A NOTE ON COMPLEX $\mathcal{P}_1$ SPACES.

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According to a rather established terminology a Banach space A is a $\mathcal{P}_1$ space if for every pair of Banach spaces X and Y with $Y \subset X$ and for every bounded linear operator $T: Y \to A$ there exists a norm preserving extension $\tilde{T}: X \to A$. It was shown by L. Nachbin in [3] that if A is a real Banach space, then A is a $\mathcal{P}_1$ space if and only if every family of closed balls in A with the binary intersection property (that is any two balls of the family have a common element) has a non empty intersection. In a recent paper [1] we have shown that in the case of complex Banach spaces the following intersection property corresponds in several ways to the binary intersection property: A family $\{B(a_j, r_j)\}_{j \in J}$ of closed balls in a normed space A has the <u>weak intersection property</u> if for any linear functional $\varphi \in A^*$ with norm ≤ 1 the family $\{B(\varphi(a_j), r_j)\}_{j \in J}$ of balls in $\mathbb{C}$ (or in $\mathbb{R}$) has a non empty intersection. It is easy to see [1, Corollary 1.10] that in the case of real normed spaces the weak intersection property is equivalent with the binary intersection property. Hence the real version of the following theorem is exactly the theorem of Nachbin mentioned above.

THEOREM. Let A be a complex Banach space. Then A is a $\mathcal{P}_1$ space if and only if any family of closed balls in A with the weak intersection property has a non empty intersection.

The proof of this theorem will be given after we have proved the following three lemmata. In what follows

$$H^3 = \{(z_j) \in \mathbb{C}^3 : \sum_{j=1}^{3} z_j = 0\}.$$

LEMMA 1. A family $\mathcal{F}$ of closed balls in a complex normed space A has the weak intersection property if and only if for any subfamily of three balls $\{B(a_j,r_j)\}^3_{j=1} \subset \mathcal{F}$ it is true that

$$(*) \qquad \|\sum_{j=1}^{3} z_j a_j\| \leq \sum_{j=1}^{3} |z_j| r_j \quad , \qquad (z_j) \in H^3.$$

PROOF. This follows easily from [1, Corollary 1.3 and Corollary 1.4] together with the Helly theorem on intersection of compact convex sets in $\mathbb{C}$.

LEMMA 2. A family $\mathcal{F} = \{B(a_j,r_j)\}_{j \in J}$ of closed balls in a normed space A has the weak intersection property if and only if there exists a Banach space $X \supset A$ and an $x \in X$ such that

$$(1) \qquad \|x - a_j\| \leq r_j \quad , \qquad j \in J.$$

PROOF. Assume that $X \supset A$ is a normed space and that $x \in X$ satisfies (1). Let $\varphi \in A^*$ with $\|\varphi\| \leq 1$ be given. Choose $\tilde{\varphi} \in X^*$ as a norm preserving extension of φ. It then follows from (1) that

$$|\tilde{\varphi}(x) - \varphi(a_j)| = |\tilde{\varphi}(x-a_j)| \leq \|x-a_j\| \leq r_j \quad , \qquad j \in J.$$

Hence $\mathcal{F}$ has the weak intersection property. Assume conversely that $\mathcal{F}$ has this property. As is well known, there exists a set Γ such that A can be considered as a subspace of $l_\infty(\Gamma)$.

Let $\gamma \in \Gamma$. Since the evaluation map $x \to x(\gamma)$ is a linear functional on $l_\infty(\Gamma)$ with norm ≤ 1, it follows by hypothesis that there exists a complex number x_γ such that

$$|x_\gamma - a_j(\gamma)| \leq r_j , \qquad j \in J. \tag{2}$$

Define $x \in l_\infty(\Gamma)$ by $x(\gamma) = x_\gamma$, $\gamma \in \Gamma$. We then get from (2) that

$$\|x - a_j\| = \sup_{\gamma \in \Gamma}\{|x_\gamma - a_j(\gamma)|\} \leq r_j ; \qquad j \in J.$$

LEMMA 3. Let X,Y and A be normed spaces with $Y \subset X$. Let $T: Y \to A$, be a bounded linear operator and let $x \in X \setminus Y$. Then the family $\{B(Ty,\|T\|\cdot\|x-y\|): y \in Y\}$ has the weak intersection property.

PROOF. Let $y_1,y_2,y_3 \in Y$ and let $(z_j) \in H^3$. Then

$$\left\|\sum_{j=1}^{3} z_j T y_j\right\| \leq \|T\|\cdot\left\|\sum_{j=1}^{3} z_j(y_j - x)\right\| \leq \sum_{j=1}^{3} |z_j|\ \|T\|\cdot\|x-y_j\|.$$

The desired conclusion now follows from Lemma 1.

PROOF OF THE THEOREM. Assume that A is a $\mathcal{P}_1$ space. Let $\mathcal{F} = \{B(a_j,r_j)\}_{j\in J}$ be a family of balls in A with the weak intersection property. Choose, according to Lemma 2, the Banach space $X \supset A$ such that (1) is fulfilled. Since A is a $\mathcal{P}_1$ space, there exists a projection P from X onto A such that $\|P\| = 1$. It follows that $Px \in A$ and that

$$\|Px - a_j\| = \|P(x-a_j)\| \leq \|x-a_j\| \leq r_j , \qquad j \in J$$

Hence $\mathcal{F}$ has a non empty intersection.

Assume conversely that any family of balls in A with the weak intersection property has a non empty intersection. Let X and Y be Banach spaces with $Y \subset X$ and let $T: Y \to A$ be a bounded linear operator. In order to show that T admits a norm preserving extension $\tilde{T}: X \to A$, it is sufficient, by a standard application of the Zorns lemma, to assume that $\dim X/Y = 1$. Let $x \in X \setminus Y$. By a basic lemma of Nachbin (see [2, Lemma 5.2]), the operator T admits a norm preserving extension $\tilde{T}: X \to A$ if and only if

$$(3) \qquad \bigcap_{y \in Y} B(Ty, \|T\| \cdot \|x-y\|) \neq \emptyset.$$

But it follows from Lemma 3 and from our assumption that (3) is fulfilled.

REFERENCES

[1] O. Hustad, Intersection properties of balls in complex Banach spaces whose duals are L_1 spaces, Preprint Series, Inst.Math., University of Oslo, No. 9, (1973).

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[3] L. Nachbin, A theorem of the Hahn-Banach type for linear transformations, Trans.Amer.Math.Soc. 68 (1950) 28-46.

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