THE WIGHTMAN AXIOMS AND THE MASS GAP FOR STRONG INTERACTIONS OF EXPONENTIAL TYPE IN TWO-DIMENSIONAL SPACE-TIME

by

Sergio Albeverio
Instituto di Fisica Teorica, INFN
Università di Napoli
and Institute of Mathematics
University of Oslo

and

Raphael Høegh-Krohn
Institute of Mathematics
University of Oslo
Blindern, Oslo 3
Norway

A B S T R A C T

We construct boson models in two space-time dimensions which satisfy all the Wightman axioms with mass gap. The interactions are exponential and no restriction on the size of the coupling constant is made. The Schwinger functions for the space cut-off interaction are shown to be non-negative and to decrease monotonically to their unique, non zero, infinite volume limit, as the space cut-off is removed. The correspondent Wightman functions satisfy all the Wightman axioms. The mass gap of the space cut-off Hamiltonian is non decreasing as the space cut-off is removed and the Hamiltonian for the infinite volume limit has mass gap at least as large as the bare mass. The infinite volume Schwinger functions and the mass gap depend monotonically on the coupling constant and the bare mass. The coupling of the first power of the field to the first excited state is proven.

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1. Introduction

Nelson's introduction [1] of Markoff fields and euclidean methods in constructive quantum field theory has had a strong influence on recent studies of some models ([2], [3], [4]) and these methods played also a certain role in Glimm-Spencer's proof [5] of the mass gap and the uniqueness of the infinite volume limit for weakly coupled $P(\varphi)_2$ models, completing the verification og Wightman's axioms for these models [6]. [1]) One of the advantages of the Markoff fields approach is to make available for quantum fields methods used successfully in the study of the thermodynamic limit in statistical mechanics. A first direct use of this connection was made in [3] , where Kirkwood-Salzburg equations for dilute gases where applied to the study of the infinite volume limit of weakly coupled ultraviolet cut-off non-polynomial interactions. For strongly coupled models other methods of statistical mechanics have been used quite recently. These are the so called correlation inequalities, in particular the Griffiths inequalities for Ising ferromagnetic systems [7] (see also e.g [8]). Guerra, Rosen and Simon [9] and Nelson [10] introduced a framework which makes possible to apply the correlation inequalities to the case of quantum fields, (the so called lattice approximation for the $P(\varphi)_2$ models). The consequences of these correlation inequalities include a result of Nelson [10] on the existence of the infinite volume limit for a class of $P(\varphi)_2$ interactions with Dirichlet boundary conditions, without restrictions to weak coupling, and a result of Simon [11] on the coupling to the first excited state. Uniqueness of the vacuum in these models was not obtained by this method, but for certain polynomials of at most fourth degree and Dirichlet boundary conditions yet another method inspired by statistical mechanics (Lee-Yang Theorems) has been successful [12]. The question of the mass gap remains however open in these strong coupling models.

In this paper we shall use euclidean Markoff fields and correlation inequalities for the study of the infinite volume limit of scalar bosons with even exponential self-coupling in two space-time dimensions, without cut-offs and restrictions on the size of the coupling constant, and we prove all Wightman axioms including uniqueness of the vacuum and the mass gap.

A class of the models including the present ones have been studied previously by one of us [13]. Their space cut-off Hamiltonians $H_r = H_o + V_r$ have been shown to exist as essentially self-adjoint operators. Moreover the uniqueness of the ground state of H_r, with eigen value E_r, was proved as well as the finiteness of the spectrum in the interval $[E_r, E_r + m - \varepsilon]$, where $\varepsilon > 0$ is arbitrary and m is the bare mass. Infinite volume limit points were then given using a compactness argument (along the lines of [14]), but the question of the uniqueness of the limit was not tackled yet.
In this paper we establish the existence and uniqueness of the infinite volume limit, ty proving that the Schwinger functions of the space cut-off interactions are non-negative and monotonically decreasing as the space cut-off function is increasing. We then verfy that the infinite volume Schwinger functions satisfy all the axioms [15] for euclidean fields and using a result of Osterwalder and Schrader [15], we then have that all the Wightman axioms are satisfied in these models. Furthermore we have that the spectrum of the space cut-off Hamiltonian is void in the open interval $(E_r, E_r + m)$ and the mass gap is monotonically non decreasing as the space cut-off is removed. Moreover the mass gap of the physical Hamiltonian H is of size larger or equal m and is a monotone non decreasing function of the coupling constant and the bare mass m. We also prove that the first excited states of the physical Hamiltonian are in the odd subspace of the physical Hilbert space and are coupled to the vacuum by the first power of the field.

2. Exponential interactions in two space-time dimensions.

In [13] a class of models of a boson field with exponential self-coupling in two space-time dimensions was considered. The space cut-off Hamiltonian for the space cut-off interactions of these models is given by

$$H_r = H_o + \int_{|x|\leq r} :V(\varphi(x)):dx \,, \tag{2.1}$$

where H_o is the free energy for a scalar bosonfield with positive mass $m > 0$, in two space-time dimensions, and $V(s)$ is a non negative real function of the form

$$V(s) = \int e^{\alpha s} \, d\nu(\alpha) \,, \tag{2.2}$$

where ν is a bounded positive measure with compact support in the open interval $(-\sqrt{2\pi}, \sqrt{2\pi})$. The precise set of functions (2.2) for which the method of [13] hold is actually given by all positive measures ν with support in the closed interval $[-\sqrt{2\pi}, \sqrt{2\pi}]$ and such that

$$\iint (2\pi - st)^{-1} \, d\nu(s) \, d\nu(t) < \infty \tag{2.3}$$

The Wick ordered function of the field $:V(\varphi(x)):$ is defined as the positive bilinear form on the free Fock space $\mathcal{F}$ given by

$$:V(\varphi(x)): = \int e^{\alpha\varphi_+(x)} \cdot e^{\alpha\varphi_-(x)} \, d\nu(\alpha) \,, \tag{2.4}$$

where $\varphi_+(x)$ and $\varphi_-(x)$ are the creation and annihilation parts of the free field $\varphi(x)$. In [13] it was proved that

$$V_r = \int_{|x|\leq r} :V(\varphi(x)): dx \tag{2.5}$$

is a positive self-adjoint operator, and moreover that

$$H_r = H_o + V_r \tag{2.6}$$

is essentially self-adjoint on a domain contained in the intersection of the domains of definition for H_o and V_r. It was also proved that H_r has a lowest simple isolated eigenvalue E_r, which is separated from the essential spectrum of H_r by the distance m. Since V_r is a strictly local perturbation the time translations exist as a one parameter group of C^*-isomorphisms of the C^*-algebra of local operators.

Let Ω_r be the normalized eigenvector belonging to the eigenvalue E_r. The estimate

$$(\Omega_r, H_o \Omega_r) \leq 2r \|v\| \tag{2.7}$$

was proved and it was indicated how this estimate could be used to construct an infinite volume vacuum as a state on the C^*-algebra of local operators, invariant under space and time translations. An infinite volume vacuum was obtained as a weak limit point in the state space of convex combinations of the states given by Ω_r and its translates. Summarizing briefly the results of the present paper for the vacuum, we prove, for the case $V(s) = V(-s)$, that E_r is separated from the <u>rest</u> of the spectrum of H_r by the distance m, and that the state ω_r given by Ω_r has a unique limit point ω as $r \to \infty$ and this is invariant under space and time translations as well as under homogeneous Lorentz transformations. Moreover in the renormalized Hilbert space given by the infinite volume vacuum ω the infinitesimal generator of the time translations i.e. the Hamiltonian H has zero as a simple lowest eigenvalue with eigenvector given by ω, and the rest of the spactrum of H is contained in the interval $[m, \infty)$.

3. The Markoff field and the Schwinger functions.

Following Nelson [1] we introduce the free Markoff field in two dimensions $\xi(x)$, $x \in \mathbb{R}^2$, which is the generalized Gaussian stochastic field with mean zero and covariance given by

$$E(\xi(x)\xi(y)) = G(x-y) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i(x-y)p} \frac{dp}{\mu(p)^2} , \qquad (3.1)$$

where $\mu(p)^2 = p^2 + m^2$, $p^2 = p_1{}^2 + p_0{}^2$. Since $G(x)$ is not a bounded function, $\xi(x)$ itself is not a stochastic variable, but only a generalized stochastic variable, so that $\xi(h) = \int_{\mathbb{R}^2} \xi(x)h(x)dx$ is a Gaussian stochastic variable with mean zero and variance $(hGh) = \int h(x)\, G\,(x-y)\, h\,(y) dx\, dy$, for all real distributions h such that this integral is finite.

Let $(X, d\mu)$ be the probability space on which all the stochastic variables $\xi(h)$ are measurable functions. Let L_p be the space of $L_p(X, d\mu)$-integrable functions. The distributions h such that $(h\,Gh) < \infty$ are the elements of the real Sobolev space $H_{-1}(\mathbb{R}^2)$ and one verifies easily that distributions of the form $\delta(x_0-t) \otimes f(x_1) = h(x_1,x_0)$, where $f \in H_{-\frac{1}{2}}(\mathbb{R})$, are in $H_{-1}(\mathbb{R}^2)$. Let E_0 be the conditional expectation with respect to the σ-algebra generated by the stochastic variables of the form $\xi_0(f) = \int_{\mathbb{R}} \xi(x,o)f(x)dx$ for $f \in H_{-\frac{1}{2}}(\mathbb{R})$. Since E_0 is a conditional expectation it is an orthogonal projection in L_2 . There is then a natural identification of the free Fock space $\mathcal{F}$ with E_0L_2 such that the free vacuum Ω_0 is identified with the function 1 and such that the self adjoint operator $\varphi(f) = \int_{\mathbb{R}} \varphi(x)f(x)dx$ is identified with the multiplication operator given by the function $\xi_0(f)$.

Since the covariance function (3.1) is euclidean invariant in

$\mathbb{R}^2$ so is the measure $d\mu$ and hence we have a strongly continuous unitary representation on L_2 of the euclidean group in $\mathbb{R}^2$. Let U_t be the unitary representation in L_2 of the translations of the form $(x_1,x_o) \to (x_1,x_o+t)$ in $\mathbb{R}^2$. One verifies [1] now that in the representation of the free Fock space as E_oL_2 one has the following representation of the semigroup e^{-tH_o}. Let F be functions in E_oL_2, then

$$e^{-tH_o} F = E_o U_t F \qquad (3.2)$$

Theorem 3.1. (Feynman-Kac-Nelson formula).

Let V be a real function in E_oL_2 such that V is bounded below by a constant function, and consider V as a self adjoint multiplication operator on E_oL_2.

If $H = H_o + V$ is essentially self adjoint, then for any $F \in E_oL_2$,

$$e^{-tH} F = E_o \, e^{-\int_o^t U_\tau V d\tau} U_t F ,$$

where the integral in the exponent is the strong $L_2(X,d\mu)$ integral.

Proof: Since $H = H_o + V$ is essentially self-adjoint on $D(H_o) \cap D(V)$ and bounded below, and $-H_o$ as well as $-V$ are infinitesimal generators of strongly continuous semigroups e^{-tH_o} and e^{-tV}, we have by the Kato-Trotter theorem on perturbations of infinitesimal generators that the closure of $-H$ is the infinitesimal generator of a strongly continuous semigroup e^{-tH} and

$$e^{-tH} = \text{strong} \lim_{n\to\infty} [e^{-t/n H_o} e^{-t/n V}]^n . \qquad (3.3)$$

By (3.2) we have that, for any $F \in E_oL_2$,

$$\left[e^{-t/n H_o} e^{-t/n V}\right]^n F = E_o\left[e^{-\frac{1}{n}\sum_{k=1}^{n} U_{\frac{kt}{n}} V} U_t F\right] . \tag{3.4}$$

Since U_t is strongly continuous on L_2 and V is in $E_o L_2 \subset L_2$ we have that $U_t V$ is strongly continuous and uniformly bounded in L_2. Hence $\frac{1}{n}\sum_{k=1}^{n} U_{\frac{kt}{n}} V$ converges to $\int_0^t U_\tau V\, d\tau$ in L_2. Hence there is a subsequence n_j which converges pointwise almost everywhere in X with respect to the measure $d\mu$. Since V is bounded below and U_t comes from a transformation on X we have that $-\frac{1}{n}\sum_{k=1}^{n} U_{\frac{kt}{n}} V$ are bounded below uniformly in n. Therefore $\exp(-\frac{1}{n_j}\sum_{k=1}^{n_j} U_{\frac{kt}{n_j}} V)$ is uniformly bounded and converges pointwise. Hence by dominated convergence the right hand side of (3.4) converges to the right hand side of the formula in the theorem. By (3.3) the corresponding left hand side converges. This proves the theorem.

Corollary to theorem 3.1.

Let $V \in E_o L_2$ and real but not necessarily bounded below. Let $V_k(x) = V(x)$ if $V(x) \geq -k$ and $V_k(x) = -k$ if not, and assume that $H_k = H_o + V_k \geq -C$ where C does not depend on k. Then H_k tends strongly in the generalized sense to a self-adjoint operator H which is bounded below and the formula of the theorem still holds.

Proof: Since $-C \leq H_{k'} \leq H_k$ for $k \leq k'$ we have that $(H_k+c+1)^{-1} \leq (H_{k'}+c+1)^{-1} \leq 1$. Hence $(H_k+c+1)^{-1}$ is a monotone sequence of positive operators uniformly bounded by 1, so that $(H_k+c+1)^{-1}$ converges strongly and since the limit is bounded below by $(H_k+c+1)^{-1}$ for a fixed k we have that the limit is the resolvent $(H+c+1)^{-1}$ of a self adjoint operator H. By the

Trotter theorem on approximation of semigroups we have that e^{-tH_k} converges strongly to e^{-tH}. Hence the left hand side of the formula in the theorem converges. Since U_t comes from a transformation on X we have that $(U_\tau V_{k'})(x) \leq (U_\tau V_k)(x)$ for $k \leq k'$ and all $x \in X$. Hence $\exp(-\int_0^t U_\tau V_k d\tau)$ increases monotonically for all $x \in X$ with k. Hence for $F \geq 0$ the right hand side of the formula converges by the monotone convergence theorem. This proves the corollary. ▨

We shall now apply the Feyman-Kac-Nelson formula to the exponential interactions, but first we need to introduce the Wick powers of the free Markoff field.

