On the classification of complex Lindenstrauss-spaces

by

Gunnar Hans Olsen

Abstract

We prove the Lindenstrauss-Wulbert classification scheme for complex Banach spaces whose duals are L_1-spaces, and give some characterizations of the different classes by means of the unit ball in dual space. The work leans heavily on [8] and the real theory I am indebted to B. Hirsberg and A. Lazar for a preprint of [12] . Finally I would like to thank E. Alfsen and Å. Lima for making literature available and for helpful comments.

1. Preliminaries and notations.

Any unexplained notation in this paper will be standard or that of Alfsen's book [1]. Otherwise we will use the following notations:

T: unit circle in $\mathbb{C}$

V: a complex Banach-space

K: the unit ball in V^* with w^*-topology

M(K): The Banach space of complex regular Borel measure on K with total-variation as norm

$M_1(K)$: those measures in M(K) with norm ≤ 1

$M_1^+(K)$: probability measures on K. When F is a convex set then $\partial_e F$ will denote the extrempoints in F. If μ is a measure then $|\mu|$ is the total variation of μ. A measure μ is said to be maximal or a boundary measure if $|\mu|$ is maximal in Choquet's ordering. The set of maximal (Probability-) measures on K is denoted by $(M_1^+(\partial_e K))$ $M(\partial_e K)$.

We shall now repeat some results and definitions from [8].

A function $f \in C_{\mathbb{C}}(K)$ is said to be <u>T-homogeneous</u> if $f(\alpha k) = \alpha f(k)$ for all $\alpha \in T$, $k \in K$. The class of T-homogenous functions in $C_{\mathbb{C}}(K)$ is denoted by $C_{hom}(K)$. If $f \in C_{\mathbb{C}}(K)$, then the function

$$[\hom_T f](k) = \int \alpha^{-1} f(\alpha k)\, d\alpha \,, \quad k \in K$$

where $d\alpha$ is the unit Haar-measure on T, is continuous and T-homogeneous. It is now verified that $\hom_T$ is a norm-decreasing projection of $C(K)$ onto $C_{hom}(K)$.

Taking the adjoint of this projection on $M(K)$

$$\hom_T \mu = \mu \circ \hom_T \,,$$

we get a norm-decreasing w*-continuos projection of $M(K)$ onto a linear subspace denoted by $M_{hom}(K)$.

A measure $\mu \in M_{hom}(K)$ is called T-homogeneous and satisfies $\sigma_\alpha \mu = \alpha\mu$ where $\sigma_\alpha : K \to K$ is the homoeomorphism $k \rightsquigarrow \alpha k$ $\alpha \in T$, $k \in K$.

Each $v \in V$ can in a canonical way be regarded as an affine T-homogeneous w*-countinious function on K. Conversely by a result of Banach-Dieudonne ([1] corollary I.1.13), each affine T-homogeneous function can be extended to a w*-continious complex-linear functional on V^*, ie. to an element of V. We may therefore identify V with the affine functions in $C_{hom}(K)$.

If $\mu \in M(K)$ then the <u>resultant</u> of μ is defined to be the unique point $r(\mu) \in V^*$ satisfying

$$r(\mu)(v) = \mu(v) \quad \text{for all } v \in V .$$

If $\mu \in M_1^+(K)$ then it can be proved that $r(\mu)$ coincides with the barycenter of μ. (See [8] for a proof). Moreover it is readily verified that $r: M(K) \to V^*$ is a w*-continious norm-decreasing linear surjection.

Let X be a topological space and $\mu \in M^+(K)$. A function $f: K \to X$ is measurable if for every $\epsilon > 0$ there is a compakt set $D \subseteq K$ such that $\mu(K \setminus D) < \epsilon$ and $f|D$ is continious. If $X = \mathbb{R}$ or $\mathbb{C}$ then this definition coincides with the customary one by virtue of Lusin's theorem.

Let $\mu \in M(K)$. Then there is a compleks $|\mu|$-measurable function φ on K with $|\varphi| = 1$ a.e. μ such that $\mu = \varphi|\mu|$. This representation is called the <u>polardokomposition</u> for μ and is unique up to zero sets.

Since $\varphi: K \to \mathbb{C}$ is $|\mu|$-measurable it follows that $\omega: K \to K$ defined by

$$\omega(p) = \varphi(p) \cdot p$$

also is measurable. Hence by Lusin's theorem $\omega(|\mu|)$, defined by
$$\omega(|\mu|)(f) = \int f\circ\omega \, d|\mu| \quad f \in C_{\mathbb{C}}(K) ,$$
is a regular Borel-measure. (This definition is due to Phelps). Clearly $\|\omega(|\mu|)\| \leq \|\mu\|$ and the other statements in the following lemma are proved in [8]

<u>Lemma 1</u>

Let $\mu \in M(K)$, then

a) $r(\mathrm{hom}_T \mu) = r(\mu)$

b) $r(\omega(|\mu|)) = r(\mu)$

c) $\|\omega(|\mu|)\| = \|\mu\|$

d) $\mathrm{hom}_T \omega(|\mu|) = \mathrm{hom}_T \mu$

e) If μ is maximal, then so are $\omega(|\mu|)$ and $\mathrm{hom}_T \mu$.

Lemma 2

Let $\mu_1, \mu_2 \in M(K)$ and put $\mu = \mu_1 + \mu_2$. If $\|\mu\| = \|\mu_1\| + \|\mu_2\|$ then μ_1 and μ_2 admit the same polardecomposition, i.e. there is a complex measureable function φ on K with $|\varphi| = 1$ a.e $|\mu|$ such that $\mu_1 = \varphi\cdot|\mu_1|$ $\mu_2 = \varphi\cdot|\mu_2|$.

Proof

Since $\|\mu\| = \|\mu_1\| + \|\mu_2\|$ we easily get $|\mu| = |\mu_1| + |\mu_2|$. In particular $|\mu_1|$, $|\mu_2| \ll |\mu|$, so by the Rador-Nikodyn theoren there is non-negative measurable functions f_1, f_2 such that $|\mu_1| = f_1\cdot|\mu|$, $|\mu_2| = f_2\cdot|\mu|$. Let $\mu = \varphi|\mu|$, $\mu_1 = \varphi_1|\mu|$, $\mu_2 = \varphi_2\cdot|\mu|$ be the polardecompoitions. Then

$$\begin{aligned}
\varphi\cdot|\mu| &= \varphi_1\cdot|\mu_1| + \varphi_2\cdot|\mu_2| \\
\varphi(f_1+f_2)\cdot|\mu| &= (\varphi_1\cdot f_1)\cdot|\mu| + (\varphi_2\cdot f_2)\cdot|\mu| \\
\varphi(f_1+f_2) &= \varphi_1\cdot f_1 + \varphi_2\cdot f_2 \quad \text{a.e. } |\mu| \\
\varphi = \varphi_1 = \varphi_2 & \quad \text{a.e. } |\mu|
\end{aligned}$$

The above lemma immediately gives

Corollary 3

Let μ_1, $\mu_2 \in M(K)$ and put $\mu = \mu_1+\mu_2$.
If $\|\mu\| = \|\mu_1\| + \|\mu_2\|$, then
$\omega(|\mu|) = \omega(|\mu_1|) + \omega(|\mu_2|)$

2 Complex Lindenstrauss-spaces and complex affine selections.

A complex Banach-space W is called an L-space if $W \cong L^1_{\mathbb{C}}(Q,B,m)$ for some measure-space (Q,B,m).

A complex Lindenstrauss-space is a complex Banach-space whose dual is an L-space.

Theorem 4

If W is an L-space and $\pi: W \to W$ a projection with norm one, then $\pi(W)$ is an L-space

Proof

see [8]

Corollary 5

If V is a Lindenstrauss-space $\pi: V \to V$ a projection with norm one, then $\pi(V)$ is a Lindenstrauss-space.

Proof

Let π^* be the adjoint projection. Then the restriction map $\gamma: V^* \to (\pi V)^*$ takes $\pi^*(V^*)$ isometrically onto $(\pi V)^*$ and π^* is a projection with norm one.

