Spectra of ergodic transformations

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1. Introduction. In ergodic theory there are several results on the spectrum of the unitary operator on L^2 defined by an ergodic transformation. In the present paper we shall prove similar results for ergodic groups of *-automorphisms of a von Neumann algebra. Our setting is mostly the following: $\mathcal{R}$ is a von Neumann algebra, G a group, and α representation of G as an ergodic group of *-automorphisms on $\mathcal{R}$. We assume α is implemented by a unitary representation U of G. After a discussion of the eigenvectors and eigenvalues for U in the first part, we specialize to the situation when G is locally compact and abelian, and U is strongly continuous in the second. Then the work of Arveson [2] and Connes [3] on the spectrum for α is available. If by Stone's theorem $U_g = \int_{\hat{G}} \overline{(\gamma,g)} dP_\gamma$, the spectrum of U is the same as the support of the projection valued measure dP. We show that if $\ker U = \ker \alpha$ then $\mathrm{Sp}U$ equals the dual group of $G/\ker U$, and that this result is applicable to the GNS representation of an invariant state (e.g. to asymptotically abelian C*-algebras). If we specialize to the case when G is the integers Z or the reals $\mathbb{R}$ then usually $\mathrm{Sp}U = \hat{G}$. If they are different then there exists a faithful normal G-invariant state, and $\mathcal{R}$ is abelian. As a consequence it follows that in ergodic theory, when $\mathcal{R}$ is abelian, and W is the unitary operator defined by an ergodic transformation, then the spectrum of W is the whole unit circle, unless $\mathcal{R}$ is finite dimensional as a complex vector space.

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2. Eigenvalues. If $\mathcal{R}$ is a von Neumann algebra we denote by Aut $\mathcal{R}$ the group of *-automorphisms of $\mathcal{R}$. If G is a group and α a representation of G in Aut $\mathcal{R}$ we say an operator V in $\mathcal{R}$ is an eigenoperator for α if $V \neq 0$ and for each $g \in G$ there is a complex number (γ, g) such that $\alpha_g(V) = (\gamma, g)V$. We say α represents G as an ergodic group of automorphisms (or just that G is ergodic) if for $A \in \mathcal{R}$, $\alpha_g(A) = A$ for all $g \in G$, implies $A = \lambda I$ for some scalar λ, where I is the identity operator.

Lemma 2.1. Let $\mathcal{R}$ be a von Neumann algebra, G a group and α a representation of G as an ergodic group of *-automorphisms of $\mathcal{R}$. Then we have

1) If V is an eigenoperator for α, and $\alpha_g(V) = (\gamma, g)V$ then γ is a character on G, called the eigenvalue for V.
2) The eigenvalues form a subgroup of the character group on G.
3) Two eigenoperators with the same eigenvalue are scalar multiples of the same unitary operator.
4) Let $\mathcal{M}$ be the weak closure of the vector space spanned by the eigenoperators. Then $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{R}$, called the eigenalgebra for G in $\mathcal{R}$.

Proof. With V as in 1) we have for $g, h \in G$, $(\gamma, gh)V = \alpha_{gh}(V) = \alpha_g(\alpha_h(V)) = \alpha_g((\gamma, h)V) = (\gamma, h)(\gamma, g)V$, so γ is a character, and 1) follows. Furthermore, we have $\alpha_g(V^*) = \alpha_g(V)^* = ((\gamma, g)V)^* = (\gamma^{-1}, g)V$, so that V^* is an eigenoperator with eigenvalue γ^{-1}. Let U be an eigenoperator with eigenvalue μ. Then $\alpha_g(UV) = (\gamma\mu, g)UV$, so $\gamma\mu$ is an eigenvalue if $UV \neq 0$. Thus 2) follows as soon as we have shown 3). But $V^*V \neq 0$ and $VV^* \neq 0$, so the above shows that V^*V and VV^* are eigenoperators with eigenvalue 1, the identity character. Thus $\alpha_g(V^*V) = V^*V$, so V^*V is a scalar operator, since G is ergodic. Similarly VV^* is a scalar operator. Since $\|V^*V\| = \|VV^*\|$, $VV^* = V^*V$, hence V is a scalar multiple of a unitary operator. Similarly, if U is an eigenoperator with eigenvalue γ, we see that U^*V is a scalar, so 3) follows.

For each eigenvalue γ choose a unitary eigenoperator V_γ with eigenvalue γ. By 3) $V_\gamma V_\mu = r(\gamma,\mu)V_{\gamma\mu}$ where $r(\gamma,\mu)$ is a complex number of modulus one. Thus $V_\gamma V_\mu \in \mathcal{M}$. Since $V_\gamma^* = cV_{\gamma^{-1}}$ by the above proof, $V_\gamma^* \in \mathcal{M}$. Thus 4) follows.