Let P_n be the closed linear subspace of $L_2(X,d\mu)$ generated by functions of the form $\xi(f_1)\ldots\xi(f_k)$, $k \leq n$. We then have that $\bigcup_n P_n$ is dense in L_2 and $P_{n-1} \subseteq P_n$. We then introduce the Hermite polynomials $\mathcal{H}_n$ of degree n as the orthogonal complement of P_{n-1} in P_n: $\mathcal{H}_n = P_n \ominus P_{n-1}$. Since $\bigcup_n P_n$ is dense in L_2 we get that

$$L_2(X,d\mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n . \qquad (3.5)$$

Following Segal [16] we define the Wick ordered monomial $:\xi(h_1)\ldots\xi(h_n):$ as the orthogonal projection of $\xi(h_1)\ldots\xi(h_n)$ on $\mathcal{H}_n$. A direct computation gives that

$$\| :\xi(h)^n: \|^2 = n!\ (hGh)^n . \qquad (3.6)$$

Hence $\mathcal{H}_n$ is isometric to the n-th symmetric tensor product of the Sobolev-space H_{-1}. Let g be a smooth function. We define the n-th Wick power $:\xi(x)^n:$ of the field $\xi(x)$ by

$$:\xi^n:(g) = \int_{\mathbb{R}^2} :\xi(x)^n: g(x)dx , \qquad (3.7)$$

where $:\xi^n:(g)$ is the element in $\mathcal{H}_n$ given by

$$(:\xi^n:(g), :\xi(h_1)\ldots\xi(h_n):) = n! \int g(x) \prod_{j=1}^{n} G(x-y_j)h_j(y_j)dy_j dx. \quad (3.8)$$

(3.8) defines a linear functional on $\mathcal{H}_n$, but computing the norm of this linear functional we get

$$\|:\xi^n:(g)\|^2 = n! \int g(x)\, G\,(x-y)^n g(y)dxdy\ , \quad (3.9)$$

which shows that for g smooth this linear functional is bounded and hence an element in $\mathcal{H}_n$. We have actually proved that $:\xi^n:(g)$ is in $\mathcal{H}_n \subset L_2(X,d\mu)$ for all distributions g such that

$$(gG^n g) = \int g(x)\, G\,(x-y)^n g(y)dxdy \quad (3.10)$$

is finite.

We now define the Wick exponential $:e^{\alpha\xi}:(g)$ by the series

$$:e^{\alpha\xi}:(g) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :\xi^n:(g)\ , \quad (3.11)$$

whenever the series converges in L_2. Since the n-th term of (3.11) is in $\mathcal{H}_n$ we have by (3.5) that the L_2 convergence corresponds to the convergence of the series

$$\|:e^{\alpha\xi}:(g)\|_2^2 = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} (g\,G^n g)\ . \quad (3.12)$$

Since $G(x-y)$ is a positive function the series in (3.12) converges if and only if the integral

$$(g\ e^{\alpha^2 G} g) = \int g(x)\, e^{\alpha^2 G(x-y)}\, g(y)dxdy \quad (3.13)$$

is finite, and in this case we get

$$\| : e^{\alpha\xi} : (g)\|_2^2 = (g e^{\alpha^2 G} g) . \tag{3.14}$$

Since $G(x) = \frac{1}{2\pi} K_0(m|x|)$, where $|x|^2 = x_1{}^2 + x_0{}^2$ and K_0 is a modified Bessel function, we have that $G(x)$ goes exponentially to zero as $|x| \to \infty$, that $G(x)$ is bounded for $|x| \geq \varepsilon > 0$ and that $G(x) + \frac{1}{2\pi} \ln|x|$ is bounded for $|x| \leq 1$. Hence $e^{\alpha^2 G(x)} - 1$ is an integrable function in $\mathbb{R}^2$ for $|\alpha| < 2\sqrt{\pi}$, and since $G(x)$ is positive so is $e^{\alpha^2 G(x)} - 1$. Let

$$C_\alpha = \int_{\mathbb{R}^2} (e^{\alpha^2 G(x)} - 1) dx ,$$

then we get from (3.13) that $(g e^{\alpha^2 G} g)$ is bounded by $\|g\|_1^2 + C_\alpha \|g\|_2^2$. Hence we have the estimate for the L_2-norm of the Wick ordered exponential

$$\| : e^{\alpha\xi} : (g)\|_2^2 \leq \|g\|_1^2 + C_\alpha \|g\|_2^2 , \tag{3.15}$$

for $|\alpha| < 2\sqrt{\pi}$.

Let $d\nu(\alpha)$ be a positive measure of compact support in the open interval $(-2\sqrt{\pi}, 2\sqrt{\pi})$ and let

$$V(s) = \int e^{\alpha s} d\nu(\alpha) .$$

We then define the corresponding Wick ordered function of the free Markoff field by

$$: V : (g) = \int : e^{\alpha\xi} : (g) d\nu(\alpha) . \tag{3.16}$$

That $:V:(g)$ defines a function in $L_2(X, d\mu)$ follows from the fact that the series in (3.11) converges uniformly in L_2 for α in a compact subinterval of $(-2\sqrt{\pi}, 2\sqrt{\pi})$. By going through the arguments above we actually find that, if $g \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$ and

$$C_V = \iiint (e^{\alpha\beta G(x)} - 1) d\nu(\alpha) d\nu(\beta) dx$$

is finite, then $:V:(g)$ is in $L_2(X,d\mu)$ and

$$\| :V:(g)\|_2^2 \leq \|\nu\|^2 \|g\|_1^2 + C_V \|g\|_2^2 \; . \tag{3.17}$$

Consider now the operator V_r given on the Fock space by (2.5). In [13] it was proved that in the $E_o L_2$ representation of the Fock space V_r is a multiplication operator by a non negative function in $E_o L_2$ and we shall denote this function also by V_r. Since $H_o + V_r$ is essentially self adjoint [13] we may apply theorem 3.1.

Since in the $E_o L_2$ representation the self adjoint operator $\varphi(f) = \int_{\mathbb{R}} \varphi(x_1) f(x_1) dx_1$ goes over into the multiplication by the function $\xi_o(f) = \int_{\mathbb{R}} \xi(x_1,0) f(x_1) dx_1$, we have that

$$\int :e^{\alpha\varphi(x)}: f(x) dx \rightarrow :e^{\alpha\xi}:(f \otimes \delta) \tag{3.18}$$

and this is in $E_o L_2(X,d\mu)$ for $|\alpha| < \sqrt{2\pi}$, which follows from (3.14) and the calculations given in [13]. Choosing $f = \chi_r$, where $\chi_r(x_1) = 1$ for $|x_1| \leq r$ and zero if not, and integrating (3.18) with respect to a measure that satisfies the condition (2.3), we get that

$$V_r = :V:(\chi_r \otimes \delta) \; . \tag{3.19}$$

Now since $U_t \xi(g) = \xi(g_t)$, where $g_t(x_1,x_o) = g(x_1,x_o-t)$, we get immediately that $U_t :e^{\alpha\xi}:(g) = :e^{\alpha\xi}:(g_t)$ and therefore also that

$$U_t V_r = :V:(\chi_r \otimes \delta_t) \; ,$$

where δ_t is the unite point measure at t. Therefore

$$\int_0^t U_\tau V_r \, d\tau = :V:(\chi_r^t) \; , \tag{3.20}$$

where $\chi_r^t = 1$ if $|x_1| \leq r$ and $0 \leq x_0 \leq t$ and zero if not.

Hence in our case the formula in theorem 3.1 takes the form

$$e^{-tH_r} F = E_0 \, e^{-:V:(\chi_r^t)} U_t F \; , \tag{3.21}$$

for F in $E_0 L_2$.

Let now $F_1, \ldots, F_n$ be in $E_0 L_\infty(X, d\mu)$, then by repeated applications of (3.21) we get that, for $s \leq t_1 \leq \ldots \leq t_n \leq t$ and $\Omega_0(x) \equiv 1$,

$$(\Omega_0, e^{-(t_1-s)H_r} F_1 \, e^{-(t_2-t_1)H_r} F_2 \ldots e^{-(t_n-t_{n-1})H_r} F_n \, e^{-(t-t_n)H_r} \Omega_0)$$

$$= E[F_1^{t_1} \ldots F_n^{t_n} \, e^{-:V:(\chi_r^{s,t})}] \; , \tag{3.22}$$

where $F_i^{t_i} = U_{t_i} F_i$, $i = 1, \ldots, n$ and $\chi_r^{s,t}(x) = 1$ for $|x_1| \leq r$ and $s \leq x_0 \leq t$, and zero if not.

Now let $f_1, \ldots, f_n$ be real functions in the Sobolev space $H_{-\frac{1}{2}}(\mathbb{R})$, and let $F_i = \rho(\varphi(f)) = \rho(\xi_0(f))$ where ρ is a bounded function. Since $\varphi(f) = \xi_0(f)$ is in $L_p(X, d\mu)$ for all $p < \infty$ [ ], and $:V:(\chi_r^t) \geq 0$ so that $\exp(-:V:(\chi_r^t))$ is in L_∞, we get, by letting $\rho(\alpha)$ be an approximation to the function α, that

$$(\Omega_0, e^{-(t_1-s)H_r} \varphi(f_1) e^{-(t_2-t_1)H_r} \varphi(f_2) \ldots e^{-(t_n-t_{n-1})H_r} \varphi(f_n) e^{-(t-t_n)H_r} \Omega_0)$$

$$= E[\xi_{t_1}(f_1) \ldots \xi_{t_n}(f_n) \, e^{-:V:(\chi_r^{s,t})}] \tag{3.23}$$

$$= \int \ldots \int_{\mathbb{R}^n} E[\xi(x_1, t_1) \ldots \xi(x_n, t_n) e^{-:V:(\chi_r^{s,t})}] f_1(x_1) \ldots f_n(x_n) dx_1 \ldots d_{x_n} \; ,$$

where $\xi_t(f) = \xi(f \otimes \delta_t)$ and the integral over $\mathbb{R}^n$ is understood in the weak sense.

For $g \geq 0$ and $g \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$ we define the measure

$$d\mu_g = (E(e^{-:V:(g)}))^{-1} e^{-:V:(g)} d\mu . \tag{3.24}$$

Since $:V:(g) \geq 0$, $d\mu_g$ is a new probability measure on X , which is absolutely continuous with respect to $d\mu$. For any measurable function F on X , we shall denote

$$\langle F \rangle_g = \int F \, d\mu_g . \tag{3.25}$$

The Schwinger functions for the interaction cut-off in space and time by g , are the distributions defined by the formula

$$S_g(h_1 \dots h_n) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} S_g(y_1, \dots, y_n) h(y_1) \dots h(y_n) dy_1 \dots dy_n = \langle \xi(h_1) \dots \xi(h_n) \rangle g . \tag{3.26}$$

By formula (3.23) we see that

$$\int_{\mathbb{R}} \dots \int_{\mathbb{R}} S_{\chi_r}^{s,t}(x_1,t_1) \dots, x_n t_n) f_1(x_1) \dots f_n(x_n) dx_1 \dots dx_n \tag{3.27}$$

$$= (\Omega_0, e^{-(t-s)H_r} \Omega_0)^{-1} \cdot (\Omega_0, e^{-(t_1-s)H_r} \varphi(f_1) e^{-(t_2-t_1)H_r} \varphi(f_2) \dots$$

$$\dots e^{-(t_n-t_{n-1})H_r} \varphi(f_n) e^{-(t-t_n)H_r} \Omega_0) .$$

From [13] we know that H_r has a simple lowest eigenvalue E_r , which is separated from the rest of the spectrum of H_r . Let Ω_r be the corresponding normalized eigenvector. We shall see that we can use this fact to prove that $S_{\chi_r}^{s,t}$ converges to a limit S_{χ_r} as $s \to -\infty$ and $t \to \infty$. To prove this we shall need the following lemma

<u>Lemma 3.1</u>: For any $p > 1$ and $q > 1$ there is a T depending only on p and q such that e^{-tH_r} is a contraction from $E_o L_p$ to $E_o L_q$.

<u>Proof</u>: The corresponding lemma with H_o instead of H_r was proved by Glimm [17 , lemma 5.1]. By the Kato-Trotter product formula we have

$$e^{-tH_r} F = \lim_{n\to\infty} e^{-t/n\, H_o} e^{-t/n\, V_r} \cdots e^{-t/n\, H_o} e^{-t/n\, V_r} F \,. \quad (3.28)$$

By (3.21) e^{-tH_r} maps non-negative functions into non-negative functions, hence $e^{-tH_r} F \leq e^{-tH_r} |F|$ for any function F in $E_o L_2$. Since $V_r \geq 0$ and $e^{-t/n\, H_o}$ maps non-negative functions into non-negative functions, we get from (3.28) that $e^{-tH_r}|F| \leq e^{-tH_o}|F|$. Hence $e^{-tH_r} F \leq e^{-tH_o}|F|$, and therefore also $-e^{-tH_r} F = e^{-tH_r}(-F) \leq e^{-tH_o}|F|$; so that $|e^{-tH_r} F| \leq e^{-tH_o}|F|$. Thus $\|e^{-tH_r} F\|_q \leq \|e^{-tH_o}|F|\,\|_q$, which by the result of Glimm is smaller or equal to $\|F\|_p$. This proves the lemma. ▨

Since $e^{-tH_r}\Omega_r = e^{-tE_r}\Omega_r$ we get that Ω_r is in L_p for all $p < \infty$. Furthermore, since E_r is an isolated eigenvalue, we have that $e^{-t(H_r - E_r)}\Omega_o$ converges strongly to $\Omega_r(\Omega_r, \Omega_o)$ in L_2. Since $e^{-s(H_r-E_r)}$ is a contraction from L_2 to L_p for any p, where s depends on p, by the lemma 3.1, we have that $e^{-s(H_r-E_r)}$ maps strong convergence in L_2 into strong convergence in L_p. Therefore since

$$e^{-t(H_r-E_r)}\Omega_o = e^{-s(H_r-E_r)} \cdot e^{-(t-s)(H_r-E_r)}\Omega_o$$

and $e^{-(t-s)(H_r-E_r)}\Omega_o$ converges strongly to $\Omega_r(\Omega_r, \Omega_o)$ in L_2, we also get that $e^{-t(H_r-E_r)}\Omega_o$ converges strongly to $\Omega_r(\Omega_r, \Omega_o)$

in L_p for all $p < \infty$.