In [8] Effros proved that complex Lindenstrauss space may be characterized by:

If μ, $\nu \in M_1^+ \partial_e K)$ and $r(\mu) = r(\nu)$, then: $\hom_T \mu = \hom_T \nu$

This theorem will be fundamental in the following, and we shall refer to it as Effros'characterization.

A map $\varphi: K \to M_1(K)$ is said to be a complex affine selection if φ is affine, $\varphi(\alpha\, k) = \alpha\, \varphi(k)$ and $r(\varphi(k)) = k$; $k \in K, \alpha \in T$.

φ is called T-homogenous if $\varphi(k) = \hom_T \varphi(k)$, $k \in K$.

Theorem 6

V is a Lindenstrauss space if and only if there is a complex affine selection on K . Moreover if a complex affine selection exist, then there is a unique T-homogenous complex affine selection φ on K and $\varphi(k)$ is maximal for all $k \in K$.

Proof

Necessity

Put $\varphi(x) = \hom_T \nu_x$ where ν_x is a maximal measure in $M_1^+(K)$ with $r(\nu_x) = x$. φ is well-defined by Effros' characterisation, and from the proof of that it also follows that φ is a complex affine selection.

Sufficiency

Assume $\varphi: K \to M_1(K)$ is a complex affine selection. Let $\bar{\varphi}: V^* \to M(K)$ be defined by $\bar{\varphi}(k) = \|k\| \ \varphi(\frac{k}{\|k\|})$. Then $\bar{\varphi}$ is complex linear and extends φ so $\|\bar{\varphi}\| \leq 1$. Since r is a norm-decreassing projection, we get

$\|k\| = \|r(\bar{\varphi}(k))\| \leq \|\bar{\varphi}(k)\| \leq \|k\|$, $k \in K$.

Hence $\bar{\varphi}$ is a isometry.

Let now $\pi: M(K) \to \bar{\varphi}(V^*)$ be defined by $\pi(\mu) = \bar{\varphi}(r(\mu))$. Then π is a projection with norm one onto $\bar{\varphi}(V^*)$, and since $M(K)$ is an L-space it follows from theorem 4 that $\bar{\varphi}(V^*)$ is an L-space. Hence V^* is an L-space, which implies V is a Lindenstrauss space.

Uniqueness

Let $x \in K$ with $\|x\| = 1$. From Lemma 1 it follows:

$1 = \|x\| = \|r(\omega(|\varphi(x)|))\| \leq \|\omega(|\varphi(x)|)\| \leq \|\varphi(x)\| \leq 1$,

so $\omega|\varphi(x)| \in M_1^+(K)$.

Let $\nu \in M_1^+(K)$ with $r(\mu) = x$, $f: K \to \mathbb{R}$ continous and convex, and $\epsilon > o$. Choose simple probability measure $\sum_{i=1}^{n} \alpha_i \ \epsilon_{y_i}$ such that ([1] proposition I.2.3)

(2.1) $\nu_x(f) \leq (\sum_{i=1}^{n} \alpha_i \ \epsilon_{y_i})(f) + \epsilon$, $\sum_{i=1}^{n} \alpha_i \ y_i = x$

Since φ is affine, we get $\varphi(x) = \sum_{i=1}^{n} \alpha_i \ \varphi(y_i)$. Moreover

$1 = \|\varphi(x)\| \leq \sum_{i=1}^{n} \alpha_i \ \|\varphi(y_i)\| \leq \sum_{i=1}^{n} \alpha_i = 1$, so by corollary 3

$$\omega(|\varphi(x)|) = \sum_{i=1}^{n} \alpha_i \ \omega \ (|\varphi(y_i)|)$$

Now by lemma 1

$$\sum_{i=1}^{n} \alpha_i \epsilon_{y_i} < \sum_{i=1}^{n} \alpha_i \omega(|\varphi(y_i)|)$$

Since f is convex, we get from (2.1):

$$\nu_x(f) \leq [\sum_{i=1}^{n} \alpha_i \omega(|\varphi(y_i)|)](f) + \epsilon = [\omega(|\varphi(x)|)](f) + \epsilon$$

Hence $\omega(|\varphi(x)|)$ is maximal and is <u>the only maximal probability measure with barycenter x</u>.

By lemma 1 $\mathrm{hom}_T\, \omega(|\varphi(x)|)$ is maximal. But if φ is T-homogenous, we get from lemma 1

$$\mathrm{hom}_T\, \omega(|\varphi(x)|) = \mathrm{hom}_T\, \varphi(x) = \varphi(x)$$

Theorem follows now from the relation $\varphi(x) = \|x\| \varphi\left(\frac{x}{\|x\|}\right)$, $x \in K$.

The proof above also shows

<u>Corollary 7</u>

If V is a Lindenstrauss-space then every $k \in K$ with norm one can be represented by a unique maximal probability measure

Now by [1] theorem II.3.6

<u>Cor llary 8</u>

If V is a Lindenstrauss-space and F is a w*-closed face in K, then F is a compakt simplex.

<u>Remark</u>

The above corollary may of course be proved by a direct argument, since a face-cone in an L-space must be a lattice-cone.

<u>Theorem 9</u>

The following statements are equivalent

i) V Lindenstrauss-space with $\partial_e K \cup \{0\}$ w*-closed.

ii) There is a continous complex affine selection $\varphi: K \to M_1(K)$

iii) For each $f \in C_{hom}(K)$ there exists $v \in V$ such that $f|\partial_e K = v|\partial_e K$.

Proof

Put $\varphi(x) = \hom_T \mu_x$, where μ_x is a maximal probability measure with $r(\mu_x) = x$. Then, as in the proof of theorem 6, φ is a complex affine T-homogeneous selection. We first prove $\varphi(K)$ is compact.

Let $\{\mu_\gamma\} \subset \varphi(K)$ be a net which converges to $\mu \in M_1(K)$. Let $f \in C_{\mathbb{C}}(K)$. Then , since each μ_γ is T-homogeneous:

$\mu(f) = \lim \mu_\gamma(f) = \lim [\hom_T \mu\gamma] (f)$

$= \lim \mu_\gamma(\hom_T f) = \mu(\hom_T f) = \hom_T \mu \ (f)$,

which proves μ is T-homogeneous. By lemma 1 each μ_γ is maximal, and since $\partial_e K \cup \{0\}$ is closed it follows from [1] that $\operatorname{supp}(\mu) \subseteq \partial_e K \cup \{0\}$.

But since μ is T-homogeneous, $\mu(\{0\}) = 0$, hence μ is maximal ([1] proposition I.4.5)

Let $k \in \partial_e K$. Then by lemma 1: $\nu = \omega(|\mu|) + \frac{1}{2}(1 - \|\omega(|\mu|)\|)(\epsilon_k+\epsilon_{-k})$ is a maximal probability measure. By lemma 1 and since μ is T-homogenous we get

$$\varphi(r(\mu)) = \hom_T \nu = \hom_T (\omega(|\mu|)) = \hom_T \mu = \mu$$

Thus $\mu \in \varphi(K)$, which implies $\varphi(K)$ is compact.

The map $\mu \to r(\mu)$ is 1-1 from $\varphi(K)$ onto the compact $\varphi(K)$, thus the inverse map is continious, i.e. φ is continious.

ii) $\Rightarrow$ iii)

If φ is a complex affine continious selection on K , then so is $\hom_T \circ\varphi$. Hence we may assume that φ is T-homogeneous. By ii) the map $x \to [\varphi(x)](f)$ $x \in K$, is continious, affine and T-homogeneous for all $f \in C_{\mathbb{C}}(K)$. But if f is T-homogenous it follows from theorem 6 , Effros' characterization and [1] corollary I.2.4:

$$f(x) = [\varphi(x)](f) \quad \text{for all } x \in \partial_e K .$$

iii) $\Rightarrow$ i)

When $f \in C_{hom}(K)$ then by iii) and Bauer's Maximum Principle ([1] theorem I.5.3) there is a unique function v_f in V such that

(2.2) . $f|\partial_e K = v_f|\partial_e K$ and $\|f\| \geq \|v_f\|$.

