The scattering matrix for some non-polynomial interactions II

by

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A B S T R A C T

We continue the study of the infinite volume limit of quantum field theoretical models in n-dimensional space-time with interaction densities which are bounded functions of an ultraviolet cut-off boson field. The truncated off-shell scattering amplitudes are (in contrast to the on-shell ones) the limits of the correspondent space cut-off quantities. They are analytic in the energy variables outside the union of certain real hyperplanes and have the crossing symmetry. Remarks are given on the restriction of the off-shell scattering amplitudes to the physical mass shell.

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1. Introduction

As in two preceding papers [1], [2], we study quentum field theoretical models in $n \geq 4$ - dimensional space-time [1)] with non polynomial boson self-interaction. The infinite volume models are obtained as limits of the corresopndent ones with a space cut-off in the interaction. The Hamiltonian of the space cut-off interaction is

$$H_1 = H_0 + \lambda \int_{|\vec{x}|\leq 1} v(\varphi_\varepsilon(\vec{x}))d\vec{x} \ ,$$

where H_0 is the free energy of the free time zero boson field $\varphi(\vec{x})$ of mass $m > 0$, and $\varphi_\varepsilon(\vec{x}) = \int \chi_\varepsilon(\vec{x}-\vec{y})\varphi(\vec{y})d\vec{y}$, with $\vec{x} \in \mathbb{R}^{n-1}$, and $\chi_\varepsilon(\vec{x}) \in C_0^\infty(\mathbb{R}^{n-1})$, $\chi_\varepsilon(\vec{x}) \geq 0$, $\chi_\varepsilon(\vec{x}) = \chi_\varepsilon(-\vec{x})$.

$v(\alpha)$ is a real valued function of the form $v(\alpha) = \int e^{i\alpha s}d\nu(s)$, where $d\nu(s)$ is a bounded measure of bounded support on the real line. λ is a coupling constant. For λ real, H_1 is a self-adjoint operator, bounded from below, with the same domain as H_0 in the Fock space $\mathcal{F}$ of the free boson field $\varphi(\vec{x})$.

In [2] we proved the existence and uniqueness of the infinite volume vacuum Ω for all $|\lambda| < \lambda_0, \lambda_0 > 0$. Moreover we constructed the imaginary and real time Wightman functions and proved the relative cluster properties. We obtained thus the physical Hilbert space $\mathcal{H}$ with a strongly continuous unitary representation of the space and time translation group and hence, in particular, the Hamiltonian $H \geq 0$ of the infinite volume models. We also proved analyticity in λ of the imaginary time Wightman functions and of the infinite volume limit of the energy density.

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1) The results of [2] are valid for all $n \geq 1$.

In [1] we started the study of the scattering in these models. For the space cut-off interactions we constructed the S-matrix in terms of asymptotic fields and proved that it is analytic in the coupling constant λ and equal to the sum of the linked cluster expansion, which in turns corresponds to the usual expression of the S-matrix in terms of Feynman graphs. This S-matrix for the space cut-off interaction was also given ([1]) in terms of so called scattering functions, for which the existence of the infinite volume limit was proved. [2)]

We remarked moreover that the limits for $l \to \infty$ of the off-shell scattering amplitudes exist. In this paper we continue the study of these infinite volume off-shell scattering amplitudes and prove results on their analytic dependence on the energy variables, on the position of the corresponding cuts and on the crossing symmetry. More specifically, in section 2 we introduce the Fourier transforms of the infinite volume scattering functions constructed in [1]. We prove that they have a simple expression in terms of Fourier-laplace transforms of the correlation functions of [2], the so called spactral density functions, which are analytic in the coupling constant and exhibit explicitely a large analyticity domain in the complex energy variables, restricted only by the spectrum of the Hamiltonian H .

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2) The scattering functions are the analytic continuation of the correlation functions discussed in [2], which arise quite naturally in the Markovian euclidean version of the models and have the interpretation of classical correlation functions for a gas of variably charged particles in $\mathbb{R}^n$. Similar correlation functions were introduced for related euclidean models in [3], where also a relation of these and euclidean Bogoliubov off-shell scattering elements is given. Ideas on the relation between the vacuum functional and the euclidean (space-time cut-off) S-matrix are contained in several Kiev publications: [4].

In section 3 the analyticity results in the energy variables are applied to the infinite volume truncated off-shell scattering amplitudes, which are proven to be analytic outside the union of certain real hyperplanes. From this we have then the crossing-symmetry of the off-shell scattering amplitudes. [3)]

In contrast to the off-shell scattering matrix the on-shell S-matrix is not the limit of the corresponding quantity for the space cut-off interaction, but will be obtained by restricting the off-shell scattering amplitude to the physical mass shell, given by the eigenvalues $m(\vec{p})$ of H in the subspace of fixed momentum $\vec{P} = \vec{p}$. This is discussed in 3.3.