Theorem 2.2. Let $\mathcal{R}$ be a von Neumann algebra, G a group and α a representation of G as an ergodic group of *-automorphisms of $\mathcal{R}$. Suppose ω_{x_o} is a G-invariant vector state on $\mathcal{R}$ with x_o a vector which is cyclic for the eigenalgebra $\mathcal{M}$ for G in $\mathcal{R}$. Then $\mathcal{R} = \mathcal{M}$, and ω_{x_o} is a faithful normal trace on $\mathcal{R}$.

Proof. Let $\omega = \omega_{x_o}$. Since G is ergodic ω is the unique normal G-invariant state on $\mathcal{R}$, see e.g. [8], and also ω is faithful. Let V be a unitary eigenoperator with eigenvalue γ. Let $A \in \mathcal{R}$. Then for $g \in G$,

$$\begin{aligned} \omega(V^*\alpha_g(A)V) &= \omega(\alpha_{g^{-1}}(V^*)A\,\alpha_{g^{-1}}(V)) \\ &= \omega((\gamma^{-1},g^{-1})V^*A(\gamma,g^{-1})V) \\ &= \omega(V^*AV)\,. \end{aligned}$$

Thus the state $A \to \omega(V^*AV)$ is G-invariant, so by the uniqueness of ω, $\omega(V^*AV) = \omega(A)$ for all $A \in \mathcal{R}$. In particular $\omega(AV) = \omega(VA)$, and ω is in the centralizer $\mathcal{R}_\omega$ for ω.

Since ω is faithful, x_0 is separating for $\mathcal{R}$, hence by our assumption, x_0 is separating and cyclic for both $\mathcal{R}$ and $\mathcal{M}$. From Tomita theory, see [17], there is a conjugation J and a positive self-adjoint operator Δ - the modular operator for x_0 - such that $J\Delta^{\frac{1}{2}} A x_0 = A^* x_0$ for all $A \in \mathcal{R}$, and $J\mathcal{R}J = \mathcal{R}'$. Now $\mathcal{M} \subset \mathcal{R}_\omega$, hence $\Delta V = V\Delta$ for all $V \in \mathcal{M}$ [17, Lemma 15.8]. Since x_0 is a trace vector for $\mathcal{M}$, we have that the conjugation defined by x_0 relative to $\mathcal{M}$ is given by $Vx_0 \to V^*x_0$, and the modular operator is the identity operator. But if $V \in \mathcal{M}$ then $J\,Vx_0 = JV\,\Delta^{\frac{1}{2}} x_0 = J\,\Delta^{\frac{1}{2}} Vx_0 = V^*x_0$, so J and Δ extend the corresponding operators for $\mathcal{M}$. Thus by [17, Theorem 12.1] $\mathcal{R}' \subset \mathcal{M}' = J\mathcal{M}J \subset J\mathcal{R}J = \mathcal{R}'$, and $\mathcal{M} = \mathcal{R}$. The proof is complete.

<u>Remark 2.3</u>. The above theorem is trivial if $\mathcal{R}$ is abelian. Indeed, then $\mathcal{M}$ is abelian and has a cyclic vector. Thus $\mathcal{M}$ is maximal abelian, so $\mathcal{M} = \mathcal{R}$.

<u>Corollary 2.4</u>. If in Theorem 2.2 the group of eigenvalues is cyclic, then $\mathcal{R}$ is abelian.

<u>Proof</u>. Let γ be a generator for the group of eigenvalues, and let V_γ be a unitary eigenoperator with eigenvalue γ. Let $V_{\gamma^n} = V_\gamma^n$. Then $V_\gamma{}^n$ is an eigenoperator with eigenvalue γ^n, so $\mathcal{M}$ is abelian, hence so is $\mathcal{R}$.

It might be expected that we always have $\mathcal{M}$ and hence $\mathcal{R}$ abelian when G is abelian. In order to explain the difficulty we give an example.

Let G be the four group, $G = \{1,a,b,c\}$, where $a^2 = b^2 = c^2 = 1$, $ab = c$, $ac = b$, $bc = a$. Let U be the unitary representation of G on $\mathbb{C}^4$ defined by

$$U_a = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad U_b = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

Let

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then $UV = -VU$. Let $\mathcal{R}$ the vector space spanned by I, U, V, UV. Then $\mathcal{R}$ is a non abelian von Neumann algebra. Furthermore,

$$U_a V U_a^* = -V, \quad U_b V U_b^* = -V, \quad U_a U U_a^* = U, \quad U_b U U_b^* = -U .$$

Thus U_G defines a group of $*$-automorphisms of $\mathcal{R}$. Since the only diagonal operators in $\mathcal{R}$ are the scalars, the group is ergodic on $\mathcal{R}$. Note that I, V, U, UV are all eigenoperators for U_G, so $\mathcal{R} = \mathcal{M}$. If x_o is the vector

$$x_o = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

then $U_a x_o = U_b x_o = U_c x_o = x_o$, and x_o is cyclic for $\mathcal{R}$. We thus have the situation in Theorem 2.2 with $\mathcal{R}$ non abelian.