Assume $\mu, \nu \in M_1^+(\partial_e K)$ with $r(\mu) = r(\nu) = k$

Let $f \in C_{hom}(K)$. Then by (2.2):

$\mu(f) = \mu(v_f) = v_f(k) = \nu(v_f) = \nu(f)$. Hence

$hom_T \nu = hom_T \mu$, so by Effros' characterization is V a Lindenstrauss-space.

It remains to prove that $\partial_e K \cup \{0\}$ is closed.

By (2.2) it suffices to prove

$$(2.3) \quad \partial_e K \cup \{0\} = \bigcap_{f \in C_{hom}(K)} \{x \in K | f(x) = v_f(x)\} .$$

a) Assume $x \in K$ and $\|x\| < 1$

Let $g: \cup\{\alpha x | \alpha \in T\} \to \mathbb{C}$ be defined by $g(\alpha x) = \alpha$. Then g is continious. Extend g by Tietze to $\tilde{g}: K \to C$ with $\|\tilde{g}\| = \|g\|$. Put $f = hom_T \tilde{g}$. Then $f(x) = 1$ and $\|f\| = 1$.

Hence

$$\begin{aligned} f(x) = 1 = \|f\| \geq \|v_f\| &\geq |v_f(\tfrac{x}{\|x\|})| \\ &= \tfrac{1}{\|x\|} |v_f(x)| > |v_f(x)| \quad . \end{aligned}$$

b) Assume $x \in K$ with $\|x\| = 1$ and there is no $v \in V$ such that $\|v\| = 1$ and $v(x) = 1$.

Construct f as above. Then $f(x) = 1 \neq v_f(x)$

c) Assume $x \in K$, $\|x\| = 1$, $x \notin \partial_e K$ and there is $v \in V$ such that $v(x) = 1 = \|v\|$.

Then $F = \{y \in K \mid v(y) = 1\}$ is a w*-closed face in K . Since $x \notin \partial_e K$ there is $y, z \in F$ such that

$$x = \tfrac{1}{2}y + \tfrac{1}{2}z \qquad y, z \neq x .$$

By the Hahn-Banach teorem there is a real convex continious function g_F on F such that

$g_F(y) = g_F(z) = 1 \quad , \quad g_F(x) = 0 .$

Define g on $\bigcup_{\alpha\in T} \alpha F$ by $g(\alpha k) = \alpha g_F(k) \quad \alpha \in T \quad , \quad k \in F .$

g is well defined since F is a face. Extend g to $\tilde{g} \in C_C(K)$ by Tietze with $\|\tilde{g}\| = \|g\|$ and put $f = \mathrm{hom}_T \tilde{g}$. Then $f|F = g_F$

Let μ be a maximal probability measare on K with $r(\mu_x) = x$. Since F is a face , supp $(\mu_x) \subseteq F$ and μ_x is seen to be maximal on F . Hence

$$v_f(x) = \int_K v_f \, d\mu_x = \int_F v_f \, d\mu_x = \int_F g_F d\mu_x$$

By corollary 8 F is a simplex so [1] theorem II.3.7 gives

$$v_f(x) = \int_F g_F \, d\mu_x = g_F(x) = \tfrac{1}{2}(\hat{g}_F(y) + \hat{g}_F(z))$$

$$\geq 1 > 0 = f(x) .$$

($\hat{g}_F$ denotes the upper envelope of g_F , see [1] p. 4)

(2.3) now follows from a) , b) and c) and the proof is complete.

Notes

Theorem 6 was proved for simplexes by Namcoka and Phelps, and for real Lindenstrauss-spaces by Ka-Sing Lau [18] , and Fakhoury in a weaker form [24]. However, as pointed to us by Hirsberg, there exist a very simple proof in the simplex-case, and it is this idea we have used in the uniqueness-part. Ka-Sing Lau [18] also proved theorem 9 in the real case. We have proceeded in the same way, but the proof is somewhat simplified.

---

3 Complex C_σ-spaces

A compact Hansdorf space X is called a T_σ-space if there exists a map $\sigma \colon TxX \to X$ such that

i) σ is continious

ii) $\sigma(\alpha, \sigma(\beta,x)) = \sigma(\alpha\beta , x) \quad \alpha , \beta \in T , x \in X$

iii) $\sigma(1, x) = x$

Let X be a T_σ-space . Then each $\alpha \in T$ defines a homeomorphism $\sigma_\alpha : X \to X$ by $\sigma_\alpha(x) = \sigma(\alpha, x)$, $x \in X$ (σ_α and $\sigma_{\alpha-1}$ are continious by i), ii) and iii) imply that $\sigma_\alpha \circ \sigma_{\alpha-1}$ is the identity on X)

A function $f \in C_C(x)$ is said to be <u>σ-homogenious</u> if $f(\sigma_\alpha x) = \alpha f(x)$ for all $\alpha \in T$, $x \in X$. The class of σ-homogeneous functions in $C_C(X)$ is denoted by $C_\sigma(X)$

A complex <u>C_σ-space</u> is a complex Banach-space which is isometric to $C_\sigma(X)$ for some T_σ-space X .

If $f \in C_C(X)$ then the function

(3.1) $[\pi_\sigma f](p) = \int \alpha^{-1} f(\sigma_\alpha p) d\alpha$, $p \in X$

where $d\alpha$ is the unit Haar-measure is seen to be continious and σ-homogeneous. π_σ is easily shown to be a normdecreasing projection of $C_C(x)$ onto $C_\sigma(X)$.

Hence by corollary 5 is complex C_σ-spaces Lindenstrauss-spaces.

If Y is locally-compact Hansdorf space then <u>$C_o(Y)$</u> will denote the continious functions on Y vanishing at infinit.

<u>Proposition 10</u>

If Y is locally-compact Hausdorf space then $C_o(Y)$ is a C_σ-space.

Proof.

Let $X = (T \times Y) \cup \{\omega\}$ be the one point compactifisation, and define

$\sigma : T \times X \to X$ by

$$\sigma(\alpha, x) = \begin{cases} (\alpha\, \alpha o, y) & \text{if } x = (\alpha o , y) \in T \times X \\ \omega & \text{if } x = \omega \end{cases}$$

i) σ is easly seen to be continious

ii) Let $x = (\alpha_o , y) \in T \times Y$, $\beta \in T$. Then

$\sigma(\alpha , \sigma(\beta , x)) = \sigma(\alpha , \sigma(\beta , (\alpha_o , y)) = \sigma(\alpha, (\beta\, \alpha_o , y))$

$= \sigma(\alpha\ \beta\ \alpha_o\ ,\ y) = \sigma\ (\alpha\ \beta\ ,\ (\alpha_o,\ y)) = \sigma(\alpha\ \beta\ ,\ x)$.

Moreover:

$$\sigma(\alpha, \sigma(\beta\ ,\ \omega)) = \sigma(\alpha\ ,\ \omega) = \omega = \sigma(\alpha\ \beta\ ,\ \omega).$$

iii) is verified in a similar way as II) .

Hence X is a T_σ-space . Each $f \in C_o(Y)$ can in a canonical way be regared as a continious function on $(\{1\} \times Y) \cup \{\omega\}$ vanishing at ω . Extend f to $\tilde{f}$ on X by $\tilde{f}(\alpha, y) = \alpha f(y)$, $(\alpha, y) \in T \times Y$. Then $\tilde{f}$ is continious and σ-homogenous. The map $f \rightsquigarrow \tilde{f}$ defined above is seen to be an isometry of $C_o(Y)$ into $C_\sigma(X)$. Since each $g \in C_\sigma(X)$ satisfies $g(\omega) = o$, the above map is surjective , i.e. $C_o(Y)$ is a C_σ-space.

Let now X be a T_σ-space and $V = C_\sigma(X)$. A subset $Z \subset X$ is called <u>σ-symmetric</u> if $x \in Z$ implies $\sigma_\alpha(x) \in Z$ for all $\alpha \in T$. Observ that if Z is σ-symmetric then $X \setminus Z$ is σ-symmetric as well.