Since $\varphi(f)$ is in L_p for all $p < \infty$ and e^{-tH_r} is a contraction on L_p [, lemma 3.3], we may use the above result on formula (3.27) and we get that $S_{\chi_r}^{s,t}$ converges as $s \to -\infty$ and $t \to \infty$ and the limit S_{χ_r} is given by

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} S_{\chi_r}(x_1 t_1, \ldots, x_n t_n) f_1(x_1) \ldots f_n(x_n) dx_1 \ldots dx_n$$
$$= (\Omega_r, \varphi(f_1) e^{-(t_2-t_1)(H_r-E_r)} \varphi(f_2) \ldots e^{-(t_n-t_{n-1})(H_r-E_r)} \varphi(f_n)\Omega_r). \tag{3.29}$$

From the ergodicity of e^{-tH_r} [, section 4] we have that $\Omega_r(x) > 0$ for almost all $x \in X$, hence polynomials of the time zero field, i.e. polynomials in $\varphi(f_1) \ldots \varphi(f_n)$ for arbitrary n and $f_1, \ldots f_n$ applied to Ω_r are dense in the Fock space $E_o L_2$. It follows therefore from (3.29) that S_{χ_r} uniquely determines Ω_r and H_r. In the following section we shall show that Ω_r and H_r have unique limits as r tends to infinity by showing that S_{χ_r} has unique limit as r tends to infinity.

4. <u>The lattice approximation and the correlation inequalities.</u>

In order to prove correlation inequalities for the Schwinger functions we shall extend to our case a method developed by Guerra, Rosen and Simon [9] and Nelson [10] for the $P(\varphi)_2$ interactions. The method uses the following lattice approximation for the Markoff fields.

Let $\delta > 0$ be a fixed real number and denote by L^2_δ the lattice of points $n \cdot \delta$, where n runs over the set $\mathbb{Z}^2$ of ordered pairs of integers. The correspondence of the Laplacian Δ in $\mathbb{R}^2$ is the finite difference operator Δ_δ acting on functions f over L^2_δ as

$$\Delta_\delta f(n\delta) = \delta^{-2}[4f(n\delta) - \sum_{|n'-n|=1} f(n\delta)] \; . \tag{4.1}$$

Define the Fourier transforms from $l_2(L^2_\delta)$ to $L_2([-\frac{\pi}{\delta}, \frac{\pi}{\delta}] \times [-\frac{\pi}{\delta}, \frac{\pi}{\delta}]) = L_2(T^2_\delta)$ by

$$\hat{h}(k) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \delta^2 h(\delta n) e^{-ikn\delta}$$

Since T^2_δ is the dual group of L^2_δ one verifies that $h \to \hat{h}$ is a unitary mapping from $l_2(L^2_\delta)$ onto $L_2(T^2_\delta)$, and the inverse mapping is given by

$$h(\delta n) = \frac{1}{2\pi} \int_{T^2_\delta} \hat{h}(k) e^{ikn\delta} dk \tag{4.2}$$

and the Plancherel's formula

$$\sum_{n \in \mathbb{Z}^2} \delta^2 |h(\delta n)|^2 = \int_{T^2_\delta} |\hat{h}(k)|^2 dk \tag{4.3}$$

By Fourier transformation $-\Delta_\delta + m^2$ acts as a multiplication by the function $\mu_\delta(k)^2$ in $L^2(T^2_\delta)$ where

$$\mu_\delta(k)^2 = \delta^{-2}[4-2\cos(\delta k_1) - 2\cos(\delta k_2)] + m^2 . \tag{4.4}$$

Set for $x \in \mathbb{R}^2$

$$f_{\delta,n}(x) = (2\pi)^{-2} \int_{T_\delta^2} e^{ik(x-n\delta)} \frac{\mu(k)}{\mu_\delta(k)}\, dk \tag{4.5}$$

and define the lattice approximation of the random field by

$$\xi_\delta(\delta n) = \xi(f_{\delta,n}) \tag{4.6}$$

Then $\xi_\delta(\delta n)$ are Gaussian stochastic variables with mean zero and covariance given by

$$E(\xi_\delta(\delta n)\xi_\delta(\delta n')) = G_\delta(\delta n-\delta n') = (2\pi)^{-2} \int_{T_\delta^2} e^{i(\delta n-\delta n')k} \frac{dk}{\mu_\delta(k)^2} . \tag{4.7}$$

From (4.7) we see that G_δ as a convolution operator on $l_2(L_\delta^2)$ is the inverse of $-\Delta_\delta+m^2$, and since $-\Delta_\delta$ is the nearest neighbour difference operator given by (4.1) we see that $\xi_\delta(\delta n)$ is a discrete Markovian field in the sense of Spitzer [18a] (see also Dobrushin [18b]).

For any g in $C_0^\infty(\mathbb{R}^2)$, we define

$$\xi_\delta(g) = \sum_{n\in\mathbb{Z}^2} \delta^2 \xi_\delta(\delta n) g(n\delta) \tag{4.8}$$

and we shall similarily define the lattice approximation to the Wick powers of the field by

$$:\xi_\delta^r:(g) = \sum_{n\in\mathbb{Z}^2} \delta^2 :\xi_\delta(\delta n)^r: g(n\delta) , \tag{4.9}$$

and the approximation to the Wick exponential for $|\alpha| < \frac{4}{\sqrt{\pi}}$ is given by the $L_2(X,d\mu)$ convergent series

$$:e^{\alpha\xi_\delta}:(g) = \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} :\xi_\delta^m:(g) . \qquad (4.10)$$

That this series actually converges in $L_2(X,d\mu)$ and approximates the Wick exponential for $|\alpha| < \frac{4}{\sqrt{\pi}}$ and smooth g is proven below.

Guerra, Rosen and Simon [9b), theorem IV, 1] have the following result:

<u>Lemma 4.1</u>.

For g in $C_0^\infty(\mathbb{R}^2)$, $:\xi_\delta^r:(g)$ converge to $:\xi^r:(g)$ in $L_p(X,d\mu)$ for $1 \leq p < \infty$ as $\delta \to 0$.

We shall need a corresponding result for the Wick ordered exponentials, and we have

<u>Theorem 4.1</u>.

For g in $C_0^\infty(\mathbb{R}^2)$ and $\varepsilon > 0$, $:e^{\alpha\xi_\delta}:(g)$ converge to $:e^{\alpha\xi_\delta}:(g)$ in $L_2(X,d\mu)$ uniformly in α for $|\alpha| \leq \frac{4}{\sqrt{\pi}} - \varepsilon$ as $\delta - 0$.

<u>Proof</u>. By lemma 4.1 it is enough to prove that the series in (4.10) converge in $L_2(X,d\mu)$ uniformly in δ and α for $|\alpha| \leq \frac{4}{\sqrt{\pi}} - \varepsilon$. The L_2 norm of $:\xi_\delta^r:(g)$ is given by

$$E[(:\xi_\delta^r:(g))^2] = r! \sum_{n,n'} \delta^4 g(n\delta)(G_\delta(n\delta - n'\delta))^r g(n'\delta). \qquad (4.11)$$

Let G_δ^r be the convolution operator on $l_2(L_\delta^2)$ with the kernel $(G_\delta(n\delta-n'\delta))^r$. We remark that $\mu_\delta(k)^{-2}$ is a positive definite function because from (4.4) we have that

$$\mu_\delta(k)^{-2} = \tfrac{1}{2}\left(\frac{m^2}{2} + 2\delta^{-2}\right)^{-1} \sum_{r=0}^{\infty} \left(\delta^2\left(\frac{m^2}{2} + 2\delta^{-2}\right)\right)^{-r} (\cos(\delta k_1) + \cos(\delta k_2))^r ,$$

where the series converges absolutely since $\left(\delta^2\left(\frac{m^2}{2} + 2\delta^{-2}\right)\right)^{-1} < \frac{1}{2}$, and $\cos(\delta k_1) + \cos(\delta k_2)$ is a positive definite function.

Hence $G_\delta(n\delta - n'\delta)$ is a non-negative function, and therefore the kernel of G_δ^r is non-negative, and the norm of G_δ^r as an operator on $l_2(L_\delta^2)$ is simply given by

$$\|G_\delta^r\| = \sum_n \delta^2 (G_\delta(n\delta))^r \ . \tag{4.12}$$

Using now that the integral of a function is the value at zero of its Fourier transform, we get from (4.17) that $\|G_\delta\| = m^{-2}$. Similarly we have that

$$\|G_\delta^r\| = m^{-2}(2\pi)^{-2(r-1)} \int_{|k_i| \leq \pi/\delta} \cdots \int \mu_\delta(k_1)^{-2} \mu_\delta(k_2 - k_1)^{-2} \cdots \mu_\delta(k_m - k_{m-1}) \prod_{j=1}^{m-1} dk_j \tag{4.13}$$

where $|k| = \max\{|k^1|, |k^2|\}$. Lemma IV.2 of [9b] gives the following estimate

$$\mu_\delta(k)^{-1} \leq \frac{\pi}{2} \mu(k)^{-1} \tag{4.14}$$

for $|k| \leq \pi/\delta$. Since $\mu(k)^{-2}$ is a positive function, we get from (4.13) and (4.14) the following estimate for the norm of G_δ^r:

$$\|G_\delta^r\| \leq \frac{2}{\pi} m^{-2} (2\pi)^{-2(r-1)} \int \cdots \int \mu(k_1)^{-2} \mu(k_2 - k_1)^{-2} \cdots \mu(k_{r-1} - k_{r-2})^{-2} \prod_{j=1}^{r-1} dk_j \tag{4.15}$$

where the integrations now run over all of $\mathbb{R}^2$.

Let

$$G(x-y) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i(x-y)k} \frac{dk}{\mu(k)^2} \ , \tag{4.16}$$

and let G^r be the convolution operator on $L_2(\mathbb{R}^2)$ with the kernel $(G(x-y))^r$. In the same way as we derived the formula (4.13) we get the analogous formula for the norm of G^r. Hence we have proved the estimate

$$\|G^r_\delta\| \leq (\tfrac{\pi}{2})^{2(r-1)} \|G^r\| . \tag{4.17}$$

The next step in the proof is to give an estimate for the norm of G^r. Let K be the convolution operator on $L_2(\mathbb{R}^2)$ with the kernel $K(x-y) = e^{\alpha^2 G(x-y)}-1$. Since $\mu(k)^{-2} = \frac{1}{m^2+k^2}$ is a positive definite operator $G(x-y)$ is non-negative, and this gives that $K(x-y)$ is non-negative. Hence the norm of K is given by $\int_{\mathbb{R}^2} K(x)dx$, and K is a bounded operator if $K(x)$ is an integrable function. Since $G(x)$ tends to zero exponentially as $|x| \to \infty$ and $G(x)$ is bounded for $|x| > 1$, we have that $\int_{|x|>1} K(x)dx < \infty$. Actually $G(x) = \frac{1}{2\pi} K_0(m|x|)$, where $|x|^2 = x_1^2 + x_0^2$ and K_0 is the modified Bessel function, so that $G(x) + \frac{1}{2\pi}\ln|x|$ is bounded for $|x| \leq 1$. Thus $K(x)$ is integrable if and only if

$$\int_{|x|\leq 1} e^{-\alpha^2 \frac{1}{2\pi}\ln|x|} dx = \int_{|x|\leq 1} |x|^{-\frac{\alpha^2}{2\pi}} dx$$

is finite. This gives the condition $|\alpha| < 2\sqrt{\pi}$. We have therefore proved, that for $|\alpha| < 2\sqrt{\pi}$, K is a bounded operator on $L_2(\mathbb{R}^2)$ and that $K(x)$ is an integrable function.

We shall now compute the integral of $K(x)$ in a different manner. By the definition of $K(x)$ we have that

$$K(x) = \sum_{n=1}^{\infty} \frac{\alpha^{2n}}{n!} (G(x))^n .$$

Since $G(x)$ is non-negative and $K(x)$ is integrable for $|\alpha| < 2\sqrt{\pi}$, we have by monotone convergence that

$$\int K(x)dx = \sum_{n=1}^{\infty} \frac{\alpha^{2n}}{n!} \int (G(x))^n dx$$

and the series converge. Hence

$$\|K\| = \sum_{n=1}^{\infty} \frac{\alpha^{2n}}{n!} \|G^n\| = \int K(x)dx \tag{4.18}$$

for $|\alpha| < 2\sqrt{\pi}$. Since the series is convergent the terms must be bounded which gives the inequality $\alpha^{2n}\|G^n\| \leq n!$ for all $|\alpha| < 2\sqrt{\pi}$. So that

$$\|G^n\| \leq (4\pi)^{-n} \cdot n! \ , \tag{4.19}$$

which together with (4.17) gives the uniform estimate

$$\|G_\delta^r\| \leq (4\pi)^{-r}(\tfrac{\pi}{2})^{2(r-1)} r! \ . \tag{4.20}$$

We are now in the position to prove that the series (4.10) converges in $L_2(X,d\mu)$ uniformly in δ. Since the sum in (4.10) is a direct sum in $L_2(X,d\mu)$ it is enough to prove that

$$\sum_{m=0}^{\infty} \frac{\alpha^{2m}}{(m!)^2} E\left[(:\xi_\delta^m:(g))^2\right] \tag{4.21}$$

converges uniformly in δ. By (4.11) we have that

$$E\left[(:\xi_\delta^r:(g))^2\right] \leq r!\|g_\delta\|_2^2 \ \|G_\delta^r\| \ , \tag{4.22}$$

for $r \geq 1$ and it is bounded by $\|g_\delta\|_1^2$ for $r = 0$, where

$\|g_\delta\|_2^2 = \sum_{n\in Z^2} \delta^2 |g(n\delta)|^2$ and $\|g_\delta\|_1 = \sum_{n\in Z^2} \delta^2 |g(n\delta)|$.

From (4.20), (4.21) and (4.20) we see that for $|\alpha| < \frac{4}{\sqrt{\pi}}$, g and $|g|^2$ Riemann integrable, the series (4.21) converges uniformly in δ. Moreover 4.21 converges also uniformly in α for $|\alpha| \leq \frac{4}{\sqrt{\pi}} - \epsilon$ for any $\epsilon > 0$.