The assumption in Theorem 2.2 says in a general way that α has pure point spectrum. The next result explains this in more detail.

<u>Theorem 2.5</u>. Let $\mathcal{R}$ be a von Neumann algebra, G a group and α a representation of G as an ergodic group of $*$-automorphisms of $\mathcal{R}$. Assume α is implemented by a unitary representation U

of G, and that there is a unit vector x_0 cyclic for $\mathcal{R}$ such that $U_g x_0 = x_0$ for all $g \in G$. Then there is a bijection between eigenoperators V for α and eigenvectors for U given by $V \to Vx_0$. V and Vx_0 have the same eigenvalue, hence the eigenvalues for U form a group.

Proof. For $A' \in \mathcal{R}'$ let $\beta_g(A') = U_g A' U_g^{-1}$, $g \in G$. Then $\beta_g \in \operatorname{Aut} \mathcal{R}'$. If $\beta_g(A') = A'$ for all g then $A'x_0 = U_g A' U_g^{-1} x_0 = U_g A' x_0$. But α is ergodic on $\mathcal{R}$, so x_0 is the unique eigenvector with eigenvalue 1. Thus $A'x_0 = \lambda x_0$ for some complex number λ. Since x_0 is cyclic for $\mathcal{R}$ it is separating for $\mathcal{R}'$. Thus $A' = \lambda I$, and β is ergodic on $\mathcal{R}'$. Let x be an eigenvector for U, say $U_g x = (\gamma, g)x$ for all $g \in G$. Then ω_x is G-invariant on $\mathcal{R}'$, hence by uniqueness of ω_{x_0}, $\omega_x = \omega_{x_0}$ on $\mathcal{R}'$. Thus there is an operator V in $\mathcal{R}$ such that $x = Vx_0$ (define V by $VA'x_0 = A'x$). But if $g \in G$ then $\alpha_g(V)x_0 = U_g V x_0 = U_g x = (\gamma, g)x = (\gamma, g)Vx_0$. Since x_0 is separating for $\mathcal{R}$, since α is ergodic, V is an eigenoperator with eigenvalue γ. Conversely, if V is an eigenoperator for α with eigenvalue γ and $x = Vx_0$, then $U_g x = U_g V U_g^{-1} x_0 = \alpha_g(V)x_0 = (\gamma, g)Vx_0 = (\gamma, g)x$, so x is an eigenvector with eigenvalue γ. An application of Lemma 2.1 completes the proof.

The above theorem is a generalization of the so-called proper value theorem in ergodic theory [6, p.34]. We shall now digress somewhat and give a quick proof of the so-called discrete spectrum theorem in ergodic theory [6, p.46], generalized to arbitrary groups. If G is a group and U a unitary representation of G on a Hilbert space $\mathcal{H}$, we say U has <u>pure point spectrum</u> if $\mathcal{H}$ has an orthonormal basis of eigenvectors for U. Let for $i = 1,2$ $\mathcal{R}_i$ be a von Neumann algebra, G a group, and α_i a representation of G in $\mathrm{Aut}\,\mathcal{R}_i$ such that α_i is implemented by a unitary representation U^i on the underlying Hilbert space $\mathcal{H}_i$. We say $(\mathcal{R}_1,U^1)$ <u>is isomorphic to</u> $(\mathcal{R}_2,U^2)$ if there is a unitary operator W of $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $W\mathcal{R}_1W^* = \mathcal{R}_2$, and $WU_g^1W^* = U_g^2$ for all g.

<u>Theorem 2.6</u>. Let G be a group. Let for $i = 1,2$, $\mathcal{R}_i$ be an abelian von Neumann algebra, α_i be a representation of G as an ergodic group of $*$-automorphisms of $\mathcal{R}_i$ implemented by a unitary group U^i. Assume U^1 and U^2 both have pure point spectra and invariant vectors cyclic for $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. Then $(\mathcal{R}_1,U^1)$ is isomorphic to $(\mathcal{R}_2,U^2)$ if and only if U^1 and U^2 have the same groups of eigenvalues.