Let ρ embed X into K in the canonical way. Then ρ is continious

<u>Lemma 11</u>

$$\partial_e K = \{\rho(x) |\ \sigma_\alpha\ (x) \neq x \text{ for all } \alpha \in T\setminus\{1\}\ x\ \cdot X\}$$

and $\rho(x) \subseteq \partial_e K \cup \{0\}$

Proof

First we observ that $\alpha\rho(x) = \rho(\sigma_\alpha x)$ when $\alpha \in T$, $x \in X$ and $\rho(x) = 0$ if $\sigma_\alpha(x) = x$ for some $\alpha \in T\setminus\{1\}$. Hence by [5] p 441 lemma 6

$$\partial_e K \subseteq \{\rho(x)\ |\ \sigma_\alpha(x) \neq x \text{ for all } \alpha \in T\ \{1\}\ ,\ x \in X\}\ .$$

Let $x \in X$ and assume $\sigma_\alpha(x) \neq x$ for all $\alpha \in T\setminus\{1\}$. We shall prove that $\rho(x) \in \partial_e K$. We use a σ-symmetric modification of the argument given for that lemma of [5] .

Assume

(3.2) $\rho(x) = \frac{1}{2} k_1 + \frac{1}{2} k_2$, $k_1, k_2 \in K$.

Let fo $\in C_\sigma(X)$ with $\|f\| \leq 1$ and assume f vanish on a open neigbourhood $N(x)$ of x . Since f_o is σ-homogenous we may assume $N(x)$ is σ-symmetric. Let $h: \{\sigma_\alpha(x) \mid \alpha \in T\} \cup \{X\ N(x)\} \to \mathbb{C}$ be defined by $h(\sigma_\alpha x) = \alpha$, $\alpha \in T$, $h(y) = 0$ if $y \in X \setminus N(x)$.

Extend h by Tietze to $\tilde{h}$ on X with $\|\tilde{h}\| = \|h\|$ and put $g = \Pi\sigma(\tilde{h})$. Then

$$g(x) = 1 \quad , \quad g(y)=0 \quad \text{if} \quad y \notin N(x) \quad \text{and} \quad \|g\| \leq 1$$

Thus by (3.2)

$1 = g(x) = \rho(x)(q) =$

$= \frac{1}{2}(k_1(g) + k_2(g)) \leq \frac{1}{2} (|k_2(g)| + |k_2(g)|) \leq 1$

Hence $k_1(g) = k_2(g) = 1$.

Similarly we get $k_1(g+f_o) = k_2(g+f_o) = 1$

Hence

(3.3) $k_1(f_o) = k_2(f_o) = 0$

Let $f_1 \in C_\sigma(X)$ with $\|f\| \leq 1$ and $f(x) = 0$

For each integer $n \geq 2$ there is an open σ-symmetric neighbourhood $N_n(x)$ such that

$$|f_1(y)| \leq {}^1/n \quad \text{if} \quad y \in N_n(x)$$

Let $M_n(x)$ be an open set containing x such that

$$M_n(x) \subseteq M_n(x) \subseteq N_n(x) .$$

Since $N_n(x)$ is σ-symmetric:

$$\bigcup_{\alpha \in T} \sigma_\alpha(M_n(x)) \subseteq \bigcup_{\alpha \in T} \sigma_\alpha\, M_n(x) \subseteq N_n(x)$$

and $\bigcup_{\alpha \in T} \sigma_\alpha\, M_n(x) = \sigma(T \times M_n(x))$ is closed.

As above we may construct $g_n \in C_\sigma(X)$ such that

$$\|g_n\| \leq {}^1/n \quad , \quad g_n(y) = 0 \quad \text{if} \quad y \notin N_n(x)$$

and $g_n(y) = f_1(y)$ if $y \in \sigma(T \times M_n(x))$

We get

$$f_1 - g_n \to f_1 \quad \text{uniformly and} \quad \|f_1 - g_1\| \leq 1$$

Now since $f_1 - g_n$ vanishes on $\sigma(T \times M_n(x))$, we get by (3.3):

$$0 = \lim k_1 (f_1 - g_n) = k_1(f_1)$$

$$0 = \lim k_2 (f_1 - g_n) = k_2(f_1)$$

Hence $\rho(x)(f) = 0$ implies $k_1(f) = k_2(f) = 0, f \in C_\sigma(X)$.

By [5] lemma 3.10 there are $\alpha_1, \alpha_2 \in C$ such that $k_1 = \alpha_1 \rho(x)$, $k_2 = \alpha_2 \rho(x)$. But $\|k_1\|, \|k_2\| \leq 1$ so $|\alpha_1|, |\alpha_2| \leq 1$ and by (3.2) we get $\alpha_1 = \alpha_2 = 1$ i.e. $\rho(x) = k_1 = k_2$ and the proof of lemma is complete.

Theorem 12

V is a C_σ-space if and only if V is Lindenstrauss-space and $\partial_e K \cup \{0\}$ is closed.

Proof

If V is a C_σ-space, then V is a Lindenstrauss-space and $\partial_e K \cup \{0\}$ is closed by virtue of lemma 11. Conversely assume V is a Lindenstrauss-space with $X = \partial_e K \cup \{0\}$ closed. X may be organized to a T_σ-space by scalarmultiplication. Then theorem 9 iii) completes the proof.

A complex C_Σ-space is a Banachspace which is isometric to a $C_\sigma(X)$ for some T_σ-space X, where σ_α has no fixed points if $\alpha \in T \setminus \{1\}$.

Now as in the proof for Proposition 10 we get

Proposition 13 If X is a compact Hausdorf-space, then $C_C(X)$ is a C_Σ-space.

The next theorem may be proved by a method similar to that used in proving theorem 12.

Theorem 14

V is a C_Σ space if and only if V is a Lindenstrauss-space and $\partial_e K$ is closed

Remark

Theorem 14 also proved proposition 13, just as theorem 12 proves proposition 10, by virtue of [5] p. 441 lemma 6.

Notes

The real C_σ-spaces were introduced and studied by Jerison [16]. His results are presented in Day's book [4] p. 87 - 93. The real version of theorem 12 was suggested by Effros [7], and proved by Ka-Sing Lau [18]. Theorem 14 is due to Lindenstrauss-Walbert. We have proceeded as in [18] .

4 Complex simplex spaces

Let (Q, B, m) be a measure space and assume $V^* = L^1_C(Q, B, m)$ Let $\varphi \in L^\infty_C(Q, B, m)$ with $|\varphi| = 1$ a.e. m

Then

(4.1) $S = \{\varphi\cdot p \mid p \in K,\ p \geq 0 \text{ a.e } m,\ \|p\| = 1\}$.

is seen to be a maximal (with respect to inclusion) face in K . Conversly since the norm must be additive on a face-cone [2] , we get that all maximal faces in K are on the form given in (4.1).

If $p \in \partial_e K$, then it is not hard to see that $p = a\chi_A$, where $a \in C$ and χ_A is the characteristic function of an atom $A \in B$. Thus if S is a maximal face in K and $p \in \partial_e K$, then $\alpha p \in S$ for some $\alpha \in T$. Hence

(4.2) $V \cong V|S$.

A complex Lindenstrauss-space V is called a <u>complex simplex-space</u> if there is a maximal face $S \subset K$ such that $\text{conv}\,(S \cup \{0\})$ is w*-closed.
(Observ that this definition coincides with Effros' in the real case [6])

<u>Lemma 15</u>

S is a split-face (See [1] p. 133) in $\text{conv}\,(S \cup -iS)$.