As a consequence of Theorem 4.1 we have that, for g in $C_0^\infty(\mathbb{R}^2)$ and $\epsilon > 0$, and $d\nu(\alpha)$ a finite positive measure with support in $[-\frac{4}{\sqrt{\pi}}+\epsilon, \frac{4}{\sqrt{\pi}}-\epsilon]$ $:V_\delta:(g) = \int :e^{\alpha\xi_\delta}:(g)d\nu(\alpha)$ converges in $L_2(X,d\mu)$ to

$$:V:(g) = \int :e^{\alpha\xi}:(g)d\nu(\alpha), \text{ as } \delta \to 0 .$$

This implies that there exists a subsequence $\delta_{n_j}, \delta_{n_j} \to 0$ for $n_j \to \infty$ such that $:V_\delta:(g)$ converges to $:V:(g)$ almost everywhere with respect to the measure $d\mu$. Hence $e^{-:V_\delta:(g)}$ converges almost everywhere to $e^{-:V:(g)}$ and since $e^{-:V_\delta:(g)}$ is uniformly bounded by the constant 1 we get that $e^{-:V_\delta:(g)}$ converges to $e^{-:V:(\delta)}$ in $L_p(X;d\mu)$ for all $p \leq \infty$, as $\delta \to 0$. Since the products $\xi_\delta(g_1)\dots\xi_\delta(g_r)$ are in L_p for all $1 \leq p < \infty$ and converge by Lemma 4.1 to $\xi(g_1)\dots\xi(g_r)$ in L_p, we have that, as $\delta \to 0$

$$S_{\delta,g}(g_1\dots g_r) \equiv \int \xi_\delta(g_1)\dots\xi_\delta(g_r)d\mu_{\delta,g} \text{ converges to} \tag{4.23}$$

$$S_g(g_1\dots g_r) = \int \xi(g_1)\dots\xi(g_r)d\mu_g \text{ , where}$$

$d\mu_{\delta,g}$ is the normalized measure $d\mu_{\delta,g} = e^{-:V_\delta:(g)}\left(\int e^{-:V_\delta:(g)}d\mu\right)^{-1}$.

We have thus proven the following

<u>Theorem 4.2.</u>

For any $g \geq 0$, $g \in C_0^\infty(\mathbb{R}^2)$ and $h_1,\dots,h_r \in C_0^\infty(\mathbb{R}^2)$ the lattice approximation $S_{g,\delta}(h_1\dots h_r)$ of the space-time cut-off Schwinger functions converge as $\delta \to 0$ to the space-time cut-off Schwinger function $S_g(h_1\dots h_r)$.

We shall now prove correlation inequalities for the $S_{g,\delta}$. Applying Theorem 4.2 we then will get the same inequalities also for S_g. The correlation inequalities for the $S_{g,\delta}$ are a consequence of some general inequalities which we are going to state.

First we need a definition. A ferromagnetic measure on $\mathbb{R}^n$ is any finite measure of the form $d\lambda(\underline{x}) = F_1(x_1)\dots F_n(x_n)e^{-\frac{1}{2}\underline{x}A\underline{x}}\,d\underline{x}$, where $\underline{x}$ stands for the n-tuple $x_1,\dots,x_n$, $F_i(x_i)$ are continuous, bounded, positive function on $\mathbb{R}$ and A is a positive definite matrix with non positive off-diagonal elements. A ferromagnetic measure is called even if $F_i(x) = F_i(-x)$ for all $i = 1,2,\dots,n$ ([9b]). One has:

Lemma 4.2 [9b]. If $d\lambda$ is an even ferromagnetic finite measure on R^n then the following two inequalities of Griffiths type hold:

$$\langle x_1^{j_1}\dots x_r^{j_r}\rangle_\lambda \geq 0$$

$$\langle x_1^{j_1+k_1}\dots x_r^{j_r+k_r}\rangle_\lambda \geq \langle x_1^{j_1}\dots x_r^{j_r}\rangle_\lambda\langle x_1^{k_1}\dots x_r^{k_r}\rangle_\lambda ,$$

where $\langle\ \rangle_\lambda$ means expectation with respect to the normalized measure $(\int d\lambda)^{-1}d\lambda$.

Moreover for any two continuous, polynomially bounded functions F,G such that $F(x_1,\dots,x_n) \leq F(y_1,\dots,y_n)$, $G(x_1,\dots,x_n) \leq G(y_1,\dots,y_n)$ whenever $x_i \leq y_i$ for all $i = 1,\dots,n$, one has the inequality of FKG-type

$$\langle FG\rangle_\lambda \geq \langle F\rangle_\lambda \langle G\rangle_\lambda .$$

Remark: For a proof and discussion of above inequalities we refer to [9b]. The proof uses conditions isolated in connection with statistical mechanical problems by Ginibre [8a] and Fortuyn-Kasteleyn-Ginibre [19]. ▨

Lemma 4.2 can be applied, as remarked in [9] for the case of the $P(\varphi)_2$ interactions, to prove correlation inequalities for the Schwinger functions. Consider $S_{g,\delta}(h_1,\dots,h_r)$, with fixed $g,\delta,h_1,\dots,h_r$. $S_{g,\delta}(h_1,\dots,h_r)$ is expressed in terms of expectations involving only finitely many stochastic gaussian variables $\eta_1,\dots,\eta_N$, where $\eta_j = \xi_\delta(n_j\delta)$ for some n_j such that $n_j\delta$ belongs to the lattice L^2_δ and to the union of the supports of g and $h_1,\dots,h_r$. Thus the expectations can be expressed through expectations with respect to the finite dimensional measure space $(\mathbb{R}^N, d\mu^{(N)})$, where $d\mu^{(N)}$ is the restriction of $d\mu$ to $\mathbb{R}^N$. In [9b, Sect. 4,5] it is proven that $d\mu^{(N)}$ is ferromagnetic. Since moreover

$$e^{-:V_\delta:(g)} = \prod_j e^{-\delta^2 :V(\eta_j):g(n_j\delta)},$$

where the product is over finitely many j, we have that $d\mu_{\delta,g}$ is ferromagnetic as a measure in $\mathbb{R}^N$. Furthermore if $V(\alpha)=V(-\alpha)$, then $d\mu_{\delta,g}$ is even.

From Lemma 4.2 we have then, for $h \geq 0$, $i = 1,\dots,r$:

$$S_{\delta,g}(h_1,\dots,h_r) \geq 0 \qquad (4.24)$$

$$S_{\delta,g}(h_1,\dots,h_{r+s}) \geq S_{\delta,g}(h_1,\dots,h_r)\, S_{\delta,g}(h_{r+1},\dots,h_{r+s}) \qquad (4.25)$$

and

$$\begin{aligned} &\langle F(\xi_\delta(h_1)\dots\xi_\delta(h_r))\, G(\xi_\delta(g_1)\dots\xi_\delta(g_s))\rangle_{\delta,g} \geq \\ &\langle F(\xi_\delta(h_1)\dots\xi_\delta(h_r)\rangle_{\delta,g}\, \langle G(\xi_\delta(g_1)\dots\xi_\delta(g_s))\rangle_{\delta,g}, \end{aligned} \qquad (4.26)$$

for any $g_i, h_j \in C_0^\infty(\mathbb{R}^2)$ and any F,G like in Lemma 4.2.

$\langle\ \rangle_{\delta,g}$ stands for expectation with respect to the measure $d\mu_{\delta,g}$. By Theorem 4.2 the $S_{g,\delta}$ converges as $\delta \to 0$ to the corresponding functions S_g, and thus (4.24), (4.25) hold also for S_g. Moreover if F and G are taken to be bounded we get by Lemma 4.1 and Theorem 4.1 that $F(\xi_\delta(h_1)\ldots\xi_\delta(h_r)) \to F(\xi(h_1)\ldots\xi(h_r))$ in $L_2(X,d\mu)$ and similarly for G, which then completes the proof of the following.

<u>Theorem 4.3</u>. The Schwinger functions of the space-time cut-off exponential interactions (2.1), with $V(\alpha) = V(-\alpha)$ $g,h_1,\ldots,h_r \in C_0^\infty(\mathbb{R}^2)$, $g,h_1,\ldots,h_r \geq 0$ satisfy the following correlation inequalities:

$$S_g(h_1,\ldots,h_r) \geq 0$$

$$S_g(h_1,\ldots,h_{r+s}) \geq S_g(h_1,\ldots,h_r)\ S_g(h_{r+1},\ldots,h_s)\ .$$

Moreover one has:

$$\langle F\ (\xi(h_1)\ldots\xi(h_r))\, G\ (\xi(g_1)\ldots\xi(g_r))\,\rangle_g \geq$$

$$\langle F\ (\xi(h_1)\ldots\xi(h_r))\,\rangle_g\ \langle G\ (\xi(g_1)\ldots\xi(g_r))\,\rangle_g\ ,$$

where $\langle F \rangle_g \equiv \int F\, d\mu_g$, with

$$d\mu_g \equiv E(d^{-:V:(g)})^{-1}\ e^{-:V:(g)} d\mu\ ,$$

for any continuous, bounded functions F,G such that $F(x_1,\ldots,x_n) \leq F(y_1,\ldots,y_n)$, $G(x_1,\ldots,x_n) \leq G(y_1,\ldots,y_n)$ whenever $x_i \leq y_i$ for all $i = 1,\ldots,n$. ▨

5. The monotonicity of the Schwinger functions.

The Schwinger functions $S_g(h_1,\dots,h_r)$ are functionals of the space-time cut-off function g for $g \geq 0$ and $g \in C_0^\infty(\mathbb{R}^2)$. For any functional $\Psi(g)$ depending on a function $g \geq 0$, $G \in C_0^\infty$, we define, for any function $\gamma \geq 0$, $\gamma \in C_0^\infty$, the directional derivative at the point g as

$$D_\gamma \Psi(g) = \lim_{\epsilon \searrow 0} \epsilon^{-1} [\Psi(g+\epsilon\gamma) - \Psi(g)] , \tag{5.1}$$

whenever this limit exists.

Lemma 5.1.

For any $g \geq 0$ and $g \in C_0^\infty(\mathbb{R}^2)$ and any $\gamma \geq 0$ and $\gamma \in C_0^\infty(\mathbb{R}^2)$, the directional derivative of $S_g(h_1,\dots,h_r)$ exists and is given by

$$\begin{aligned} D_\gamma S_g(h_1,\dots,h_r) &= \langle :V:(\gamma)\rangle_g \langle \xi(h_1)\dots\xi(h_r)\rangle_g \\ &+ \langle \xi(h_1)\dots\xi(h_r):V:(\gamma)\rangle_g , \end{aligned} \tag{5.2}$$

where $\langle \ \rangle_g$ stands for the expectation with respect to the measure

$$d\mu_g = \left[E(e^{-:V:(g)})\right]^{-1} e^{-:V:(g)} d\mu .$$

Proof: Let F be any function in $L_2(X,d\mu)$ and g and γ be non-negative functions in $C_0^\infty(\mathbb{R}^2)$. We set $\Psi(g) = E(F\, e^{-:V:(g)})$, and we shall see that the directional derivative $(D_\gamma\Psi)(g)$ exists. Consider the expression

$$\frac{1}{\epsilon}\left[\Psi(g+\epsilon\gamma) - \Psi(g)\right] + E\left[F:V:(\gamma)\ e^{-:V:(g)}\right] , \tag{5.3}$$

which makes sense since F and $:V:(\gamma)$ are in $L_2(X,d\mu)$ and

$:V:(g) \geq 0$. If we prove that (5.3) tends to zero as $\varepsilon \searrow 0$ through positive values we have proved that the directional derivative $(D_\gamma \Psi(g)$ exists and that

$$(D_\gamma \Psi)(g) = -E\left[F{:}V{:}\,(\gamma)\ e^{-:V:(g)}\right] . \qquad (5.4)$$

Now (5.3) is equal to

$$E\left[F\ e^{-:V:(g)}\left\{:V:(\gamma) - \frac{1}{\varepsilon}(1 - e^{-\varepsilon :V:(\gamma)})\right\}\right] , \qquad (5.5)$$

and since $0 \leq \alpha - \frac{1}{\varepsilon}(1 - e^{-\varepsilon\alpha}) \leq \alpha$ for $\varepsilon > 0$ and $\alpha \geq 0$, we have that the absolute value of the integrand in (5.5) is dominated by $|F|\ :V:(\gamma)$ which is in L_1 since both F and $:V:(\gamma)$ are in L_2. Hence by dominated convergence we get that (5.5) tends to zero as $\varepsilon \searrow 0$ through positive values, since $\alpha - \frac{1}{\varepsilon}(1 - e^{-\varepsilon\alpha})$ tends to zero as $\varepsilon \searrow 0$. To justify the inequality $0 \leq \alpha - \frac{1}{\varepsilon}(1 - e^{-\varepsilon\alpha})$ for $\varepsilon > 0$ and $\alpha \geq 0$, we consider the function $\eta(\varepsilon,\alpha) = \alpha - \varepsilon^{-1}(1 - e^{-\varepsilon\alpha})$ for $\varepsilon \neq 0$ and $\eta(0,\alpha) = 0$. $\eta(\varepsilon,\alpha)$ is continuously differentiable with $\frac{\partial\eta}{\partial\alpha} = 1 - e^{-\varepsilon\alpha} \geq 0$ for $\varepsilon \geq 0$ and $\alpha \geq 0$ and since $\eta(\varepsilon,0) = 0$ we see that $\eta(\varepsilon,\alpha) \geq 0$ for all $\varepsilon \geq 0$, $\alpha \geq 0$. But

$$S_g(h_1,\ldots,h_r) = \left[E(e^{-:V:(g)})\right]^{-1} E(\xi(h_1)\ldots\xi(h_r)\ e^{-:V:(g)})$$

and we get, using also the fact that the quotient of two on-sided differentiable functions is again a one-sided differentiable function, that $D_\gamma S_g(h_1,\ldots,h_r)$ exists. Then the formula (5.4) gives the one in the lemma. ▨

We shall now see that, for g,γ, and $h_1,\ldots,h_r$ all non-negative, $D_\gamma S_g(h_1,\ldots,h_r)$ is non-positive.

We recall that

$$:V:(\gamma) = \int :e^{\alpha\xi}:(\gamma)d\nu(\alpha) \ . \tag{5.6}$$

Let, for $\varepsilon > 0$, $\chi_\varepsilon(x) = \varepsilon^{-2}\chi(\varepsilon^{-1}x)$, where $\chi(x) \geq 0$ and $\chi(x)$ is in $C_0^\infty(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2}\chi(x)dx = 1$. Then we define

$$\xi_\varepsilon(x) = \int \chi_\varepsilon(x-y)\xi(y)dy \ , \tag{5.7}$$

and

$$:V_\varepsilon:(\gamma) = \iint :e^{\alpha\xi_\varepsilon(x)}: d\nu(\alpha)\gamma(x)dx. \tag{5.8}$$

We shall see that $:V_\varepsilon:(\gamma)$ converges to $:V:(\gamma)$ in $L_2(X,d\mu)$ as $\varepsilon \to 0$.