<u>Proof</u>. The necessity is trivial. Assume U^1 and U^2 have the same group H of eigenvalues. Since $\mathcal{R}_1$ is abelian we can choose a $*$-representation π of $\mathcal{R}_1$ onto the complex numbers. Let by Theorem 2.5 V_γ be a unitary eigenoperator for α_1 in $\mathcal{R}_1$ for each $\gamma \in H$. Let $\tilde{V}_\gamma = \overline{\pi(V_\gamma)}V_\gamma$. Then $\tilde{V}_\gamma$ is also an eigenoperator with eigenvalue γ. Since by Lemma 2.1 $V_\gamma V_\mu = r(\gamma,\mu)V_{\gamma\mu}$ with $r(\gamma,\mu)$ a complex number of modulus 1, an easy computation shows that $\gamma \to \tilde{V}_\gamma$ is a representation of H into $\mathcal{R}_1$. Replacing V_γ by $\tilde{V}_\gamma$ and similarly for $\mathcal{R}_2$, we may assume V^1 and V^2 are unitary representions of H into $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively such that $V_\gamma{}^i$ is an eigenoperator with eigenvalue γ. Let x_o^i be the cyclic invariant vector for U^i and let $x_\gamma{}^i = V_\gamma{}^i x_o{}^i$, $i = 1,2$. By Theorem 2.5 $\{x_\gamma{}^i\}$ is a complete set of eigenvectors for U^i, and by hypothesis $\{x_\gamma{}^i\}$ is an orthonormal basis for $\mathcal{H}_i$. Let $P_\gamma{}^i$ be the one dimensional projection onto $x_\gamma{}^i$. Then by spectral theory $U_g{}^i = \sum_{\gamma\in H}(\gamma,g)P_\gamma{}^i$ Define a unitary operator W of $\mathcal{H}_1$

onto $\mathcal{H}_2$ by $Wx_\gamma{}^1 = x_\gamma{}^2$. Then an easy computation shows $WP_\gamma{}^1W_\gamma{}^* = P_\gamma{}^2$, and $WV_\gamma{}^1W^*x_\mu{}^2 = WV_\gamma{}^1x_\mu{}^1 = WV_\gamma{}^1V_\mu{}^1x_o{}^1 = WV_{\gamma\mu}{}^1x_o = Wx_{\gamma\mu}{}^1 = x_{\gamma\mu}{}^2 = V_\gamma{}^2x_\mu{}^2$. Thus $WV_\gamma{}^1W^* = V_\gamma{}^2$, so by Theorem 2.2 (cf. Remark 2.3) $W\mathcal{R}_1W^* = \mathcal{R}_2$. The proof is complete.

Remark 2.7. The assumption that $\mathcal{R}_1$ and $\mathcal{R}_2$ are abelian is necessary in the above theorem. Indeed, if we in the example after Corollary 2.4 replace $\mathcal{R}$ by the linear span of

$$V^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

I, U, and UV^1, then we have an abelian von Neumann algebra $\tilde{\mathcal{R}}$, where U is as in the example. Let U_a, U_b, U_c be as in the example. Then it is easy to see that we have a representation of the four group as an ergodic group of $*$-automorphisms of $\tilde{\mathcal{R}}$. x_o is cyclic for $\tilde{\mathcal{R}}$ as well as for $\mathcal{R}$ and the eigenvalues are the same. Since $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are not $*$-isomorphic the conclusion of the theorem does not hold.

3. Spectra. In this section we shall study the situation when $\mathcal{R}$ is a von Neumann algebra, G a locally compact abelian group, α a representation of G as an ergodic group of $*$-automorphisms of $\mathcal{R}$, and U a strongly continuous unitary representation implementing α. By Stone's theorem there is a projection valued measure P on the dual $\hat{G}$ of G such that

$$U_g = \int_{\hat{G}} \overline{(\gamma, g)}\, dP_\gamma \,.$$

Our main interest will be the support of the measure P. This set will be the same as the spectrum of U as defined below. In the case G is cyclic generated by g, then the support of P equals the spectrum of the unitary operator U_g.

Following Arveson [2] and Forelli [5] we define two representations π_U and π_α of $L^1(G)$ into the bounded operators on the underlying Hilbert space $\mathcal{H}$ and $\mathcal{R}$ respectively by

$$\pi_U(f)x = \int_G f(t)U_t x\,dt$$

and

$$\pi_\alpha(f)A = \int_G f(t)\alpha_t(A)dt ,$$

where $f \in L^1(G)$ and dt is the Haar measure on G. If $f \in L^1(G)$ we let $Z(f) = \{\gamma \in \hat{G} : \hat{f}(\gamma) = 0\}$. We let $\mathrm{Sp}\,\alpha$ (resp. $\mathrm{Sp}\,U$) be $\cap Z(f)$, where f runs through the set of functions in $L^1(G)$ such that $\pi_\alpha(f) = 0$ (resp. $\pi_U(f) = 0$). Note that if $x,y \in \mathcal{H}$ then

$$(\pi_U(f)x,y) = \int_{\hat{G}} \hat{f}(\gamma)d(P_\gamma x,y) ,$$

so $\mathrm{Sp}\,U = \mathrm{supp}\,P$.

If $x \in \mathcal{H}$ and $A \in \mathcal{R}$ we let $\mathrm{Sp}_U x = \cap Z(f)$, where $\pi_U(f)x = 0$, and $\mathrm{Sp}_\alpha A = \cap Z(f)$, where $\pi_\alpha(f)A = 0$. If E is a closed set in $\hat{G}$ we denote by $M^\alpha(E)$ (resp. $M^U(E)$) the set of $A \in \mathcal{R}$ (resp. $x \in \mathcal{H}$) such that $\mathrm{Sp}_\alpha A \subset E$ (resp. $\mathrm{Sp}_U x \subset E$). As remarked in [2] it is easy to see that $M^U(E) = P(E)$. Note that if γ is an isolated point in $\mathrm{Sp}\,\alpha$ then γ is an eigenvalue, and every operator in $M^\alpha(\{\gamma\})$ is an eigenoperator with eigenvalue γ^{-1}. The proof of the next lemma is almost a direct copy of a similar proof in [9] based on one in [2].