Proof
Assume $\lambda_1 x_1 + (1-\lambda_1)(-ix_2) = \lambda_2 y_1 + (1-\lambda_2)(-iy_2)$,
where $x_i, y_i \in S$, $0 \le \lambda \le 1$, $i = 1, 2$.
Since S is a maximal face in K, there is $\varphi \in V^{**}$ such that $\varphi|S \equiv 1$. Thus $\lambda_1 = \lambda_2 = \lambda$.
Let μ_i, $\nu_i \in M_1^+(\partial_e K)$ $i = 1, 2$, with
$r(\mu_1) = x_1$, $r(\mu_2) = -ix_2$, $r(\nu_1) = y_1$, $r(\nu_2) = -iy_2$
Since S is a face and $S_o = (S \cup \{0\})$ is w*-compact, we get

(4.3) $$\text{supp}\,(\mu_1),\ \text{supp}\,(\nu_1) \subseteq S_o$$
$$\text{supp}\,(\mu_2),\ \text{supp}\,(\nu_2) \subseteq -iS_o$$

Since the barycenter-map is normdecreasing, we also get

(4.4) $$\mu_i(\{0\}) = \nu_i(\{0\}) = 0, \quad i = 1,2.$$

Let now $f \in C_{\mathbb{R}}(S_o)$ with $f(0) = 0$. Extend f to a T-homogenous function $\tilde{f}$ on K. By Effros' characterization we get

$$\lambda\mu_1(\tilde{f}) + (1-\lambda)\,\mu_2(\tilde{f}) = \lambda\nu_1(\tilde{f}) + (1-2)\,\nu_2(\tilde{f})$$

But $\tilde{f}$ is real on S_o, imaginary on $-iS_o$, so by (4.3):

$$\mu_1(f) = \nu_1(f)$$

But by (4.4) this holds for any $f \in C_R(S_o)$. Hence $\nu_1 = \mu_1$, which gives $x_1 = y_1$ and the proof is complete.

<u>Corollary 16</u>

Any $z \in Z_o = \text{conv}\,(S \cup -i\,S \cup \{0\})$ may be written uniquely in the form:

$$z = \alpha_1 x_1 + \alpha_2 (-ix_2) + \alpha_3 0$$

where $\alpha_i \geq 0$, $i = 1, 2, 3$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$, $x_1, x_2 \in S$

Lemma 17

Let a be a real, affine w*-continious on $S_o = \text{cow}(S \cup \{0\})$ with $a(0) = 1$. Then a amy be extended to a real affine w*-continious c on Z_o such that $c|-iS_o \equiv 0$.

Proof

Let $c: Z_o \to R$ be defined by

$$c(z) = \alpha_1(x_1) \quad , \quad z \in Z_o \quad ,$$

where $z = \alpha_1 x_1 + \alpha_2(-ix_2) + \alpha_3 \cdot 0$ is the unique decomposition from corollary 16.

c is easily verified to be affine. To see that c is continious, let $\{z^\gamma\} \subseteq Z_o$ be a net converging to $Z \in Z_o$. Decompose

$$Z^\gamma - \alpha_1{}^\gamma x_1{}^\gamma + \alpha_2{}^\gamma (-ix_2{}^\gamma) + \alpha_3{}^\gamma 0$$

$$Z = \alpha_1 x_1 + \alpha_2 (-ix_2) + \alpha_3 0$$

by corollary 16. By compactness we may assume $\{x_1{}^\gamma\}$, $\{x_2{}^\gamma\}$, $\{\alpha_1{}^\gamma\}$, $\{\alpha_2{}^\gamma\}$ all converge. Let y_1, y_2, β_1, β_2 be the limit-points.

Then

$$z^\gamma \to \beta_1 y_1 + \beta_2(-iy_2) = \beta_1 \|y_1\| \left(\frac{y_1}{\|y_1\|}\right) + \beta_2 \|y_2\| \left(-i\frac{y_2}{\|y_2\|}\right) + \beta' \cdot 0$$

where $\quad \beta' = 1 - (\|y_1\| \beta_1 + \|y_2\| \beta_2)$

(The case $\|y_1\| = 0$ or $\|y_2\| = 0$ goes similar)

Now since the decomposition in corollary 16 is unique, we get

$$\alpha_1 = \beta_1 \|y_1\| \quad , \quad x_1 = \frac{y_1}{\|y_1\|}$$

Hence

$$c(z^\gamma) = \alpha_1^\gamma \; a(x_1^\gamma) \to \beta_1 \; a(y_1) =$$

$$\beta_1 \|y_1\| \; a\left(\frac{y}{\|y_1\|}\right) = \alpha_1 \; a(x_1) = c(z) \; ,$$

which proves c is continious. Clearly c extends a and $c|-iS_o \equiv 0$, so the proof is complete.
When H is compact convex, then $A(H)$ ($A_o(H)$) will denote the space of complex affine continious functions on H (vanishing in a fixed extrempoint x_o in H).

Theorem 18

The following statements are equivalent

i) V is a simplex-space

ii) $V \cong A_o(S_o)$ for some simplex S_o

iii) $V \cong A$, where A is closed linear subspace of $C_C(X)$, X compact Hausdorf, such that A is self-adjoint and ReA is a real simplex space.

Proof

i) => ii)

Assume V is a Lindenstrauss space with a maximal face $S \subset K$ such that

$$S_o = \text{conv}\,(S \cup \{0\}) \text{ is } w^*\text{-compact.}$$

We have by (4.2)

$$(4.5) \qquad V \cong V|S_o \subseteq A_o(S_o) \qquad (x_o = 0)$$

Let $a \in A_o(S_o)$ and put $b_1 = Rea$, $b_2 = Ima$.
Then b_1, b_2 are real affine w^*-continious functions on S_o with $b_1(0) = b_2(0) = 0$, and may therefor by corollary 17 be extended to affine w^*-continious functions $\tilde{b}_1$, $\tilde{b}_2$ on Z_o such that

$$(4.6) \qquad \tilde{b}_1|-iS_o = 0\,, \tilde{b}_2|-iS_o = 0$$

By [1] corollary I.1.5 there are sequences $\{b_1^n\}$, $\{b_2^n\}$ of w^*-continious real linear functionals on V^* such that

$$b_1^n \to b_1\,,\ b_2^n \to b_2 \text{ uniformly on } Z_o$$

Let a_1^n, $b_1^n \in V$ $n = 1, 2, \dots$, be defined by

$$a_1^n(x) = b_1^n(x) - i\, b_1^n(ix), \; x \in V^*$$
$$a_2^n(x) = b_2^n(x) - i\, b_2^n(ix), \; x \in V^*$$

Then by (4.2) and (4.6) $a_1^n + i\, a_2^n$ converges to an element $c \in V$ satisfying $c|S_o = a$.

By [1] theorem II.3.6 and corollary 7 is S_o a simplex, so the proof of ii) is complete.

ii) => iii) trivial.

iii) => ii)

Let $p \in (\mathrm{Re}A)^*$ and put

$$\tilde{p}(a) = p(\mathrm{Re}a) + i\, p(\mathrm{Im}a)$$

Then $\tilde{p} \in A^*$ with $\|\tilde{p}\| = \|p\|$ and p has only this extension in A^*, so we may regard $(\mathrm{Re}A)^*$ as a subset of A^*.

Let $S_o = \{p \in A^* | p(a) \geq 0 \text{ all } a \in [\mathrm{Re}A]^+\}$ and $\psi: A \to A_o(S_o)$ be defined by $[\psi(a)](p) = p(a)$, $p \in S_o$, $a \in A$.

Then ψ is an isometry since S_o contains the evaluations.

Theorem 2.2 in [6] implies that ψ is onto and S_o is a simplex.

ii) => i)

By the Hustad-Hirsberg theorem ([11] and [13]) each $p \in A(S_o)^*$ may be represented by a measure $\mu \in M(\partial_e S_o)$ such that $\|\mu\| = \|p\|$. Moreover, since S_o is a simplex this representation is unique. Hence

$$A(s_o)^* \cong M(\partial_e S_o) \tag{4.7}$$

and the latter is proved to be an L-space in [8] (See proof of theorem 4.3)

Let $S = \cup \{F | F \text{ face in } S_o, \; F \cap \{x_o\} = \emptyset\}$

Then S is a G_δ ([1] proposition II.6.5). Let $e: M(\partial_e S_o) \to M(\partial_e S_o)$ be defined by

$$e(\mu)(c) = \mu(C \cap S), \; C \text{ Borel in } S_o.$$

Then e is seen to be an L-projection in the sense of [2].