Throughout this paper we shall always use the same notations and definitions as in [1].

3) This is the correspondent property of the one which is called crossing symmetry in relativistic covariant theories: see e.g. [5].

2. The infinite volume scattering functions in momentum space.

2.1. The scattering functions and correlation functions.

In section 6 of [1] we introduced the infinite volume scattering functions $\sigma^k(x_1s_1,\dots,x_ks_k)$, as limits of the correspondent finite-volume scattering functions. By Theorem 6.1 of [1] the infinite volume scattering functions are given by

$$\sigma^k(t_1\vec{x}_1s_1,\dots,t_k\vec{x}_ks_k) = (-i\lambda)^k(\Omega, e^{is_1\varphi_\varepsilon(\vec{x}_1)}e^{-i(t_2-t_1)H}\dots e^{-i(t_k-t_{k-1})H}e^{is_k\varphi_\varepsilon(\vec{x}_k)}\Omega), \quad (2.1)$$

where $x_i = (t_i,\vec{x}_i)$ $(i=1,\dots,k)$, t_i being a time variable and $\vec{x}_i$ a space variable, running over $\mathbb{R}^{n-1}$, where $n-1$ is the number of space dimensions. The s_i run over the support of the measure $d\nu(s)$ defined in section 1. H is the physical Hamiltonian for the infinite volume theory, and Ω is the unique eigenvector in the physical Hilbert space corresponding to the isolated simple lowest eigenvalue zero of H . It follows from (2.1) that, for real coupling constant λ , σ^k is uniformly bounded and analytic for $\operatorname{Im}(t_{i+1}-t_i) < 0$, $i = 1,\dots,k-1$. By Theorem 6.2 of [1] σ^k is related to the infinite volume correlation functions by

$$\sigma^k(x_1s_1,\dots,x_ks_k) = (i)^k\rho^k(\tilde{x}_1s_1,\dots,\tilde{x}_ks_k) , \quad (2.2)$$

where $\tilde{x} = (ix_0,\vec{x})$, with $x = (x_0,\vec{x})$.
From (2.1) we see that $\sigma_k(t_1\vec{x}_1s_1,\dots,t_k\vec{x}_ks_k)$ is a uniformly bounded continuous function of all its variables. As in (5.5) of [1] we define the infinite volume time ordered scattering functions by

$$\check{\sigma}^k(x_1s_1,\dots,x_ks_k) = \sigma^k(x_1s_1,\dots,x_ks_k) , \quad (2.3)$$

for $t_1\leq\dots\leq t_k$, and the requirement that $\check{\sigma}^k(x_1s_1,\dots,x_ks_k)$

is symmetric under permutations of its variables. $\check{\sigma}^k$ is then again a uniformly bounded function, which is continuous in $\vec{x}_1, \ldots, \vec{x}_k$ and $s_1, \ldots, s_k$ and piecewise continuous in $t_1, \ldots, t_k$. Like σ^k it is also translation invariant in space and time.

We define the infinite volume scattering functions in momentum space by

$$\mathcal{S}^k(p_1 s_1, \ldots, p_k s_k) = \int \ldots \int e^{i \sum_{j=1}^{k} p_j x_j} \check{\sigma}^k(x_1 s_1, \ldots, x_k s_k) dx_1 \ldots dx_k , \quad (2.4)$$

where the Fourier transform (2.4) is understood in the sense of tempered distributions.

Let π be any permutation of $1, \ldots, k$. Then we define

$$\mathcal{S}^k_\pi(p_1 s_1, \ldots, p_k s_k) = \int_{\pi(t_1) \leq \ldots \leq \pi(t_k)} \ldots \int e^{i \sum_{j=1}^{k} p_j x_j} \check{\sigma}^k(x_1 s_1, \ldots, s_k s_k) dx_1 \ldots dx_k . \quad (2.5)$$

It is obvious that

$$\mathcal{S}^k(p_1 s_1, \ldots, p_k s_k) = \sum_\pi \mathcal{S}^k_\pi(p_1 s_1, \ldots, p_k s_k) , \quad (2.6)$$

where the summation runs over all the permutations. It follows from (2.5) and the symmetry of $\check{\sigma}^k(x_1 s_1, \ldots, x_k s_k)$ with respect to permutations of the indices that

$$\mathcal{S}^k_\pi(p_1 s_1, \ldots, p_k s_k) = \mathcal{S}^k_o(p_{\pi(1)} s_{\pi(1)}, \ldots, p_{\pi(k)} s_{\pi(k)}), \quad (2.7)$$

where

$$\mathcal{S}^k_o(p_1 s_1, \ldots, p_k s_k) = \int_{t_1 \leq \ldots \leq t_k} \ldots \int e^{i \sum_{j=1}^{k} p_j x_j} \sigma^k(x_1 s_1, \ldots, x_k s_k) dx_1 \ldots dx_k \quad (2.8)$$