<u>Lemma 3.1</u>. Let $\mathcal{R}$ be a von Neumann algebra, G a locally compact abelian group and α a representation of G into Aut $\mathcal{R}$ implemented by a strongly continuous unitary representation U. Let E and F be closed subsets of $\hat{G}$, $A \in M^{\alpha}(E)$, $x \in M^{U}(F)$. Then $Ax \in M^{U}(E+F)$.

<u>Proof</u>. If D is a closed subset of $\hat{G}$ let $R^{U}(D)$ (resp. $R^{\alpha}(D)$) denote the closed subspace of $\mathcal{H}$ (resp. $\mathcal{R}$) generated by the ranges of all operators $\pi_U(f)$ (resp. $\pi_\alpha(f)$) with $f \in L^1(G)$ and supp $\hat{f} \subset D$. By [2, Proposition 2.2], $M^U(D) = \bigcap_N R^U(D+N)$ (resp. $M^\alpha(D) = \bigcap_N R^\alpha(D+N)$) where N runs through the compact neighborhoods of the identity in $\hat{G}$. (We denote multiplication in G and $\hat{G}$ additively in what follows). In order to prove the lemma let N be a compact neighborhood of 0 in $\hat{G}$. It suffices to show $Ax \in R^U(E+F+N)$. For this let M be a compact neighborhood of 0 in $\hat{G}$ such that $M+M \subset N$. It suffices by density arguments to consider the case when $A = \pi_\alpha(g)B$ with $B \in \mathcal{R}$ and supp $\hat{g} \subset E+M$, and $x = \pi_U(f)y$ with supp $\hat{f} \subset F+M$, $y \in \mathcal{H}$. We must show $\pi_\alpha(g)B\,\pi_U(f)y \in R^U(E+F+M)$. Computing we see

$$\begin{aligned}
\pi_\alpha(g)B\,\pi_U(f)y &= \iint_{G\times G} g(t)f(u)\,\alpha_t\,(B\,U_u\,y)\,du\,dt \\
&= \iint g(t)f(u)U_t B\,U_{u-t}\,y\,du\,dt \\
&= \iint g(t)f(w+t)U_t B\,U_w\,y\,dw\,dt \\
&= \int \left(\int g(t)f_w(t)U_t\,dt\right)B\,U_w\,y\,dw \\
&= \int \pi_U(gf_w)B\,U_w\,y\,dw \\
&= \int z_w\,dw\ ,
\end{aligned}$$

where $f_w(t) = f(w+t)$ and $z_w = \pi_U(gf_w)B\,U_w\,y$. Since supp $gf_w \subset E+F+M+M$, $z_w \in R^U(E+F+N)$. Therefore

$\int z_w \, dw \in R^U(E+F+N)$, and the lemma follows.

Theorem 3.2. Let $\mathcal{R}$ be a von Neumann algebra, G a locally compact abelian group and α a representation of G as an ergodic group of $*$-automorphisms of $\mathcal{R}$. Suppose α is implemented by a strongly continuous unitary representation U of G, and let N be the kernel of α in G. Then we have

1) $\mathrm{Sp}\alpha$ is canonically isomorphic to $\widehat{G/N}$.
2) $\mathrm{Sp}\alpha \subset \gamma + \mathrm{Sp}\,U$ for some $\gamma \in \hat{G}$.
3) If $U_g = I$ for each $g \in N$ then $\mathrm{Sp}\,U = \mathrm{Sp}\alpha$.

Proof. By a result of Connes [3, Theorème 2.2.2] $\mathrm{Sp}\alpha$ is a closed subgroup of $\hat{G}$. By [3, Lemme 2.3.8] if $g \in G$ then $\mathrm{Sp}\alpha_g$ as an operator on $\mathcal{R}$ equals the closure of the set $\{\overline{(\gamma,g)}: \gamma \in \mathrm{Sp}\alpha\}$. Thus $g \in N$ if and only if g belongs to the annihilator of $\mathrm{Sp}\alpha$. By [11, Lemma 2.1.3] $\mathrm{Sp}\alpha$ is the annihilator of N in $\hat{G}$, hence by [11, Theorem 2.1.2] $\mathrm{Sp}\alpha$ is canonically isomorphic to $\widehat{G/N}$, proving 1).