We shall prove

(4.8) $\qquad e[M(\partial_e S_o)] \cong A_o(S_o)^*$,

which implies $A_o(S_o)$ is a Lindenstrauss-space. Let $p \in A_o(S_o)^*$ and extend to $\tilde{p}$ on $A(S_o)$ with $\|\tilde{p}\| = \|p\|$ by Hahn-Banach. Then by (4.7) there is a unique measure $\mu \in M(\partial_e S_o)$ which represent $\tilde{p}$ and satisfies $\|\mu\| = \|\tilde{p}\|$

Let $\varepsilon > 0$. Choose $a \in A_o(S)$ with $\|a\| \leq 1$ such that $|p(a)| > \|p\| - \varepsilon$. Then

$$\|\mu\| - \varepsilon = \|\tilde{p}\| - \varepsilon = \|p\| - \varepsilon < |p(a)|$$
$$= |\int a d\mu| = |\int_S a \, d\,\mu| \leq |\mu|(S)$$
$$\leq |\mu|(S) + |\mu|(\{x_o\}) = |\mu|(S_o) = \|\mu\|$$

Hence $\mu \in e[M(\partial_e S_o)]$.

Assume $\mu \in e[M(\partial_e S_o)]$ annihilates $A_o(S)$. Let $\{a_\alpha\}$ be a net of real affine w*-continious functions on S_o such that

$$a_\alpha \nearrow 1 - \hat{\chi}_{x_o}$$

(See [1] corollary I.1.4 , theorem II.6.18 and theorem II.6.22)

Let $\varepsilon > 0$. Choose α such that

$$|\mu((1 - \hat{\chi}_{x_o}) - a_\alpha)| < \varepsilon/2$$
$$|a_\alpha(0)| < \varepsilon/2\cdot\|\mu\|$$

(See [1] (2.3)) . Then

$$|\mu(1)| = |\int 1\cdot d\,\mu| = |\int_S 1\cdot d\,\mu| = |\int_S (1-\hat{\chi}_{x_o})\, d\,\mu|$$
$$< |\int a_\alpha \, d\mu| + \varepsilon/2 \leq \int (a_\alpha - a_\alpha(0))\, d\,\mu$$
$$+ |\int a_\alpha(0)\, d\mu| + \varepsilon/2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Hence μ annihilates $A(S_o)$, so by (4.7) is $\mu \equiv 0$. (4.8) follows.

Let $\rho : S_o \to e(M(\partial_e S_o))$ be the canonical map. Then by [1] lemma II.6.10

$$\rho(S) = \{\mu \in M_1^+(\partial_e S_o) \mid \mu(S) = 1\} ,$$

which is easly seen to be a maximal face in the unitball of

$e(M(\partial_e S_o))$. ($\|\mu + |\mu|\| = \|\mu\| + \|\mu\| \Rightarrow \mu \geq 0$ which follows from polardecomposition)

Since $\rho(S_o) = \text{conv}\,(\rho(S) \cup \{0\})$ is compact, the proof is complete.

Let V be a Lindenstrauss-space and assume e is an extrempoint in the unit ball in V. Put

$$S = \{p \in V^* |\ p(e) = 1 = \|p\|\} .$$

Then S is w^*-compact and let $\psi : V \to C_C(S)$ be the canonical embedding. Then it is proved in [12]

Theorem 19 (Lazar-Hirsberg)

ψ is an isometry such that $\psi(e) = 1_S$

As in the proof of i) => ii) in theorem 18 we now get

Corollary 20

If V is Lindenstrauss-space and the unit ball of V contains an extrempoint, then $V \cong A(S)$ where S is a compact simplex.

Remark

Now as in the last part of the proof ii) => i) of theorem 18, we see that a Lindenstrausspace V has an extrempoint if and only if there is a maximal w^*-closed face in K. For more information about such Lindenstrauss-spaces see [12] .

A complex Banach space V is called a complex M-space if it can be represented as follows:

There is a compact Hausdorf space X and a set of triples $(x_a, y_a, \lambda_a) \in X \times X \times [0, 1]$ such that V is the subspace of $C_C(X)$ satisfying

$$f(x_a) = \lambda_a f(y_a) , a \in \mathcal{A}, \quad f \in V$$

Clearly V is self-adjoint, and by [17] Re V is a Kakutani

M-space, moreover each selfadjoint linear subspace of $C_C(X)$ whose real component is a Kakutani M-space arises in this way. Now, by theorem 18, a complex M-space is a complex simplex-space .

Notes

The real simplex-spaces were introduced and studied by Effros in [6] . Our results are based on the ideas of [12] , and lemma 17 is closely related to proposition II.6.19 in [1].

5. Complex G-spaces.

Let X be a compact Hausdorff space. A linear subspace $V \subseteq C_C(x)$ is called a complex G-space, if V consists of those $f \in C_C(X)$ satisfying a family $\mathcal{A}$ of relations:

$$f(x_a) = \lambda_a \alpha_a f(y_a) \ ; \ x_a, y_a \in X \ , \ \alpha_a \in T$$
$$\lambda_a \in [0, 1] \ , \quad a \in \mathcal{A} \ .$$

Complex G-spaces are complex Lindenstrauss-spaces by corollary 5 and the following.

Proposition 21

If V is a G-space, then there is an M-space A such that $V \cong P(A)$ where $P: A \to A$ is a projection with $\|p\| \leq 1$

Proof

We adopt the notation in the definition.

Let $Y = T \times X$ be organized to a T_σ-space in the canonical way (See proof of Proposition 10).

Let $A \subseteq C_C(Y)$ be the closed subspace satisfying

$$F(\beta, x_a) = \lambda_a F(\alpha_a \beta \ , y_a) \ , \quad a \in \quad , \ \beta \in T$$

Then A is a complex M-space. The map $T: V \to A$ defined by

$[Tf](\alpha \ , x) = \alpha f(x) \quad , \quad (\alpha, x) \in T \times X$

is seen to be an isometry of V onto a linear subspace of A , since

$$[Tf](\beta, x_a) = \beta \ f(x_a) = \lambda_a \ \beta \cdot \alpha_a \ f(y_a) = \lambda_a [Tf] \ (\beta\alpha_a \ , y_a) \ a \in \mathcal{A},$$
$$\beta \in T$$

If $F \in A$ is σ-homogeneous, then

$F(1, x_a) = \lambda_a F(\alpha_a, y_a) = \lambda_a \alpha_a F(1, y_a)$, $a \in \mathcal{A}$

Hence T takes V onto the σ-homogeneous functions in A.

Now the projection $P = \Pi_\sigma \mid A$ will do. In fact, let $F \in A$, then

$$P(F)(\beta, x_a) = \int \alpha^{-1} F(\alpha\beta, x_a)\, d\alpha$$
$$= \int \alpha^{-1} \lambda_a F((\alpha_a\beta)\beta, y_a)\, d\alpha = \lambda_a P(F)(\alpha_a\beta, y_a), a \in \mathcal{A}, \beta \in T$$

Lemma 22

Assume V is a Lindenstrauss-space and, let $E \subseteq \partial_e K$ be compact with $E \cap \alpha E = \emptyset$ when $\alpha \in T \setminus \{1\}$. Then $F = \overline{\text{conv}}(E)$ is a w^*-closed face in K.

Proof

By Milmans theorem ([1] p. 50) is

$$F = \{r(\mu) \mid \mu \in M_1^+(E)\}$$

and observ that a measure $\mu \in M_1^+(E)$ is maximal on K.

Assume $k_1, k_2 \in K$, $\lambda \in [0, 1]$ such that

$$k = \lambda k_1 + (1 - \lambda) k_2 \in F$$

Let $\mu \in M_1^+(E)$ with $r(\mu) = k$, $\mu_1, \mu_2 \in M_1^+(\partial_e K)$ with $r(\mu_1) = k_1$, $r(\mu_2) = k_2$.