In Sobolec space $H_{-1}(\mathbb{R}^2)$ we define the operator

$$(\mathcal{J}_\varepsilon h)(x) = \int \chi_\varepsilon(x-y)h(y)dy \ . \tag{5.9}$$

It is well known that, for $h \in C_0^\infty(\mathbb{R}^2)$, $\mathcal{J}_\varepsilon h$ converges strongly to h in the topology of $C_0^\infty(\mathbb{R}^2)$, hence also strongly in $H_{-1}(\mathbb{R}^2)$. Since $\mathcal{J}_\varepsilon$ is a convolution operator it is given by a multiplication operator in the Fourier transformed realization of $H_{-1}(\mathbb{R}^2)$. Hence the norm of $\mathcal{J}_\varepsilon$ as an operator on $H_{-1}(\mathbb{R}^2)$ is given by the supremum of the Fourier transform $\hat{\chi}_\varepsilon(p)$ of $\chi_\varepsilon(x)$. Since $\chi_\varepsilon(x) \geq 0$, $\hat{\chi}_\varepsilon(p)$ is a positive definite function and therefore $\hat{\chi}_\varepsilon(0) = \sup|\hat{\chi}_\varepsilon(p)| = \int \chi_\varepsilon(x)dx = 1$. As was pointed out in section 2

$$L_2(X,d\mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \ , \tag{5.10}$$

where $\mathcal{H}_n$ could be identified with the n-th symmetric tensor-product of $H_{-1}(\mathbb{R}^2)$ with itself. Relative to the decomposition (5.10) we define

$$\hat{\mathcal{J}}_\varepsilon = \bigoplus_{n=0}^{\infty} \mathcal{J}_\varepsilon^{(n)} . \tag{5.11}$$

where $\mathcal{J}_\varepsilon^{(n)} = \mathcal{J}_\varepsilon \otimes \ldots \otimes \mathcal{J}_\varepsilon$. Since $\|\mathcal{J}_\varepsilon\| = 1$ we get from (5.11) that $\|\hat{\mathcal{J}}_\varepsilon\| = 1$. Moreover since $\mathcal{J}_\varepsilon$ converges to 1 we get that $\mathcal{J}_\varepsilon^{(n)}$ must converge strongly to 1. From which it follows by uniform boundedness that $\hat{\mathcal{J}}_\varepsilon$ converges strongly to 1.

By expanding the Wick ordered exponential in (5.8) one gets imediately that

$$:V_\varepsilon:(\gamma) = \hat{\mathcal{J}}_\varepsilon :V:(\gamma) . \tag{5.12}$$

Hence, for any $F \in L_2(X, d\mu)$, we get that

$$\langle F:V_\varepsilon:(\gamma)\rangle_g \to \langle F:V:(\gamma)\rangle_g , \tag{5.13}$$

since

$$\langle F:V_\varepsilon:(\gamma)\rangle_g = \left[E(e^{-:V:(g)})\right]^{-1} E\left[F\, e^{-:V:(g)} \cdot :V_\varepsilon\cdot(\gamma)\right] .$$

Now we have on the other hand from (5.8) that

$$\begin{aligned} :V_\varepsilon:(\gamma) &= \iint e^{-\frac{\alpha^2}{2}(\chi_\varepsilon G \chi_\varepsilon)}\, e^{\alpha \xi_\varepsilon(x)}\, d\nu(\alpha)\gamma(x)dx \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \alpha^n\, e^{-\frac{\alpha^2}{2}(\chi_\varepsilon G \chi_\varepsilon)} d\nu(\alpha) \int \xi_\varepsilon(x)^n \gamma(x)dx \qquad (5.14) \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \int \alpha^{2k}\, e^{-\frac{\alpha^2}{2}(\chi_\varepsilon G \chi_\varepsilon)} d\nu(\alpha) \int \xi_\varepsilon(x)^{2k} \gamma(x)dx , \end{aligned}$$

where the series converges in $L_2(X, d\mu)$ since it is a sum of positive functions, if we assume that

$$V(s) = V(-s) , \tag{5.15}$$

i.e. $\nu(\alpha) = \nu(-\alpha)$. We recall that

$$(\chi_\varepsilon G \chi_\varepsilon) = \iint \chi_\varepsilon(x) G(x-y) \chi_\varepsilon(y)\, dx\, dy .$$

Assuming from now on $V(\alpha) = V(-\alpha)$, let us consider the expectation

$$\langle \xi(h_1)\ldots\xi(h_r) : V_\varepsilon : (\gamma)\rangle_g =$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \int \alpha^{2k} e^{-\frac{\alpha^2}{2}(\chi_\varepsilon G \chi_\varepsilon)} d\nu(\alpha) \langle \xi(h_1)\ldots\xi(h_r)\xi_\varepsilon(x)^{2k}\rangle_g \gamma(x) dx , \tag{5.16}$$

for $h_1 \geq 0,\ldots,h_r \geq 0$, $g \geq 0$, $\gamma \geq 0$ and g,γ in $C_0^\infty(\mathbb{R}^2)$,

where we have interchanged summation and integration using the fact that the series in (5.14) converges in $L_2(X,d\mu)$ and $\xi(h_1)\ldots\xi(h_r)$ is in $L_2(X,d\mu)$. Now since $\xi_\varepsilon(x) = \int \chi_\varepsilon(x-y)\xi(y)dy$, where $\chi_\varepsilon(x-y) \geq 0$, we get by Theorem 4.3 that (5.16) is larger or equal to

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} \int \alpha^{2k} e^{-\frac{\alpha^2}{2}(\chi_\varepsilon G \chi_\varepsilon)} d\nu(\alpha)\ \langle \xi_\varepsilon(x)^{2k}\rangle_g \gamma(x) dx .$$

$$\langle \xi(h_1)\ldots\xi(h_r)\rangle_g = \langle :V_\varepsilon: (\gamma)\rangle_g\ \langle \xi(h_1)\ldots\xi(h_r)\rangle_g .$$

Hence

$$\langle \xi(h_1)\ldots\xi(h_r) : V_\varepsilon :(\gamma)\rangle_g \geq \langle :V_\varepsilon :(\gamma)\rangle_g\ \langle \xi(h_1)\ldots\xi(h_r)\rangle_g .$$

Now by the fact that $:V_\varepsilon:(\gamma)$ converges to $:V:(\gamma)$ in $L_2(X,d\mu)$ we have that

$$\langle \xi(h_1)\ldots\xi(h_r) :V:(\gamma)\rangle_g \geq \langle :V:(\gamma)\rangle_g\ \langle \xi(h_1)\ldots\xi(h_r)\rangle_g . \tag{5.17}$$

<u>Theorem 5.1.</u>

Let $g \geq 0$ and g in $C_0^\infty(\mathbb{R}^2)$. Then for a space and time cut-off Wick ordered exponential interaction, where the interaction density is given by an even function $V(s) = \int e^{s\alpha} d\nu(\alpha)$, where ν is a finite positive measure with compact support in the open interval $(-4/\sqrt{\pi}, 4/\sqrt{\pi})$, the Schwinger functions $S_g(x_1,\ldots,x_n)$ are non-negative, locally integrable functions, which depend monotonically on g i.e.

$$0 \leq S_g(x_1,\ldots,x_n) \leq S_o(x_1,\ldots,x_n)$$

and

$$S_g(x_1,\ldots,x_n) \leq S_{g'}(x_1,\ldots,x_n) ,$$

for $g' \leq g$, where $S_o(x_1,\ldots,x_n)$ are the Schwinger functions for the free field:

$$S_o(h_1,\ldots,h_n) = E(\xi(h_1)\ldots\xi(h_n)) .$$

<u>Proof</u>: From (5.17) and Lemma 5.1 we see that, for $g \geq 0$ and $\gamma \geq 0$ and $h_1,\ldots,h_n \geq 0$, the directional derivative $D_\gamma S_g(h_1,\ldots,h_r)$ is non-positive. Hence $S_g(h_1,\ldots,h_r)$ will decrease as g increases. This gives that $S_g(h_1,\ldots,h_r) \leq S_o(h_1,\ldots,h_r)$, which proves that $S_g(x_1,\ldots,x_r)$ are locally integrable functions. That these functions are non-negative follows from theorem 4.3. This proves the theorem. ▨

<u>Theorem 5.2.</u>

If we consider space-time cut-off interactions of the form $\lambda{:}V{:}\,(g) + \mu{:}\xi^2{:}\,(g_1)$, where the function $V(s)$ satisfies the same restriction as in Theorem 5.1, then the same conclusion as in Theorem 5.1 hold. Moreover

$$S_{g,g_1,\lambda,\mu}(x_1,\ldots,x_n) \leq S_{g',g_1',\lambda',\mu'}(x_1,\ldots,x_n),$$

for $\lambda' \leq \lambda$ and $\mu' \leq \mu$, $g' \leq g$ and $g_1' \leq g_1$.

Proof: The proof of this theorem goes in the same way as the proof of theorem 5.1, with the help of the observation that $:\xi_\varepsilon^2:(g_1) = \int \xi_\varepsilon(x)^2 g_1(x)dx$ + constant.

Corollary to theorem 5.1 and theorem 5.2.

The conclusions in theorem 5.1 and theorem 5.2 also hold if we assume only that g, g' and g_1, g_1' are in $L_1(\mathbb{R}^2 \cap L_2(\mathbb{R}^2)$. Moreover the Schwinger functions depend strongly continuously on g and g_1 for g and g_1 in $L_1 \cap L_2$.

Proof: $:V:(g)$ depends linearly on g for $g \geq 0$ and $g \in C_0^\infty(\mathbb{R}^2)$, moreover

$$\|:V:(g)\|_2^2 = \iiiint e^{\alpha\beta G(x-y)} g(x)g(y)d\nu(\alpha)d\nu(\beta)dx\,dy \ . \qquad (5.18)$$

Let $$K(x-y) = \iint (e^{\alpha\beta G(x-y)}-1)d\nu(\alpha)d\nu(\beta). \qquad (5.19)$$

By the assumptions of the theorems $K(x)$ is an integrable function. So that the norm $\|K\|$ of $K(x-y)$ as a convolution operator in $L_2(\mathbb{R}^2)$ is bounded by $\int |K(x)|\,dx < \infty$. This gives, by 5.18, that

$$\|:V:(g)\|_2^2 \leq \|K\| \cdot \|g\|_2^2 + \|\nu\|^2 \|g\|_1^2 \ . \qquad (5.20)$$

Hence $:V:(g)$ is a bounded linear operator $:V:$ from $L_2(\mathbb{R}^2) \cap L_1(\mathbb{R}^2)$ into $L_2(X,d\mu)$.

Let now $g \geq 0$ be in $L_2(\mathbb{R}^2) \cap L_1(\mathbb{R}^2)$ and choose $g_n \in C_0^\infty(\mathbb{R}^2)$ such that $\|g_n - g\|_2 \to 0$ and $\|g_n - g\|_1 \to 0$. Then $:V:(g_n) \to :V:(g)$ in $L_2(X,d\mu)$. Hence there is a subsequence $g_{n'}$ such that $:V:(g_{n'}) \to :V:(g)$ almost everywhere. Therefore

$$S_{g_{n'}}(h_1,\ldots h_r) = \left[E(e^{-:V:(g_{n'})})\right]^{-1} E\left[\xi(h_1)\cdots\xi(h_r)e^{-:V:(g_{n'})}\right]$$

will converge to $S_g(h_1,\ldots,h_r)$. This also proves that $S_g(h_1,\ldots,h_r)$ is a strongly continuous function of g for $g \in L_2 \cap L_1$. Let now g and g' be in $L_2 \cap L_1$ and of compact support and $0 \leq g' \leq g$ almost everywhere. We define $g_\epsilon(x) = \int \chi_\epsilon(x-y)g(y)dy$ and $g_\epsilon'(x) = \int \chi_\epsilon(x-y)g'(y)dy$. Since $g' \leq g$ almost everywhere we have that g_ϵ and g_ϵ' are in $C_0^\infty(\mathbb{R}^2)$ and $0 \leq g_\epsilon'(x) \leq g_\epsilon(x)$, and moreover g_ϵ and g_ϵ' converge strongly to g and g' in $L_2 \cap L_1$. From theorem 5.1 we then have that, for $h_1 \geq 0,\ldots,h_r \geq 0$,

$$S_{g_\epsilon}(h_1,\ldots,h_r) \leq S_{g_\epsilon'}(h_1,\ldots,h_r) \leq S_0(h_1,\ldots,h_r) .$$

By the strong continuity in g of S_g we therefore get that

$$S_g(h_1,\ldots,h_r) \leq S_{g'}(h_1,\ldots,h_r) \leq S_0(h_1,\ldots,h_r) ,$$

for $0 \leq g' \leq g$ and g and g' of compact support. Let now g and g' be arbitrary in $L_2 \cap L_1$ such that $0 \leq g' \leq g$. Let g_N and g_N' be equal to g and g' for $|x| \leq N$ and zero of not. Then

$$S_{g_N}(h_1,\ldots,h_r) \leq S_{g_N'}(h_1,\ldots,h_r) \leq S_0(h_1,\ldots,h_r) ;$$

by the stronly continuity we therefore get

$$S_g(h_2,\ldots,h_r) \leq S_{g'}(h_1,\ldots,h_r) \leq S_0(h_1,\ldots,h_r) .$$

This proves that the conclusions of theorem 5.1 also hold for $0 \leq g \leq g'$ almost everywhere and g and g' in $L_2 \cap L_1$. In the same way one also gets the result in theorem 5.2 for g,g' and g_1,g_1' in $L_2(\mathbb{R}^2) \cap L_1(\mathbb{R}^2)$. This then proves the corollary.

We shall now prove the equation of motion for the Schwinger functions or the partial integration formula for the integral $\langle F \rangle_g$. We shall say that $F \in L_1(X, d\mu)$ has a finite dimensional base if there is a measurable function $f(y_1, \dots, y_n)$ of n real variables, and $h_1, \dots, h_n$ in $H_{-1}(\mathbb{R}^2)$ such that $F = f(\xi(h_1), \dots \xi(h_n))$. We shall say that F is differentiable if $f(y_1, \dots, y_n)$ is differentiable, and in this case we define

$$\frac{\delta F}{\delta \xi(x)} = \sum_{i=1}^{n} h_i(x) \frac{\partial f}{\partial y_i} (\xi(h_1), \dots, \xi(h_n)), \tag{5.21}$$

in the sense that

$$\int \frac{\delta F}{\delta \xi(x)} \Psi(x) dx = \sum_{i=1}^{n} \int h_i(x) \Psi(x) dx \, \frac{\partial f}{\partial y_i} (\xi(h_1), \dots, \xi(h_n)) \tag{5.22}$$

for $\Psi \in H_1(\mathbb{R}^2)$.