In order to show 2) we first remark that if E is a closed set in $\hat{G}$ and $P(E) \neq 0$, then $P(E)$ is separating for $\mathcal{R}$. Indeed, let $AP(E) = 0$. Then $P(E)A^* = 0$, hence $P(E)$ annihilates the range projection to A^*. Since $P(E)U_gA^*U_g^* = U_gP(E)A^*U_g^* = 0$, $P(E)$ annihilates the range projection to $U_gA^*U_g^*$, hence the union Q of all these range projections. But $Q \in \mathcal{R}$ and Q is invariant under α. Since α is ergodic $Q = 0$, so $A = A^{**} = 0$.

If $\gamma \in \mathrm{Sp}\,U$ then 0 belongs to the spectrum of the representation $g \to (\gamma,g)U_g$. Thus in order to show 2) we may assume $0 \in \mathrm{Sp}\,U$, and will show $\mathrm{Sp}\alpha \subset \mathrm{Sp}\,U$. By [3, Lemme 2.1.3] $P(N) = M^U(N) \neq 0$ for each compact neighborhood N of 0. Let $\gamma \in \mathrm{Sp}\alpha$. Then by [3, Lemme 2.1.3] $M^\alpha(E) \neq 0$ for each compact neighborhood

E of γ. Let $0 \neq A \in M^{\alpha}(E)$ and choose by the above paragraph $x \in M^{U}(N)$ such that $Ax \neq 0$. By Lemma 3.1 $Ax \in M^{U}(E+N)$. Since the neighborhoods $E+N$ form a fundamental system of neighborhoods for γ, $\gamma \in \operatorname{Sp} U$ by [3, Lemme 2.1.3]. Thus 2) follows.

Assume $U_g = I$ for each $g \in N$. Then the kernel of U is N, so U defines a strongly continuous unitary representation $\tilde{U}$ of G/N by $\tilde{U}_{g+N} = U_g$. By definition $\operatorname{Sp} \tilde{U} \subset \widehat{G/N}$. But we may identify $\widehat{G/N}$ with the annihilator of N in $\hat{G}$ [11, Theorem 2.1.2]. Thus the uniqueness of the measure P in Stone's theorem shows $\operatorname{Sp} \tilde{U} = \operatorname{Sp} U$. Also α defines a representation $\tilde{\alpha}$ of G/N into $\operatorname{Aut} \mathcal{R}$ in a similar manner. By 1) $\operatorname{Sp} \tilde{\alpha} = \operatorname{Sp} \alpha = \widehat{G/N}$. We thus have by 2) that $\widehat{G/N} = \operatorname{Sp} \tilde{\alpha} \subset \gamma + \operatorname{Sp} \tilde{U} \subset \widehat{G/N}$ for some $\gamma \in \widehat{G/N}$, so $\operatorname{Sp} U = \operatorname{Sp} \tilde{U} = \widehat{G/N}$. The proof is complete.

<u>Corollary 3.3.</u> In addition to the assumptions in Theorem 3.2 assume there is a cyclic vector x_0 for $\mathcal{R}$ such that $U_g x_0 = x_0$ for all $g \in G$. Then $\operatorname{Sp} U$ is canonically isomorphic to $\widehat{G/N}$.

<u>Proof</u>. If $g \in N$ and $A \in \mathcal{R}$ then $U_g A x_0 = \alpha_g(A) x_0 = A x_0$. Since x_0 is cyclic for $\mathcal{R}$, $U_g = I$. Now apply Theorem 3.2.

We next want to know what happens when $\operatorname{Sp} \alpha$ is a discrete subgroup of $\hat{G}$. By Theorem 3.2 this is equivalent to analyzing the situation when G/N is compact.

<u>Theorem 3.4.</u> Let $\mathcal{R}$ be a von Neumann algebra, G a locally compact abelian group and α a strongly continuous representation of G as an ergodic group of *-automorphisms of $\mathcal{R}$. Let N be the kernel of α and assume G/N is compact. Then $\mathcal{R}$ is equal to the eigenalgebra for G in $\mathcal{R}$, and there is a unique faithful normal finite G-invariant trace on $\mathcal{R}$.

Proof. Recall that α strongly continuous means that for each A in $\mathcal{R}$ the map $g \to \alpha_g(A)$ is continuous when $\mathcal{R}$ is given the norm topology. α induces a strongly continuous representation $\tilde{\alpha}$ of G/N as an ergodic group of $*$-automorphisms of $\mathcal{R}$. Since G/N is compact so is the image of $\tilde{\alpha}$, hence so is α_g. It is then immediate from the theorem in [16] that there is a normal G-invariant state ω on $\mathcal{R}$. Since G is ergodic ω is unique and faithful. Let π be the GNS-representation of $\mathcal{R}$ defined by ω. Then π is a normal $*$-isomorphism of $\mathcal{R}$ onto a von Neumann algebra. Replacing $\mathcal{R}$ by $\pi(\mathcal{R})$ we may assume there is a cyclic vector x_o for $\mathcal{R}$ and a strongly continuous unitary representation U of G implementing α such that $U_g x_o = x_o$ for all $g \in G$. By Theorem 3.2 $Sp\alpha$ is canonically isomorphic to $\widehat{G/N}$, which is a closed discrete subgroup of $\hat{G}$ [11,1.7.3]. By Theorem 2.5 the eigenvectors for U are given by $V_\gamma x_o$ where V_γ is a unitary eigenoperator for α with eigenvalue γ. Since $Sp\alpha$ is a discrete group the eigenvalues are exactly the elements in $Sp\alpha$. By Corollary 3.3 SpU is also a discrete group, hence $\{V_\gamma x_o : \gamma \in Sp\alpha\}$ is a complete set of eigenvectors for U. Thus x_o is cyclic for the eigenalgebra $\mathcal{M}$ for G in $\mathcal{R}$. An application of Theorem 2.2 now completes the proof.