Let $\epsilon > 0$, and choose compact C such that

$$C \cap \bigcup_{\alpha\in T} \alpha E = \emptyset$$

$$\mu_1\Big(\bigcup_{\alpha\in T} \alpha E \cup C\Big) \geq 1 - \epsilon, \quad \mu_2\Big(\bigcup_{\alpha\in T} \alpha E \cup C\Big) \geq 1 - \epsilon$$

Let f be a T-homogeneous function on K such that $f|E = 1$, $f|$, $f| \bigcup_{\alpha\in T} \alpha C = 0$, $\|f\| \leq 1$. By Effros' characterization we get:

$$1 = \mu(f) = \lambda\mu_1(f) + (1 - \lambda)\mu_2(f)$$
$$\leq \lambda \int_{\substack{\cup\alpha E \\ \alpha\in T}} f\, d\mu_1 + (1 - \lambda) \int_{\substack{\cup\alpha E \\ \alpha\in T}} f\, d\mu_2 + 2\epsilon \leq 1 + 2\epsilon .$$

Hence $\mu_1 (\bigcup_{\alpha\in T} \alpha E) = \mu_2 (\bigcup_{\alpha\in T} \alpha E) = 1$.

Assume now $\mu_1 (E) \neq 1$.

Let f be a T-homogeneous function on K with $f|E = 1$ and $\|f\| \leq 1$.

Put $E' = \bigcup_{\alpha\in T\setminus\{1\}} \alpha E$. By Effros' characterization we get

$$\begin{aligned} 1 = \mu(\mathrm{Ref}) &= \lambda \mu_1 (\mathrm{Ref}) + (1-\lambda)\mu_2 (\mathrm{Ref}) \\ &= \lambda \int_E \mathrm{Ref}\, d\mu_1 + \lambda \int_{E'} \mathrm{Ref}\, d\mu_1 + (1-\lambda)\mu_2(\mathrm{Ref}) \\ &< \lambda \int_E 1 \, d\mu_1 + \lambda \int_{E'} 1 \, d\mu_1 + (1-\lambda)\mu_2(1) = 1 , \end{aligned}$$

which is a contradiction. Hence $\mu_1(E) = 1$ which implies $k_1 \in F$ and the proof is complete.

Let $V \subseteq C_C(x)$ be a G-space.

Put

$$z = \{x \in X| \ (y, \lambda, \alpha) \in X \times [0, 1> \times T \text{ such that } f(x) = \lambda\alpha f(y) \text{ for all } f \in V\} .$$

Let $\delta : X \to K$ be the canonical map. Then we have

Lemma 23

$$\partial_e K = \bigcup_{\alpha\in T} \alpha\, \delta\, (X\setminus Z)$$

Proof

We use the same notations as in the proof of proposition 21, and when W is a Banach-space, then $B(W)$ will denote the unit ball.

Clearly no point in $\delta(Z)$ is extrem so by [5] p 441 lemma 6 is $\partial_e K \subseteq \bigcup_{\alpha\in T} \alpha\, \delta(X\ Z)$

To prove the converse inclusion let $x_o \in X\setminus Z$, $g \in A$. Then

$$\begin{aligned} P^*(\delta(1, x_o))\ (g) = \delta(1, x_o)(P(g)) &= \int \alpha^{-1} g(\alpha, x_o) d\alpha \\ &= \int \alpha^{-1}\, \delta(\alpha, x_o)(g)\ d\alpha \end{aligned}$$

Hence

(5.1) $$P^*(\delta(1, x_o)) = \int \alpha^{-1}\delta(\alpha, x_o)(\quad)\, d\alpha$$

Let $S_o = \overline{\text{conv}}\,(\{\delta(\alpha,x) |\ \alpha \in T\ ,\ x \in X\} \cup \{0\})$

Then S_o is a simplex and $A \cong A_o(S_o)$ (See section 4). By the real theory $\bigcup_{\alpha\in T} \delta(\alpha, x_o) \cup \{0\}$ ([7] Remark 8.2) is a w*-closed subset of $\partial_e S_o$. Let $f_o \bigcup_{\alpha\in T} \delta(\alpha, x_o) \breve{\cup} \{0\} \to C$ be defined by

$$f_o(\delta(\alpha, x)) = \alpha\ ,\ \alpha \in T\ ,\ f_o(0) = 0\ .$$

By [3] corollary 4.6 f_o can be extended to an element of $A_o(S_o)$ with norm one. Thus there is $f \in A$ such that

$$f(\alpha, x_o) = \alpha \quad \alpha \in T \text{ and } \|f\| = 1\ .$$

Let $E_1 = \bigcup_{\alpha\in T} \alpha^{-1}\delta(\alpha, x_o)$, $E_2 = \bigcup_{\alpha,\beta\in T} \beta\,\delta(\alpha,x_o)$

Then $E_1, E_2 \subseteq \partial_e B(A^*)$ by the real theory. Put $F = \overline{\text{conv}}\,(E_1)$, $H = \overline{\text{conv}}\,(E_2)$

by Milmans theorem: $\partial_e F = E_1$ and $\partial_e H = E_2$

Moreover $f^{-1}(1) \cap E_2 = E_1$. Hence $f^{-1}(1) \cap H = F$

Assume $P^*(\delta(1, x_o)) \notin F$. Then $P^*(\delta(1,x_o)) \notin H$ and since H is cirled et follows from the Hahn-Banach theorem that there is $g \in A$ such that

$$|P^*\delta(1, x_o)(g)| > 1\ ,\ |\delta(\alpha, x_o)(g)| < 1\quad \alpha \in T$$

which by 5.2 gives a comtradiction.

Thus $P^*(\delta(1, x_o) \in F$.

Assume $\delta(x_o) = \lambda k_1 + (1-\lambda)k_2$, $k_1, k_2 \in K$

Since $B(P^*(A^*))$ and K are affinely homeomorfic there corresponds unique $\tilde{k}_1 = P^* T^{*-1}(k_1)$, $\tilde{k}_2 = P^* T^{*-1}(k_2)$ such that

$$P^*(\delta(1, x_o)) = \lambda \tilde{k}_1 + (1-\lambda)\tilde{k}_2$$

But by lemma 23 F is a face in $B(A^*)$. Hence $\tilde{k}_1, \tilde{k}_2 \in F$.

But each $g \in P(A)$ is constant on F , so we get

$$\tilde{k}_1(g) = \tilde{k}_2(g) = P^*(\delta(1, x\partial))(g) \quad \text{for all } g \in P(A)$$

Thus $\delta(x_o) = k_1 = k_2$, i.e $\delta(x_o)$ is extrem.

Theorem 24

Let $V \subseteq C_C(x)$ be a Lindenstrauss-space. Then the following statements are equivalent

i) V is a G-space

ii) $\overline{\partial_e K} \subseteq [0, 1] \, \partial_e K$

Proof

Let $x \in Z$. Put

$\lambda_o = \inf \{\lambda \in [0, 1\rangle \mid y_\lambda \in X \, , \, \alpha_\lambda \in T \, , \, f(x) = \lambda \, \alpha_\lambda f(y_\lambda)$ all $f \in V\}$.

By compactness we may without loss of generality assume $(\lambda, y_\lambda, \alpha_\lambda)$ converges to $(\lambda_o, y_o, \alpha_o) \in [0, 1\rangle \times X \times T$. By continuity $f(x) = \lambda_o \, \alpha_o \, f(y_o)$ for all $f \in V$

If $\lambda_o = 0$, then $\delta(x) = 0 \in [0, 1] \, \partial_e K$

If $\lambda_o \neq 0$, then $\delta(y_o) \in \partial_e K$, which gives

$\delta(x) = \lambda_o \, (\alpha_o \, \delta(y_o)) \in [0, 1] \, \partial_e K$.