We shall say that F is of exponential type if f is of exponential type.

<u>Lemma 5.2</u>. (The partial integration lemma)

If F has a finite dimensional base and is differentiable and of exponential type then

$$E[\xi(x)F] = \int dy \, G(x-y) \, E\left[\frac{\delta F}{\delta \xi(y)}\right]$$

in the sense that for h in $H_{-1}(\mathbb{R}^2)$

$$E[\xi(h)F] = E\left[\iint dx \, dy \, G(x-y) h(x) \frac{\delta F}{\delta \xi(y)}\right].$$

<u>Remark</u>: This lemma may also be stated in the form of the equation of motion for the free Markoff field, namely that

$$E[(-\Delta + m^2)\xi(x)F] = E\left[\frac{\delta F}{\delta \xi(x)}\right], \tag{5.23}$$

where the equality is in the sense of distributions.

Proof: $E[\xi(h_1)F] = E[\xi(h_1)\cdot f(\xi(h_1),\dots,\xi(h_n))]$ (5.24)

We know that $G_{ij} = (h_i G h_j)$ is a strictly positive definite matrix and let A_{ij} be the inverse matrix, so that $\sum_{j=1}^{n} G_{ij}A_{jk} = \delta_{ik}$. Let $|2\pi A|$ be the determinant of the matrix $2\pi A_{ij}$. Then by (5.24)

$$E[\xi(h_1)F] = |2\pi A|^{-\frac{1}{2}} \int_{\mathbb{R}^n} y_1 f(y_1,\dots,y_n)\, e^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j}\, dy$$

$$= - |2\pi A|^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(y_1,\dots,y_n) \sum_{k=1}^{n} G_{1k} \frac{\partial}{\partial y_k}\, e^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j}\, dy \,.$$

Since $f(y_1,\dots,y_n)$ is differentiable and of exponential type we get by partial integration that this is equal to

$$|2\pi A|^{-\frac{1}{2}} \int_{\mathbb{R}^n} \sum_{k=1}^{n} G_{1k} \frac{\partial f}{\partial y_k}(y_1,\dots,y_k)\, e^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j}\, dy$$

$$= \sum_{k=1}^{n} (h_1 G h_k)\, E\left[\frac{\partial f}{\partial y_k}(\xi(h_1),\dots,\xi(h_n))\right],$$

which by (5.22) is equal to

$$E\left[\iint dx\, dy\, G(x-y) h(x) \frac{\delta F}{\delta \xi(y)} \right].$$

This proves the lemma. ▨

Let $\xi_\varepsilon(z) = \int \chi_\varepsilon(z-y)\xi(y)dy$. Since

$$:e^{\alpha\xi_\varepsilon(z)}: = e^{-\frac{\alpha^2}{2}(\chi_\varepsilon G \chi_\varepsilon)}\, e^{\alpha\xi_\varepsilon(z)} \qquad (5.25)$$

we see that $:e^{\alpha\xi_\varepsilon(z)}:$ has a finite dimensional base and is of exponential type. By (5.25) and (5.22) we get that

$$\frac{\delta}{\delta\xi(x)} : e^{\alpha\xi_\varepsilon(z)} : = \alpha : e^{\alpha\xi_\varepsilon(z)} : \chi_\varepsilon(z-x) \ . \tag{5.26}$$

Let $V'(s)$ be the derivative of the function $V(s) = \int e^{\alpha s} d\nu(\alpha)$. Integrating both sides of (5.26) with respect to $d\nu(\alpha)$ we find that $:V(\xi_\varepsilon(z)):$ has a finite dimensional base and is of exponential type and

$$\frac{\delta}{\delta\xi(x)} : V(\xi_\varepsilon(z)): = : V'(\xi_\varepsilon(z)): \chi_\varepsilon(z-x) \tag{5.27}$$

Let $g \geq 0$ and $g \in C_0^\infty(\mathbb{R}^2)$. By the previous section we then know that $\int :V(\xi_\varepsilon(z)):g(z)dz$ converges strongly in $L_2(X,d\mu)$ to $:V:(g)$ as $\varepsilon \to 0$. By the result of section 3 we know that $:V(\xi_\varepsilon(z)):$ is strongly $L_2(X,d\mu)$ continuous in z. Hence the integral $\int :V(\xi_\varepsilon(z)):g(z)dz$ is approximated strongly in $L_2(X,d\mu)$ by its Riemann approximation

$$R_{\varepsilon,\delta} = \sum_{n\in Z^2} \delta^2 : V(\xi_\varepsilon(\delta n)): g(n\delta) \ . \tag{5.28}$$

The Riemann approximation $R_{\varepsilon,\delta}$ has again a finite dimensional base and is of exponential type, and is differentiable.

Let F be a function with a finite dimensional base that is differentiable and of exponential type and consider

$$E\left[\xi(h) F \, e^{-:V:(g)}\right] . \tag{5.29}$$

By the fact that $R_{\varepsilon,\delta}$ approximate $:V:(g)$ strongly in $L_2(X,d\mu)$ and the fact that $R_{\varepsilon,\delta} \geq 0$ and strong L_2-convergence implies convergence almost everywhere of subsequences, we get that (5.29) is approximated by $E\left[\xi(h) F \, e^{-R_{\varepsilon,\delta}}\right]$. By lemma 6.1 we now get that

$$E\left[\xi(h)\, F\, e^{-R_{\varepsilon,\delta}}\right] = E\left[\iint dx\, dy\, G(x-y)h(x)\, \frac{\delta F}{\delta\xi(y)}\, e^{-R_{\varepsilon,\delta}}\right]$$
$$+ E\left[\iint dx\, dy\, G(x-y)h(x) F\, \frac{\delta R_{\varepsilon,\delta}}{\delta\xi(y)}\, e^{-R'_{\varepsilon,\delta}}\right] . \tag{5.30}$$

By (5.22), (5.28) and (5.27) we have that

$$\frac{\delta R'_{\varepsilon,\delta}}{\delta\xi(y)} = \sum_{n\in\mathbb{Z}^2} \delta^2 \; :V'(\xi_\varepsilon(\delta n)): \; g(n\delta)\chi_\varepsilon(n\delta-y) \; . \tag{5.31}$$

As $\delta \to 0$ this converges strongly in $L_2(X,d\mu)$ to

$$\int :V'(\xi_\varepsilon(z)): \; g(z)\chi_\varepsilon(z-y)dz \tag{5.32}$$

and as $\varepsilon \to 0$

$$\iiint dx\, dy\, dz\; G(x-y)h(x) \; :V'(\xi_\varepsilon(z)): \; g(z)\chi_\varepsilon(z-y) \tag{5.33}$$

converges strongly in $L_2(X,d\mu)$ to

$$\iint dx\, dy\, G(x-y)h(x)g(y) \; :V'(\xi(y)): \; .$$

By dominated convergence and the fact that $R_{\varepsilon,\delta} \geq 0$, $F\, e^{-R_{\varepsilon,\delta}}$ converges through subsequences strongly to $F\, e^{-:V:(g)}$. Hence we have proved that the last term on the right hand side of (5.30) converges through subsequences to

$$- E\left[\iint dx\, dy\, G(x-y)h(x)g(y) \; :V'(\xi(y)): F\, e^{-:V:(g)}\right] .$$

Similarily we get the convergence of the first term on the right hand side of (5.30). We have thus proved the following formula

$$E\left[\xi(h)\, F\, e^{-:V:(g)}\right] = E\left[\iint dx\, dy\, G(x-y)h(x)\, \frac{\delta F}{\delta\xi(y)}\, e^{-:V:(g)}\right]$$
$$+ E\left[\iint dx\, dy\, G(x-y)h(x)g(y) \; :V'(\xi(y)) \, : F\, e^{-:V:(g)}\right] . \tag{5.34}$$

We formulate this in the following theorem

Theorem 5.3. (The partial integration theorem)

Let $g \geq 0$ and $g \in C_0^\infty(\mathbb{R}^2)$, let $h \in H_{-1}(\mathbb{R}^2)$ and let F have a finite dimensional base, be differentiable and at most of exponential growth. Then

$$\langle \xi(h)F\rangle_g = \langle \iint dx\, dy\, G(x-y)h(x) \frac{\delta F}{\delta \xi(y)}\rangle_g$$
$$- \langle \iint dx\, dy\, G(x-y)h(x)g(y) \; :V'(\xi(y)): F\rangle_g \,.$$

This formula may be written shortly in the form

$$\langle \xi(x)F\rangle_g = \int dy\; G(x-y)\; \langle \frac{\delta F}{\delta \xi(y)}\rangle_g$$
$$- \int dy\; G(x-y)g(y)\; \langle\; :V'(\xi(y)):\; F\rangle_g \;,$$

where the meaning of the last formula is the formula above.

Remark. The formula in theorem 5.3 may also be written in the form of the equation of motion for the interacting Markoff field

$$\langle [(-\Delta+m^2)\,\xi\,(x) + g(x)\; :V'(\xi(x)):]\,F\rangle_g = \langle \frac{\delta F}{\delta \xi(x)}\rangle_g \,, \qquad (5.35)$$

where the equality is in the sense of distributions.

6. The infinite volume limit of the Schwinger functions.

For any bounded measureable set $A \subset \mathbb{R}^2$ we define $S_A(x_1 \ldots, x_r) = S_{\chi_A}(x_1, \ldots, x_r)$ and by the corollary of theorem 5.1 we then have that for $B \subseteq A$

$$0 \leq S_A(x_1, \ldots, x_r) \leq S_B(x_1, \ldots, x_r) \leq S_o(x_1, \ldots, x_r) \; . \qquad (6.1)$$

Theorem 6.1. For interactions densities satisfying the conditions of theorem 5.2, there is a positive locally integrable function $S(x_1, \ldots, x_r)$ such that

$$0 \leq S(x_1, \ldots, x_r) \leq S_o(x_1, \ldots, x_r)$$

and such that

$$S_{A_n}(x_1, \ldots, x_r) \rightarrow S(x_1, \ldots, x_r)$$

as locally L_1-integrable functions and also pointwise for all $x_1, \ldots, x_r$ such that $x_i \neq x_j$ for all $i \neq j$, if A_n has the property that for any bounded region $B \subset \mathbb{R}^2$ there is a N_B such that $B \subseteq A_n$ for all $n \geq N_B$. Moreover

$$S(x_1, \ldots, x_r) = \inf_B S_B(x_1, \ldots, x_r) \; ,$$

where the infimum is taken over all bounded measurable sets. $S(x_1, \ldots, x_r)$ is a Euclidian invariant function, and symmetric under permutation of its arguments.

Proof: Let $S(x_1, \ldots, x_r) = \inf S_B(x_1, \ldots, x_r)$. Then $\liminf S_{A_n}(x_1, \ldots, x_n) \geq S(x_1, \ldots, x_n)$, on the other hand since for any bounded B $A_n \supseteq B$ for $n \geq N_B$ we get from (6.1) that $\limsup S_{A_n}(x_1, \ldots, x_n) \leq S(x_1, \ldots x_n)$. Hence the limit exists pointwise, and by (6.1) we have dominated convergence, and since

$S_o(x_1,\ldots,x_r)$ is locally integrable we get that the limit exists in the local L_1-sense. Let T be a Euclidian transformation. Then $S_B(Tx_1,\ldots,Tx_n) = S_{TB}(x_1,\ldots,x_n)$ and the Euclidian invariance of $S(x_1,\ldots x_n)$ follows from the fact that it is given by the infimum. The fact that $S(x_1,\ldots,x_n)$ is symmetric under permutation of its arguments follows from the fact that $S_B(x_1,\ldots,x_n)$ is so. This proves the theorem. ▨

Remark: $S(x_1,\ldots x_r)$ are the infinite volume Schwinger functions and we have from theorem 6.1 that they are the limits of the space-time cut-off Schwinger functions $S_B(x_1,\ldots,x_r)$ when B converges to $\mathbb{R}^2$ in the van Hove sense. For this type of convergence (see e.g. [20]).

Theorem 6.2.

The infinite volume Schwinger functions $S_{\lambda,\mu}(x_1,\ldots,x_n)$ for the interactions of the form $\lambda :V: + \mu :\xi^2:$ satisfy the inequalities

$$S_{\lambda,\mu}(x_1,\ldots,x_n) \leq S_{\lambda',\mu'}(x_1,\ldots,x_n)$$

for $\lambda' \leq \lambda$ and $\mu' \leq \mu$.

Proof: This follows immediately from theorem 6.1 and theorem 5.2. ▨

Theorem 6.3.

The infinte volume Schwinger functions are unique limits of the corresponding space cut-off quantities i.e. for $t_1 \leq \ldots < t_n$ and $f_1,\ldots,f_n$ in $H_{-\frac{1}{2}}(\mathbb{R})$

$$\int_{\mathbb{R}}\ldots\int_{\mathbb{R}} S(y_1t_1,\ldots,y_nt_n)f(y_1)\ldots f(y_n)dy_1\ldots dy_n =$$

$$= \lim_{r\to\infty} (\Omega_r,\varphi(f_1)\, e^{-(t_2-t_1)(H_r-E_r)}\ldots\varphi(f_{n-1})e^{-(t_n-t_{n-1})(H_r-E_r)}\varphi(f_n)\Omega_r)$$

Proof: Let $\chi_r^{s,t}$ be the characteristic function of the interval $[-r,r] \times [s,t]$. In section 3, (3.29), we proved that the Schwinger functions $S_{\chi^{s,t}}$ for the space-time cut-off interaction with space-time cut-off function $\chi_r^{s,t}$, converge as $s \to -\infty$ and $t \to +\infty$ to the limit

$$\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} S_{\chi_r}(y_1t_1,\ldots,y_nt_n)f(y_1)\ldots f(y_n)dy_1\ldots dy_n = \tag{6.2}$$
$$= (\Omega_r,\varphi(f_1)\, e^{-(t_2-t_1)(H_r-E_r)}\varphi(f_2)\ldots e^{-(t_n-t_{n-1})(H_r-E_r)}\varphi(t_n)\Omega_r).$$

Since $0 \leq S_{\chi_r s,t}(x_1,\ldots,x_r) \leq S_o(x_1,\ldots,x_r)$ we get by dominated convergence, using theorem 6.1, that (6.2) tends to the left hand side of the formula in theorem 6.3 as $r \to \infty$, $s \to -\infty$, and $t \to \infty$, since $[-r,r] \times [s,t]$ converges to $\mathbb{R}^2$ in the van Hove sense. This proves the theorem.

