and (4.46) is an equation between operator valued distributions in x for fixed t. The Wightman functions at temperature $1/\beta$ for the infinite volume interaction is given by

$$W_\infty^\beta(x_1,t_1,\dots,x_n,t_n) = \omega_\beta(\varphi(x_1,t_1)\dots\varphi(x_n,t_n)) , \tag{4.47}$$

and the Wightman functions for the field in a periodic box of length β, at temperature zero ($\beta = \infty$) is given by

$$W_\beta(x_1 t_1,\dots,x_n t_n) = (\Omega_\beta,\varphi(x_1,t_1)\dots\varphi(x_n,t_n)\Omega_\beta) \tag{4.48}$$

where Ω_β is a normalized eigenvector belonging to the lowest eigenvalue $e_\beta(V)$ for the H_β, Hamiltonian in a periodix box of length β

$$H_\beta = H_o^\beta + \int_o^\beta :V(\varphi(x)): dx \tag{4.49}$$

where V is either a polynomial which is bounded below or an exponential function of the type (4.3), and H_o^β is the free Hamiltonian in a periodic box of length β. We have then that (4.47)

is analytic in $t_1, \ldots, t_n$ in the domain $0 > \mathrm{Im} t_1 > \ldots > \mathrm{Im} t_n > -\beta$ and that (4.48) is analytic in $\mathrm{Im} t_1 > \ldots > \mathrm{Im} t_n$. The values at the imaginary points $t_k = -is_k$ for $0 < s_1, \ldots < s_n < \beta$ for (4.47) and $s_1 < \ldots < s_n$ for (4.48) is called the Schwinger functions

$$S_{\infty}^{\beta}(x_1, s_1, \ldots, x_n, s_n) = W_{\infty}^{\beta}(x_1, -is_1, \ldots, x_n, -is_n) \qquad (4.50)$$

and

$$S_{\beta}^{\infty}(x_1, s_1, \ldots, x_n, s_n) = W_{\beta}^{\infty}(x_1, -is_1, \ldots, x_n, -is_n) \qquad (4.51)$$

We may now express the duality principle from theorem 4.1 in terms of Wightman - and Schwinger functions, and this gives us the following duality theorem

<u>Theorem 4.2 (The duality theorem)</u>

Let $W_{\infty}^{\beta}(x_1, t_1 \ldots x_n, t_n)$ be the Wightman functions at temperature $1/\beta$ for the infinite volume interaction, and let W_{β}^{∞} be the usual Wightman functions at temperature zero ($\beta = \infty$) for the interacting field in a periodic box of length β. Let S_{∞}^{β} and S_{β}^{∞} be the corresponding Schwinger functions, i.e. the Wightman functions at imaginary time, so that $W_{\beta}^{\infty}(x_1 t_1, \ldots, x_n t_n)$ and $S_{\beta}^{\infty}(x_1 s_1, \ldots, x_n s_n)$ is periodic with period β in $x_1, \ldots, x_n$.

Then $W_{\infty}^{\beta}(x_1, t_1, \ldots x_n, t_n)$ is analytic in $0 > \mathrm{Im} t_1 > \ldots > \mathrm{Im} t_n > -\beta$, and $W_{\beta}^{\infty}(x_1, t_1, \ldots x_n, t_n)$ is analytic in $\mathrm{Im} t_1 > \ldots > \mathrm{Im} t_n$, and for the corresponding Schwinger functions we have

$$S_{\infty}^{\beta}(x_1, s_1, \ldots, x_n, s_n) = S_{\beta}^{\infty}(s_1, x_1, \ldots, s_n, x_n) .$$

Moreover the difference between the pressure for the free and the interacting field at temperature $1/\beta$ is equal to

$$p_\beta(0) - p_\beta(V) = \beta^{-1} e_\beta(V)$$

where $e_\beta(V)$ is the lowest eigenvalue for the interacting Hamiltonian in the periodic box of length β.

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