In fact, if $\delta(y_o) \notin \partial_e K$, then by lemma 23 there is $(\lambda, y, \alpha) \in [0, 1\rangle \times X \times T$ such that

$$f(y_o) = \lambda \, \alpha \, f(y) \quad \text{all} \quad f \in V \, ,$$

which implies

$$f(x) = \lambda_o \, \alpha_o \, f(y_o) = (\lambda_o \, \lambda) \, (\alpha_o \, \alpha) f(y) \quad \text{all } f \in V \, ,$$

condradicting the definition of λ_o.

ii) follows now easly from lemma 23.

ii) => i)

Let $A \subseteq C_C(\overline{\partial_e K})$ be the space of T-homogeneous functions f satisfying

$$(5.2) \qquad f(k) = \|k\| \; f\left(\frac{k}{\|k\|}\right) \quad k \in \overline{\partial_e K}$$

Then A is a G-space. We shall prove $A \cong V$. It is then enough to prove

$$A \subseteq V| \, \overline{\partial_e K}$$

Let $f \in A$. Then Ref satisfies (5.2), and since V is a Lindestrauss-space and f is T-momogeneous, we have

(5.3) $\nu_1(\mathrm{Ref}) = \nu_2(\mathrm{Ref})$ whenever $\nu_1, \nu_2 \in M_1^+(\partial_e K)$ with $r(\nu_1) = r(\nu_2)$. Assume $k \in \partial_e K$. Then

$$\nu = \frac{(1+\|k\|)}{2}\, \epsilon_{\frac{k}{\|k\|}} + \frac{(1-\|k\|)}{2}\, \epsilon_{\frac{-k}{\|k\|}}$$

is a maximal probability-measure with $r(\nu) = k$ by ii), and

$$\nu(\mathrm{Ref}) = \mathrm{Ref}(k)$$

By (5.3) this holds for any maximal probability measure with barycenter k . Hence by [8] theorem 2.3 , Ref may be extended to an affine real w*-continious function g on K with $g(0) = 0$. Let $F:K \to C$ be defined by

$$F(x) = g(x) + ig(-ix) \qquad x \in K$$

Then $F \in V$ and $F|\overline{\partial_e K} = f$.

Remark

The G-spaces include the M-spaces and it is readily verified that a C_σ-space is a G-space.

Notes

The real G-spaces were introduced by Grothendieck in [10] . Proposition 21 was announced in [22] in the real case, but, as pointed to us by Jan Raeburn, the proof is wrong, however the same idea can be used to give a correct proof. Theorem 24 was proved by Effros in the separable, real case [7] and in general by Fakhoury [9] . It is on his idas we have based the proof of lemma 23, and the other part of theorem 24 is proved as in [7]. Lemma 22 was proved by Lazar for real Lindenstrauss-space [19] .

6. The classification scheme.

Summarizing the foregoing we get

$$\begin{tikzcd}
 & \hat{A}(S) \arrow[rr] & & \hat{A}_o(S) \arrow[dr] & \\
\hat{C}(K) \arrow[ur] \arrow[r] \arrow[dr] & \hat{C}_o(K) \arrow[r] \arrow[dr] & \hat{M} \arrow[ur] \arrow[dr] & & V^* \cong L^1_e(Q,\ B,\ m) \\
 & \hat{C}_\Sigma \arrow[r] & \hat{C}_\sigma \arrow[r] & \hat{G} \arrow[ur] &
\end{tikzcd}$$

where $\hat{A}(S)$ denotes the class of Lindenstrauss-spaces with extrempoints, $\hat{A}_o(S)$ the simplex-spaces, and so on. $A \to B$ means that the class A is included in B.

It is also possible to read the intersections between the classes.

In fact:

(6.1) $\quad \hat{G} \cap \hat{A}(S) = \hat{C}_\Sigma \cap \hat{A}_o(S) = \hat{C}(K)$

(6.2) $\quad \hat{G} \cap \hat{A}_o(S) = \hat{M}$

(6.3) $\quad C_\sigma \cap \hat{A}_o(S) = \hat{C}_o(K)$

Proof

If V is a G-space with extreme-points, then there is a maximal w^*-closed face S in K with closed extrem-boundary. Hence S is a Bauer-simplex and the first equality in (6.1) follows from [1] theorem II.4.3

If S is a maximal face in K such that $\mathrm{conv}(S \cup \{0\})$ is compact and $\partial_e K$ is closed, then $\partial_e S$ is closed, Hence S is closed and [1] theorem II.4.3 will do.

If V is a simplex-space with $\overline{\partial_e K} \subseteq [0,\ 1]\ \partial_e K$, then there is a maximal face S in K such that $\mathrm{conv}(S \cup \{0\})$ is w^*-compact and $\overline{\partial_e S} \subseteq [0,\ 1]\ \partial_e S$. Now [8] theorem 2.3 gives (6.2) as in the proof of theorem 24.

If V is a simplex-space with $\partial_e K \cup \{0\}$ closed, then there is a maximal face S in K such that $\partial_e S \cup \{0\}$ is compact. Hence $S_o = \mathrm{conv}\ (S \cup \{0\})$ is a Bauersimplex and by [1] theorem II.4.3 we get $\quad C_o\ (\partial_e S) \cong A_o(S_o) \cong V.$

Notes

The classification scheme is essentially due to Lindenstrauss-Walbert [22] , but was later on modified in [20]. For more information about complex Lindenstrauss-spaces, see Hustad's works [14] and [15], where he studies intersection properties of balls and extensions of compact operators. These topics are related to Lindenstrauss results [21] in the real case.

References

1. E.M. Alfsen — Compact convex sets and boundary integrals. Ergebnisse der math. Springer verlag 1971.

2. E.M. Alfsen and E.G. Effros — Structure in Real Banach-spaces I & II. Annals of math. 1 July 1972 p. 98 - 173.

3. E.M. Alfsen and B. Hirsberg — On dominated extensions in linear subspaces of $C_C(X)$. Pacif Journal of math. 36 1971 p. 567-584.

4. M. Day — Normed linear spaces. Ergebnisse der Mathematik. Springer verlag 1958.

5. N. Dunford, J. Schwarz — Linear operators part 1. Interscience New York 1958.

6. E. Effros — Struture in simplexes. Acta. Math. 117 - 1967 p. 103 - 121.

7. E. Effros — On a class of real Banach-spaces. Israel Journal of math. 9 - 1971, p. 430 - 457.

8. E. Effros — On a class of complex Banach-spaces. To appear

9. H. Fakhoury — Preduaux de L-espace: Notion de centre. Journal of Functional-analysis 9 - 1972; p. 182 - 207.

10. A. Grothendieck — Une Caracterisation vectorielle metrique des espaces L^1. Canadan Journal of math. 7 - 1955, p. 552-561.

11. B. Hirsberg — Representations integrales des formes linéaires. C.R. Acad. Serie A 274 1972 p. 1222.

12. B. Hirsberg and A. Lazar — Complex Lindenstrauss spaces with extreme points. To appear in Trans. Am. math. soc.

13. O. Hustad — A norm-preserving complex Choquet theorem. Math. Scand 29 - 1972 p. 272 - 278.

14. O. Hustad — Intersection properties of balls in complex Banach-spaces whose duals are L_1-spaces. Preprint series no. 9 April 1973, University of Oslo.

15. O. Hustad — A note on complex P_1-spaces. Preprint series no. 10, May 1973, Univ. of Oslo.

16. M. Jerison — Certain spaces of contious functions. Trans. Am. math. soc. 70 - 1951 p. 103 - 113.

17. S. Kakutani — Concrete representation of abstract M-spaces. Ann. of math. 42 - 1941, p. 994 - 1024.

18. Ka-Sing Lau — The dual bau of a Lindenstrauss space. To appear.

19. A. Lazar — The unit ball in conjugate L_1-spaces. Duke math. Journal, Vol 39, No. 1 1972.

20. A. Lazar - J. Lindenstrauss — Banach spaces whose duals are L_1-spaces and their representing matrices. Acta. math. 117 - 1967, p. 103 - 121.

21. J. Lindenstrauss — Extension of compact operators. Mem Amer. math. soc. 48 1964.

22. J. Lindenstrauss and D. Walbert — On the classification of the Banach spaces whose Duals are L^1 spaces. Journal of Funct. analysis, 4 - 1969, p. 332 - 349.

23. I Namioka and R. Phelps — Tensor products of compact convex sets. Pac. J. of Math. 31 - 1969 p. 469 - 480.

24. H. Fakhoury — Une caracterisation des L-espaces duax. Bull. Sc. math. 96 - 1972.