Remark:

Although we have now proved that the infinite volume Schwinger functions exist, and are equal to the limit of the corresponding space cut-off quantities in theorem 6.3, we cannot yet be sure that the limits are different from zero. But we shall now see that they are, by giving estimates from below utilizing theorem 5.2 where we shall make full use of the monotonicity of $S_{g;g_1}(x_1,\ldots,x_n)$ in g_1. What we are doing now is at least in spirit closely related to the ideas of Nelson's proof for the existence of the infinite volume limit in the $:P(\varphi)_2:$ interactions with Dirichlet boundary conditions ([10], as quoted e.g. in [9b]).

Let now C be any simple closed piecewise C^1-curve of finite length in $\mathbb{R}^2$, and let δ_C be the measure concentrated on C induced from the Lebesgue measure on $\mathbb{R}^2$, i.e. the arc-length measure on C. Let $\delta_C^{\varepsilon} = \chi_{\varepsilon} * \delta_C$. By the method of section 5 we get, because of $(\delta_C\, G^2\, \delta_C) = \iint G(x-y)^2\, \delta_C(x)\, \delta_C(y) < \infty$, that $:\xi^2:(\delta_C^{\varepsilon})$ converges in $L_2(X,d\mu)$ hence in all $L_p(X,d\mu)$ to the function $:\xi^2:(\delta_C)$. We shall also introduce the notation $\int_C :\xi^2:$ for this $L_p(X,d\mu)$ function.

By the now standard methods of polynomial interactions [21], we get that $e^{-:\xi^2:(\delta_C)}$ is in $L_p(X,d\mu)$ for all finite p and that

$$e^{-:\xi^2:(\delta_C^{\varepsilon})} \to e^{-:\xi^2:(\delta_C)} \tag{6.3}$$

in $L_p(X,d\mu)$ for all finite p. Hence for $F \in L_p(X,d\mu)$ we get that

$$E\left[F\, e^{-\sigma:\xi^2:(\delta_C^{\varepsilon})}\, e^{-:V:(g)}\right] \to E\left[F\, e^{-\sigma:\xi^2:(\delta_C)}\, e^{-:V:(g)}\right] \tag{6.4}$$

Thus for $h_1,\dots,h_n$ in $H_{-1}(\mathbb{R}^2)$ and $g \geq 0$ and in $L_2(\mathbb{R}^2) \cap L_1(\mathbb{R}^2)$ we have

$$S_{g,\sigma\delta_C^{\varepsilon}}(h_1,\dots,h_n) \to S_{g,\sigma C}\,(h_1,\dots,h_n)\ , \tag{6.5}$$

where

$$\begin{aligned} S_{g,\sigma C} &= \langle \xi(h_1)\dots\xi(h_n)\rangle_g^{\sigma C} \\ &= (\langle e^{-\sigma:\xi^2:(\delta_C)}\rangle_g)^{-1}\, \langle \xi(h_1)\dots\xi(h_n)\, e^{-\sigma:\xi^2:(\delta_C)}\rangle_g\ . \end{aligned} \tag{6.6}$$

Since $\delta_C^{\varepsilon} \in C_0^{\infty}(\mathbb{R}^2)$ it follows by this convergence and from theorem 5.2 that if $h_1 \geq 0,\dots,h_n \geq 0$ then

$$S_{g,\sigma C}(h_1,\dots,h_n) \leq S_{g',\sigma' C'}(h_1,\dots,h_n) \leq S_{g'}(h_1,\dots h_n) \tag{6.7}$$

for $g' \leq g$, $\sigma' \leq \sigma$ and $C' \subseteq C$.

By (6.7) we have that

$$\lim_{\sigma\to\infty} S_{g,\sigma C}(h_1,\dots,h_n) = S_g^{\,C}(h_1,\dots,h_n) \tag{6.8}$$

exists and we have the inequality

$$S_g^{\,C}(h_1,\dots,h_n) \le S_{g'}^{\,C'}(h_1,\dots,h_n) \le S_{g'}(h_1,\dots,h_n), \tag{6.9}$$

where $g' \le g$ and $C' \subseteq C$.

For any $F \in L_p(X,d\mu)$, for some p $1 < p < \infty$, we define

$$E^{\sigma C}(F) = \left[E(e^{-\sigma:\xi^2:(\delta_C)})\right]^{-1} E\,(F\, e^{-\sigma:\xi^2:(\delta_C)}). \tag{6.10}$$

An explicite calculation gives that for h in $H_{-1}(\mathbb{R}^2)$

$$E_{\sigma C}\left[e^{i\xi(h)}\right] = e^{\frac{1}{2}(hG_{\sigma C}h)} \tag{6.11}$$

where $(h\,G_{\sigma C}h) = \iint h(x)\,G_{\sigma C}(x-y)h(y)dx\,dy$ and $G_{\sigma C}(x-y)$ is the kernel of the inverse operator of $-\Delta_{\sigma C} + m^2$ where $-\Delta_{\sigma C}$ is the self adjoint operator corresponding to the closable positive form

$$(\Psi(-\Delta_{\sigma C})\Psi) = \sum_{i=1}^{2} \int (\frac{\partial\Psi}{\partial y_i})^2 + \sigma \int_C \Psi^2$$

$$= \sum_{i=1}^{2} \int (\frac{\partial\Psi}{\partial y_i})^2 + \sigma \int \Psi^2(y)\delta_C(y) \;.$$

Hence $E^{\sigma C}$ is the expectation with respect to the Gaussian measure with covariance function $G_{\sigma C}\cdot(x-y)$. Since $-\Delta_{\sigma C} + m^2$ converges monotonically to $-\Delta^C + m^2$ where Δ^C is the Laplace operator with zero boundary condition on C, i.e. the closure of the restriction of Δ to the $C_0^{\infty}(\mathbb{R}^2)$ functions which vanish on C, we get that $(h\;G_{\sigma C}\,h)$ converges to $(h\;G^C h)$, where $G^C(x-y)$ is the kernel of the inverse operator of $-\Delta^C + m^2$. Hence by (6.11) $E_{\sigma C}$ converges weakly to E^C, which is the expectation with respect to the Gaussian measure $d\mu^C$ with covariance function $G^C(x-y)$. Since C is a simple closed C^1-curve of finite length, it divides $\mathbb{R}^2$ into a finite number of disjoint open sets $D_1,\ldots,D_k$, so that $\mathbb{R}^2 = C \cup D_1 \cup \ldots \cup D_k$. It follows from the fact that Δ^C is the closure of the restriction of Δ to the C_0^∞ functions that vanish on C, that

$$\Delta^C = \Delta_{D_1} \oplus \ldots \oplus \Delta_{D_k} , \tag{6.12}$$

relative to the direct decomposition

$$L_2(\mathbb{R}^2) = L_2(D_1) \oplus \ldots \oplus L_2(D_k) ,$$

where Δ_{D_i} is the Laplacian with Dirichlet boundary conditions in D_i, $i = 1,\ldots k$. Hence we get that

$$G^C = G_{D_1} \oplus \ldots \oplus G_{D_n} , \tag{6.13}$$

where G_{D_i} is the inverse of $-\Delta_{D_i} + m^2$ and therefore

$$d\mu^C = d\mu_{D_1} \times \ldots \times d\mu_{D_k} , \tag{6.14}$$

where $d\mu_{D_i}$ is the Gaussian measure with covariance given by G_{D_i}.

Now we use the technique of section 4 and construct the lattice approximation for $S_{g,\sigma C}$. By the method of Guerra, Rosen and Simon ([9b] sections 4 and 5) we then get by letting $\sigma \to \infty$ a lattice approximation for $E^C[\xi(h_1)\ldots\xi(h_n)e^{-:V:(g)}]$. By removing the lattice approximation we prove in this way that

$$S_g^C(h_1,\ldots,h_n) = (E^C[e^{-:V:(g)}])^{-1} \, E^C[\xi(h_1)\ldots\xi(h_n) \, e^{-:V:(g)}] \tag{6.15}$$

Now let C be a connected curve and suppose $h_1,\ldots,h_n$ all have support in the interior of C. By (6.14) and (6.15) and the fact that $:V:(g_1+g_2) = :V:(g_1) + :V:(g_2)$ we have that $S_g^C(h_1,\ldots,h_n)$ is independent of g if g is constant on the interior of C. Let $C = C_1 \cup C_2$ where C_1 and C_2 are connected curves and let C_1 be contained in the interior of C_2. Moreover let us assume that $h_1,\ldots,h_n$ have support in the interior of C_1.

By (6.14) and (6.15) we then get that

$$S_g^C(h_1,\ldots,h_n) = S_g^{C_1}(h_1,\ldots,h_n) \ . \tag{6.16}$$

Let now $h_1 \geq 0 \ldots, h_n \geq 0$, then by (6.9)

$$S_g^C(h_1,\ldots,h_n) \leq S_g^{C_2}(h_1,\ldots,h_n) \ ,$$

hence we have

$$S_g^{C_1}(h_1,\ldots,h_n) \leq S_g^{C_2}(h_1,\ldots,h_n) \tag{6.17}$$

Therefore if C is a connected closed curve, $S_g^C(h_1,\ldots,h_n)$ converges monotonically to a limit as C expands so that finally all points are interior to C. Moreover one proves easily that E^C converges weakly to E as C expands, by the fact that G^C converges to G. This then gives that $S_g^C(h_1,\ldots,h_n)$ converges to $S_g(h_1,\ldots,h_n)$ as C expands and by (6.17) and (6.9) we then get that

$$S_g^C(h_1,\ldots,h_n) \leq S_g(h_1,\ldots,h_n) \ . \tag{6.18}$$

Since $S_g^C(h_1,\ldots,h_n)$ is independent of g as soon as g is constant on the interior of C we get the uniform lower bound

$$S_\lambda^C(h_1,\ldots,h_n) \leq S_g(h_1,\ldots,h_n) \ , \tag{6.19}$$

where $S_\lambda^C(h_1,\ldots,h_n) = S_g^C(h_1,\ldots,h_n)$ if $g(x) = \lambda$ for x in the interior of C.

By letting g converge monotonically to λ we therefore get

$$S_\lambda^C(h_1,\dots,h_n) \le S_\lambda(h_1,\dots,h_n) \qquad (6.20)$$

which then proves that $S(x_1,\dots,x_n)$ are non-trivial.

Theorem 6.4.

Let B be a bounded measurable subset of $\mathbb{R}^2$ and set

$$d\mu_B = \left[E(e^{-:V:(\chi_B)})\right]^{-1} e^{-:V:(\chi_B)} d\mu ,$$

where χ_B is the characteristic function for the set B. Then there is a measure $d\mu_*$ on the space of tempered distributions such that $d\mu_B$ converges weakly to $d\mu_*$ as B tends to $\mathbb{R}^2$ in a way such that finally it covers all bounded sets.

Moreover, for $h \in \mathcal{S}(\mathbb{R}^2)$, $E_*(e^{i\xi(h)}) = \int e^{i\xi(h)} d\mu_*$ is given by the convergent series

$$E_*(e^{i\xi(h)}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} E_*(\xi(h)^{2n}) .$$

Proof: To prove this theorem it is obviously enough to prove that, for any $h \in \mathcal{S}(\mathbb{R}^2)$, $E_B(e^{i\xi(h)}) = \int e^{i\xi(h)} d\mu_B$ converges to a positive definite continuous function on $\mathcal{S}(\mathbb{R}^2)$. This because by the Bochner-Minlos theorem there is then a measure $d\mu_*$ on the set of tempered distributions $\mathcal{S}'(\mathbb{R}^2)$ such that the limit function is given by $E_*(e^{i\xi(h)}) = \int e^{i\xi(h)} d\mu_*$, and the pointwise convergence of the characteristic functions implies weak convergence of $d\mu_B$ to $d\mu_*$. Now by the fact that the interaction density is even, $V(s) = V(-s)$, we get that $d\mu_B$ is an even measure, so that $E_B(e^{i\xi(h)}) = E_B(\cos(\xi(h)))$. Now

$$\cos(\xi(h)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\xi(h))^{2n} , \tag{6.21}$$

and we shall see that this series converges strongly in $L_1(X, d\mu_B)$, uniformly in B. By theorem 5.1 and its corollary we have

$$E_B(\xi(h)^{2n}) = S_B^{(2n)}(h,\ldots,h) \leq S_B^{(2n)}(|h|,\ldots,|h|) \leq S_o^{(2n)}(|h|,\ldots,|h|) .$$

An explicite computation gives

$$S_o^{(2n)}(h,\ldots,h) = \frac{(2n)!}{2^n n!} (hGh)^n .$$

So that

$$E_B(\xi(h)^{2n}) \leq \frac{(2n)!}{2^n n!} (|h|\, G\, |h|)^n .$$

Since

$$\sum_{n=0}^{\infty} \frac{1}{2^n n!} (|h|\, G\, |h|)^n$$

converges, we have proved that (6.21) converges absolutely and hence strongly in $L_1(X, d\mu_B)$ uniformly in B.

Therefore it is enough to prove that

$$E_B(\xi(h)^{2n}) = S_B^{(2n)}(h,\ldots,h)$$

converges and this follows from theorem 6.1. This proves the convergence of the function $E_B(e^{i\xi(h)})$, moreover we have also proved by the uniform $L_1(X, d\mu_B)$ convergence of (6.21) that the limit function has the form

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} S^{(2n)}(h,\ldots,h) , \tag{6.22}$$

where S is the infinite volume Schwinger function. The continuity of the function (6.22) for $h \in \mathcal{S}(\mathbb{R}^2)$ follows from the estimate in theorem 6.1. That it is positive definite follows from the fact that pointwise limits of positive definite functions are positive definite. This proves the theorem. ▨

7. The mass gap and the verification of the Wightman axioms.

The first part of this section is an adaption for the exponential interactions of Simon's investigations for $:P(\varphi)_2:$ [11] on the implications which the correlation inequalities have for the mass gap, namely that it is given by the decrease at infinity of the two point Schwinger functions. So let $\rho(y) = 0$ for $y \le -1$, to $\frac{1+y}{2}$ for $|y| \le 1$ and to 1 for $y \ge 1$. Then we have

Lemma 7.1. For the exponential interactions given by a density $V(s) = V(-s) = \int e^{s\alpha} d\nu(\alpha)$ where ν is a positive finite measure with compact support in the interval $(-\frac{4}{\sqrt{\pi}}, \frac{4}{\sqrt{\pi}})$ and with space-time cut-off $g \ge 0$, $g \in C_0^\infty(\mathbb{R}^2)$, the following inequalities hold

$$0 \le \langle \rho(h_1)\ldots\rho(h_j)\rho(h_{j+1})\ldots\rho(h_{j+k})\rangle_g^T$$

$$\le \frac{1}{4} \sum_{\substack{1\le n\le j \\ j+1\le m\le k}} S_g^{(2)}(h_n h_m) ,$$

for any non negative functions $h_1\ldots h_{j+k}$ in $C_0^\infty(\mathbb{R}^2)$, where $\rho(h) \equiv \rho(\xi(h))$, and

$$\langle FG\rangle_g^T = \langle FG\rangle_g - \langle F\rangle_g \langle G\rangle_g$$

where $F = \rho(h_1)\ldots\rho(h_j)$ and $G = \rho(h_{j+1})\ldots\rho(h_{j+k})$.