In ergodic theory the group G is usually the integers $\mathbb{Z}$ or the real numbers $\mathbb{R}$. This case has the technical advantage that every closed proper subgroup of the dual group is cyclic.

Theorem 3.5. Let $\mathcal{R}$ be a von Neumann algebra and G a locally compact group isomorphic to either $\mathbb{Z}$ or $\mathbb{R}$. Let α be a representation of G as an ergodic group of $*$-automorphisms of $\mathcal{R}$. Assume α is implemented by a strongly continuous unitary representation of G. Suppose $\operatorname{Sp} U \neq \hat{G}$. Then $\mathcal{R}$ is abelian, and there is a unique faithful normal G-invariant state on $\mathcal{R}$.

<u>Proof</u>. Since $\operatorname{Sp} U \neq \hat{G}$, Theorem 3.2 shows $\operatorname{Sp}\alpha \neq \hat{G}$, so is a cyclic subgroup of $\hat{G}$. Let N denote the kernel of α. Then $\operatorname{Sp}\alpha = \widehat{G/N}$ by Theorem 3.2, so G/N is isomorphic to the circle group. By Theorem 3.4 $\mathcal{R}$ is equal to the eigenalgebra for G in $\mathcal{R}$, and there is a faithful normal G-invariant trace on $\mathcal{R}$. Since the eigenvalues for α form a cyclic group the argument in Corollary 2.4 shows $\mathcal{R}$ is abelian. The proof is complete.

If N is a positive integer let l_N^∞ denote the abelian von Neumann algebra consisting of the $N \times N$ diagonal matrices. We say a single automorphism of a von Neumann algebra $\mathcal{R}$ is ergodic if its fixed points are the scalars. The following corollary is known in many special cases in ergodic theory, but to the best of my knowledge not in general.

<u>Corollary 3.6</u>. Let $\mathcal{R}$ be a von Neumann algebra not isomorphic to l_N^∞ for any positive integer. Suppose W is a unitary operator implementing an ergodic *-automorphism of $\mathcal{R}$. Then the spectrum of W is the unit circle.

<u>Proof</u>. Let $\alpha_n(A) = W^n A W^{-n}$. Then α is a representation of Z as an ergodic group of *-automorphisms of $\mathcal{R}$. If the spectrum of W is not the unit circle then by Theorem 3.5 $\mathcal{R}$ is abelian and there is a faithful normal G-invariant state on $\mathcal{R}$. Furthermore $\operatorname{Sp}\alpha$ is a proper closed subgroup of the circle group, hence is finite. But then the eigenalgebra for G in $\mathcal{R}$ is finite dimensional as a complex vector space, so by Theorem 3.4 $\mathcal{R}$ is *-isomorphic to some l_N^∞ contrary to assumption.

4. Applications. We indicate some applications of the preceding theory to some aspects of the C*-algebra formalism of quantum physics. Let $\mathfrak{A}$ be a C*-algebra and G a group of *-automorphisms of $\mathfrak{A}$. Assume $\mathfrak{A}$ is G-abelian, see[12] or [4]. This is the most general definition of asymptotically abelian. Let ρ be an ergodic G-invariant state. Then by the GNS construction there is a Hilbert space $\mathcal{H}_\rho$ a representation π_ρ of $\mathfrak{A}$ on $\mathcal{H}_\rho$, a unitary representation U of G on $\mathcal{H}_\rho$, and a unit vector x_ρ cyclic for $\pi_\rho(\mathfrak{A})$ such that $\rho(A) = (\pi_\rho(A)x_\rho, x_\rho)$, $\pi_\rho(g(A)) = U_g\pi_\rho(A)U_g^{-1}$ for $A \in \mathfrak{A}$, and $U_g x_\rho = x_\rho$. Let $\mathcal{R} = \pi_\rho(\mathfrak{A})''$ and E the projection $[\mathcal{R}'x_\rho]$, which is the support of the state ω_{x_ρ} on $\mathcal{R}$. By [4, Theorem 3] or [12, Theorem 6.3.3], x_ρ is the unique invariant vector for U, hence the proof of [14, Theorem 5.1] shows that the automorphisms $\alpha_g : EAE \to EU_gAU_g^{-1}E$ act ergodically on $E\mathcal{R}E$. Let $V_g = U_gE$. Then the preceding theory is applicable to α, V and $E\mathcal{R}E$. In particular we obtain that the eigenvalues for V form a group (Theorem 2.5), hence we have obtained generalizations of results of Kastler, Robinson and Ruelle [7], [11], and [12, p. 166].