Corollary: The same inequalities also hold with $g \ge 0$ and $g \in L_2(\mathbb{R}^2) \cap L_1(\mathbb{R}^2)$.

Proof: This lemma is proven in the same way as the corresponding result in Simon [11, theorem 3]. In fact Simon's proof depends only on the monotonicity properties of products and sums of $\rho(h_i)$ as well as the correlation inequalities of FGK-type, which we established for the even exponential interactions in theorem 4.3.

The corollary follows using the fact that the function ρ is bounded, and the fact that $\langle\ \rangle_g$ and S_g depend strongly continuously on g in $L_2(\mathbb{R}^2) \cap L_1(\mathbb{R}^2)$. This last fact is to be found in the corollary to theorems 5.1 and 5.2 and in the proof of this corollary.

Consider now the expectation $\langle\rho(h_1)\ldots\rho(h_{j+k})\rangle_g$, taking g to be the characteristic function χ_B of a bounded measurable set B of $\mathbb{R}^2$. Since $h_1,\ldots,h_{j+k}$ are in $\mathcal{S}(\mathbb{R}^2)$ we get by theorem 6.4 that $\langle\rho(h_1)\ldots\rho(h_{j+k})\rangle_g$ converge as B increases to cover all bounded sets. By theorem 6.1 $S_g(h_n,h_m)$ also converges. Hence we have

<u>Lemma 7.2</u>. For the exponential interaction with density given by an even $V(s) = V(-s)$ we have the following inequality

$$0 \leq \langle\rho(h_1)\ldots\rho(h_j)\rho(h_{j+1})\ldots\rho(h_{j+k})\rangle_*^T \leq \frac{1}{4}\sum_{\substack{1\leq n\leq j\\ j+1\leq m\leq k}} S^{(2)}(h_n,h_m) ,$$

where $\langle F\rangle_* = E_*(F)$ and $\langle FG\rangle_*^T = \langle FG\rangle_* - \langle F\rangle_*\langle G\rangle_*$
where $F = \rho(h_1)\ldots\rho(h_j)$ and $G = \rho(h_{j+1})\ldots\rho(h_{j+k})$.

<u>Lemma 7.3</u>.

Under the same assumptions on the interaction density as in the previous lemma, we have that, for the space cut-off interaction,

$$(\rho(h_1)\ldots\rho(h_j)\Omega_r,\ e^{-t(H_r-E_r)}\rho(h_{j+1})\ldots\rho(h_{j+k})\Omega_r)^T \leq$$

$$\frac{1}{4}\sum_{\substack{1\leq n\leq j\\ j+1\leq m\leq k}} S_r^{(2)}(h_n,h_m^{\,t}) ,$$

where $h_i(y,s) = f_i(y)\delta(s)$ and $h_n^{\,t}(y,s) = f_n(y)\delta_t(s)$,

$$S_r^{(2)}(h,h') = S_{\chi_r}^{(2)}(h,h') = \lim_{\substack{t\to\infty\\ s\to-\infty}} S_{\chi_r^{s,t}}(h,h') ,$$

where $\chi_r^{s,t}$ is the characteristic function for the interval $[-r,r] \times [s,t]$ and

$$(F\Omega_r, G\Omega_r)^T = (F\Omega_r, G\Omega_r) - (F\Omega_r, \Omega_r)(\Omega_r, G\Omega_r) .$$

Proof: This lemma follows from the proof of theorem 6.3 and lemma 7.1, in the same way as the previous lemma was proved. ▨

Theorem 7.1.

The lowest eigenvalue E_r of the Hamiltonian H_r for the space cut-off interaction is separated from the rest of the spectrum of H_r by the mass gap m_r, where $m_r \geq m$, the mass of the free field, and

$$m_r = \inf_{y_1, y_2} \lim_{t\to\infty} \frac{-1}{t} \log S_r^{(2)}(y_1, 0; y_2, t) .$$

Proof: The theorem follows immediately from the estimate in lemma 7.3, because Ω_r is a function in $E_0 L_2(X, d\mu)$ which is positive almost everywhere and linear combinations of $\rho(h_1)\ldots\rho(h_j)$ are dense in $E_0 L_2(X, d\mu)$ (see Simon [11, lemma 4] and furthermore $S_r^{(2)}(x_1, x_2) \leq S_0^{(2)}(x_1, x_2)$ which is finite for $x_1 \neq x_2$ and falls off like $e^{-m|x_1-x_2|}$ as $|x_1 - x_2|$ tends to infinity.

Remark: We have also that all the propositions of theorems 4 and 5 of [11] extend to the case of the above even exponential interactions. In particular m_r is equal to the infimum of the spectrum of the restriction of H_r to the odd subspace of the Foch space, i.e. the subspace $\mathcal{F}_{odd} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(2n+1)}$, where $\mathcal{F}^{(2n+1)}$ is the 2n+1-fold tensor product of $H_{-\frac{1}{2}}(\mathbb{R})$ with itself. The set of all vectors of the form $\varphi(f)\Omega_r$ for $f \geq 0$, $f \in H_{-\frac{1}{2}}(\mathbb{R})$

is coupled to the first excited state in the sense of Simon [11].

Let now $F(\xi)$ be a Borel-measurable function (i.e. measurable with respect to the σ-algebra generated by the open sets) on the space of tempered distributions, i.e. $\xi \in \mathcal{S}'(\mathbb{R}^2)$. We then write, for any $x \in \mathbb{R}^2$, $F_x(\xi) = F(\xi_x)$, where $\langle \xi_x, h\rangle = \langle \xi, h_x\rangle$ and $h_x(y) = h(y-x)$ for any $h \in \mathcal{S}(\mathbb{R}^2)$.

<u>Theorem 7.2</u>. The measure μ_* on $\mathcal{S}'(\mathbb{R}^2)$ is strongly clustering i.e. for any two Borel-measurable sets A and B in $\mathcal{S}'(\mathbb{R}^2)$

$$\mu_*(A_x \cup B) \to \mu_*(A)\cdot\mu_*(B)$$

as $|x| \to \infty$, where A_x is the set with characteristic function $(\chi_A)_x$. Moreover

$$\left|\mu_*(A_x \cup B) - \mu_*(A)\mu_*(B)\right| \leq C\ e^{-m'|x|} ,$$

for any $m' < m$, where m is the mass of the free field. In particular the infinite volume Schwinger functions have the cluster property that

$$S(x_1 \ldots x_n, x_{n+1} + x, \ldots, x_{n+m} + x) \to S(x_1 \ldots x_n) S(x_{n+1} \ldots x_{n+m}) ,$$

as $|x| \to \infty$ in $\mathbb{R}^2$. Moreover

$$|S(x_1 \ldots x_n, x_{n+1} + x, \ldots, x_{n+m} + x) - S(x_1 \ldots x_n) S(x_{n+1} \ldots x_{n+m})| \leq Ce^{-m'|x|}$$

for any $m' < m$.

<u>Proof</u>: Since

$$S(h_1, \ldots, h_n) = \int \langle \xi, h_1\rangle \ldots \langle \xi, h_n\rangle d\mu_*(\xi) ,$$

the cluster properties of the Schwinger functions would follow from those of the measure μ_*. By lemma 7.2 we know that for

$F = \rho(h_1)\ldots\rho(h_j)$ and $G = \rho(h_{j+1})\ldots\rho(h_{j+k})$, where $h_i \geq 0$, $i = 1,\ldots,j+k$ and $\rho(h) = \rho(\langle\xi,h\rangle)$

$$\left|\int FG\, d\mu_* - \int F\, d\mu_* \cdot \int G\, d\mu_*\right| \leq \frac{1}{4} \sum_{\substack{1\leq n\leq j \\ j+1\leq m\leq k}} S^{(2)}(h_n,h_m). \tag{7.1}$$

Now since $S^{(2)}(x_1,x^2) \leq S_0^{(2)}(x_1,x_2) = G(x_1-x_2)$, we get that

$$\left|\int F_x G\, d\mu_* - \int F\, d\mu_* \int G\, d\mu_*\right| \leq C\, e^{-m'|x|}\;, \tag{7.2}$$

by the known decrease of $G(x)$ at infinity.

Since $F(\xi)$ and $G(\xi)$ take values in the compact interval $[0,1]$, we get that any functions $\overline{F}$ and $\overline{G}$ of the form

$$\overline{F} = f(F_1(\xi),\ldots,F_p(\xi)),\ \overline{G} = g(G_1(\xi),\ldots,G_q(\xi)),$$

where f and g are bounded continuous functions in $\mathbb{R}^p$ and $\mathbb{R}^q$, and $G_i(\xi)$ and $F_j(\xi)$ are of the same form as $F(\xi)$ and $G(\xi)$, may be approximated uniformly by polynomials in $F_1(\xi),\ldots,F_p(\xi)$ and $G_1(\xi),\ldots,G_q(\xi)$ respectively. By the uniform approximation and (7.2) we get again that

$$\left|\int \overline{F}_x\overline{G}\, d\mu_* - \int \overline{F}\, d\mu_* \int \overline{G}\, d\mu_*\right| \leq \overline{C}\, e^{-\overline{m}|x|} \tag{7.3}$$

where $\overline{m}$ is any number such that $\overline{m} < m'$. By approximating in $L_1(d\mu_*)$-norm the characteristic functions χ_A and χ_B by functions of the form $\overline{F}$ and $\overline{G}$, we prove the theorem.

Theorem 7.3.

The infinite volume limit Schwinger functions satisfy the Osterwalder-Schrader conditions [15] for Euclidian Green's functions in the sense that they are

1. Tempered distributions.
2. Euclidian invariant.
3. They satisfy the positivity conditions.
4. They are symmetric with respect to permutations of their arguments.
5. They have the cluster properties.

<u>Proof</u>: The only condition which we have not yet verified is the condition 3. However this follows immediately from the fact that they are the pointwise limits of the corresponding space cut-off functions $S_r(x_1,\dots,x_n)$ as r tends to infinity as was proved in theorem 6.3, and the fact that

$$S_r(y_1t_1,\dots y_nt_n) = (\Omega_r, \varphi(y_1)e^{-(t_2-t_1)(H_r-E_r)}\dots\varphi(y_n)e^{-(t_n-t_{n-1})(H_r-E_r)}\Omega_r)$$

where H_r-E_r is a positive self-adjoint operator. So this proves the theorem.

From Osterwalder and Schrader's paper [15] it follows then that the Schwinger functions $S(y_1t_1,\dots y_nt_n)$ are analytic functions of the $n-1$ variables $t_2-t_1,\dots,t_n-t_{n-1}$ for $\mathrm{Re}(t_i-t_{i-1}) > 0$ $\quad i = 2,\dots,n$. Moreover their boundary values on the product of the imaginary axis $W(x_1,\dots,x_n)$ satisfy the Wightman axioms.

<u>Theorem 7.4</u>. For the even exponential interactions with interaction density given by $V(s) = V(-s)$ where $V(s) = \int e^{s\alpha}d\nu(\alpha)$ where ν is a positive bounded measure with compact support in $(-\frac{4}{\sqrt{\pi}}, \frac{4}{\sqrt{\pi}})$, the Wightman functions $W(x_1,\dots,x_n)$ defined by analytic continuation of the infinite volume Schwinger functions $S(x_1,\dots,x_n)$ satisfy all the Wightman axioms, i.e.

1. They are tempered distributions.
2. Relativistic invariant.
3. Satisfy the positive definiteness condition.
4. Local commutativity.
5. Cluster properties.
6. The spectral condition.

Moreover they are the limit of the corresponding space cut-off quantities, in the sense of analytic functions.

Let H be the infinitesimal generator of the unitary group of time translations in the representation space constructed from the Wightman functions. Then zero is an isolated eigenvalue of H and it is separated from the rest of the spectrum by an interval of length m_* where $m_* \geq m$, and m is the free mass. Moreover m_* is a monotone increasing function of the coupling constant λ, the mass perturbation μ and the free mass m. Furthermore

$$m_* = \sup[\alpha;\ e^{\alpha t}\int_{\mathbb{R}} S^{(2)}(y,t)dy \text{ is uniformly bounded in } t] ,$$

where $S^{(2)}(x_1-x_2)$ is the two point Schwinger function.

<u>Proof</u>: That the Wightman axioms follow from theorem 7.3 is proved by Osterwalder and Schrader in [15]. That the Wightman functions are limits of the corresponding space cut-off quantities follows from the fact that the space cut-off Schwinger functions are analytic for $Re(t_i-t_{i-1}) > 0 \quad i = 2,\dots,n$ and converge for t_i all real and the boundary values on the imaginary axis are the corresponding Wightman functions. The mass gap follows from lemma 7.2 in the same way as the cluster properties, and the expression for m_* follows from the relativistic invariance, (which gives that the mass gap is smallest for zero momentum) and the

fact that $0 \leq S^{(2)}(x) \leq G(x)$ so that $S^{(2)}(y,t)$ is integrable with respect to y. Moreover the monotonicity of m_* follows from the corresponding monotonicity of the two point Schwinger function $S^{(2)}(x)$ proven in theorem 6.2. This proves the theorem.

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Footnote

1) After completion of this paper we received a preprint by J. Glimm and A. Jaffe, "The n-particle cluster expansion for the $P(\varphi)_2$ quantum field models", which carries the study of the weakly coupled $P(\varphi)_2$ interactions further and includes the construction of the scattering matrix.

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