If G is locally compact abelian and the representation of G as *-automorphisms of $\mathfrak{A}$ is strongly continuous, then so is U and hence V. Thus Corollary 3.3 shows that Sp V is a closed subgroup of $\hat{G}$.

Assume $\mathfrak{A}$ is asymptotically abelian with respect to G, i.e. there is a sequence $\{g_n\}$ in G such that $\|[g_n(A),B]\| \to 0$ as $n \to \infty$ for all A,B in $\mathfrak{A}$. Suppose the state ρ is strongly clustering, i.e. $\rho(g_n(A)B) \to \rho(A)\rho(B)$ for all A,B in $\mathfrak{A}$ as $n \to \infty$. Then x_ρ is the unique eigenvector for U and hence for V [14, Corollary 4.6]. Thus SpV cannot be a discrete subgroup of $\hat{G}$ in this case, unless ρ is a pure state so V is the trivial representation. In particular, if G is either $\mathbb{Z}$ or $\mathbb{R}$ it follows that $\mathrm{Sp}\, V = \hat{G}$ or $\{1\}$.

We next show an analogous result of this last one in the C*-algebra formulation of quantum field theory. We assume there is assigned to every bounded region $\mathcal{O}$ in the four dimensional Minkowski space M a C*-algebra $\mathcal{A}(\mathcal{O})$ of operators on an infinite dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{R}(\mathcal{O}) = \mathcal{A}(\mathcal{O})^-$, and assume $I \in \mathcal{R}(\mathcal{O})$. We assume there is a strongly continuous unitary representation $a \to U(a)$ of the four dimensional translation group, identified with M, such that the following properties are satisfied.

1. The spectrum of U is contained in the closed forward light-cone.
2. If $\mathcal{O}$ is a bounded region then $\mathcal{A}(\mathcal{O} + a) = U(a)\mathcal{A}(\mathcal{O})U(a)^*$.
3. If two regions $\mathcal{O}$ and $\mathcal{O}'$ are space-like to one another then $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')'$.
4. If $\mathcal{O} \subset \mathcal{O}'$ then $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')$.
5. If $\{\mathcal{O}_n\}$ is any covering of the unbounded region $\mathcal{O} \subset M$ of bounded regions, then the von Neumann algebra generated by the family $\{\mathcal{A}(\mathcal{O}_n)\}$ is independent of the covering, and is denoted by $\mathcal{R}(\mathcal{O})$.
6. There is up to a scalar multiple a unique vacuum vector x_0 cyclic for $\mathcal{R}(M)$.

Quite often one assumes the representation U is a representation of the Lorentz group as well. In that case the following result is well known.

Corollary 4.1. Suppose we have a local field theory satisfying axioms 1) - 6). Let a be a space-like vector in M, and denote by U_a the unitary representation $t \to U(ta)$ of $\mathbb{R}$. Then $\mathrm{Sp}\, U_a = \mathbb{R}$.

Proof. As shown by Araki [1] there is a region $\mathcal{O}$ in M with non void space-like complement such that $\mathcal{O} = \mathcal{O} + \mathbb{R}a_o$ for some space-like vector a_o. Multiplying a_o by a scalar we may map a_o onto a by a Lorentz transformation, hence we may assume $\mathcal{O}$ is a region in M such that $\mathcal{O} = \mathcal{O} + \mathbb{R}a$, and such that there is a bounded non void region $\mathcal{O}'$ space-like to $\mathcal{O}$. From the proof of [13, Theorem 4.1] x_o is separating and cyclic for $\mathcal{R}(\mathcal{O})$, and $\alpha_t(A) = U_a(t)A\, U_a(-t)$, $t \in \mathbb{R}$, is an ergodic group of *-automorphisms of $\mathcal{R}(\mathcal{O})$. If $\mathrm{Sp}\, U_a \neq \mathbb{R}$ then by Theorem 3.5 $\mathcal{R}(\mathcal{O})$ is abelian, hence $\mathcal{R}(\mathcal{O}')$ is abelian. But it is well known that this is impossible, see e.g. the proof of [13, Theorem 4.1].

Remark 4.2. In [13, Theorem 4.1] it was claimed that the von Neumann algebra $\mathcal{R}(\mathcal{O})$ above is a factor of type III. S. Doplicher pointed out to the author that the use of the edge-of-the-wedge theorem in the proof of the factor part was incorrect. Using [16, Corollary 3] instead of [13, Theorem 2.2] we can still conclude that $\mathcal{R}(\mathcal{O})$ is a von Neumann algebra of type III.

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