and we have used that $\check{\sigma}^k$ and σ^k are equal for $t_1 \leq \ldots \leq t_k$. Introduce now the variables $(\tau_i, \vec{\xi}_i) = x_{i+1} - x_i$, $i = 1, \ldots, k-1$ and $(\alpha_1, \vec{\beta}_1) = p_1$, $(\alpha_2, \vec{\beta}_2) = p_1 + p_2, \ldots, (\alpha_{k-1}, \vec{\beta}_{k-1}) = p_1 + \ldots + p_{k-1}$. Then we have

$$\mathcal{S}_o^k(p_1s_1,\ldots,p_ks_k) = \delta(p_1+\ldots+p_k)\int_{\tau_i\geq 0}\ldots\int e^{i(\sum_{j=1}^{k-1}\alpha_j\tau_j+\vec{\beta}_j\vec{\xi}_j)}\sigma^k(x_1s_1,\ldots$$
$$\ldots,x_ks_k)d\tau_1\ldots d\tau_{k-1}d\vec{\xi}_1\ldots d\vec{\xi}_{k-1}\;. \tag{2.9}$$

We now introduce the functions:

$$\eta^k(\alpha_1,\vec{\beta},\ldots\alpha_{k-1}\vec{\beta}_{k-1};s_1,\ldots s_k) =$$
$$= \int_{\tau_i\geq 0}\ldots\int e^{i(\sum_{j=1}^{k-1}\alpha_j\tau_j+\vec{\beta}_j\vec{\xi}_j)}\sigma^k(x_1s_1,\ldots,x_ks_k)d\tau_1\ldots d\tau_{k-1}d\vec{\xi}_1\ldots d\vec{\xi}_{k-1}\;, \tag{2.10}$$

where $(\tau_i,\vec{\xi}_i) = x_{i+1}-x_i$, $i = 1,\ldots,k-1$. (2.9) then gives $\mathcal{S}_o^k$ in terms of η^k , so that

$$\mathcal{S}_o^k(p_1s_1,\ldots,p_ks_k) = \delta(p_1+\ldots+p_k)\eta^k(\alpha_1\vec{\beta}_1,\ldots,\alpha_{k-1}\vec{\beta}_{k-1};s_1\ldots s_k) \tag{2.11}$$

where $(\alpha_1,\vec{\beta}_1) = p_1$, $(\alpha_2,\vec{\beta}_2) = p_1+p_2,\ldots,(\alpha_{k-1},\vec{\beta}_{k-1}) = p_1+\ldots+p_{k-1}$.

Since the integration over τ_j in (2.10) is only over the positive real axis, we see that η^k is analytic in $\mathrm{Im}\,\alpha_j > 0$, $j = 1,\ldots,k-1$ as a tempered distribution in $\vec{\beta}_1,\ldots,\vec{\beta}_{k-1}$. Since by (2.1) $\sigma^k(x_1s_1,\ldots,x_ks_k)$ is analytic and uniformly bounded for $\mathrm{Im}\,\tau_i > 0$, we may continue the integration over τ_i from the right hand real half line onto the imaginary upper half line, and the integrals will be equal by the exponential decrease of the integrand. Performing this analytic continuation for all the τ_i-integrations $i = 1,\ldots,k-1$, we get by (2.2) that for $\alpha_j > 0$, $j = 1,\ldots,k-1$

$$\eta^k(\alpha_1\vec{\beta}_1,\ldots,\alpha_{k-1}\vec{\beta}_{k-1};s_1,\ldots,s_k) =$$
$$= i(-1)^{k-1}\int_{\tau_i\geq 0}\ldots\int e^{-\sum_{j=1}^{k-1}\alpha_j\tau_j}\cdot e^{i\sum_{j=1}^{k-1}\vec{\beta}_j\cdot\vec{\epsilon}_j}\rho^k(x_1s_1,\ldots,x_ks_k)d\tau_1\ldots d\tau_{k-1}d\vec{\xi}_1\ldots d\vec{\xi}_{k-1}\;, \tag{2.12}$$

where $(\tau_i, \vec{\xi}_i) = x_{i+1} - x_i$, $i = 1,\dots,k-1$.

Since $\mathcal{S}^k$ by (2.6), (2.7) and (2.11) is given in terms of η^k , (2.12) gives the time ordered scattering function in terms of the correlation function $\rho^k(x_1 s_1,\dots,x_k s_k)$. The $\eta^k(\alpha_1\vec{\beta}_1,\dots,\alpha_{k-1}\vec{\beta}_{k-1};s_1\dots s_k)$ will be called the spectral density functions.

2.2. Analyticity in the coupling constant and the energy variables of the spectral density functions.

In [2] we introduced the correlation functions $\rho^k(x_1 s_1,\dots,x_k s_k)$. By Lemma 4.1 of [2] we have that the correlation functions $\rho^k(x_1 s_1,\dots,x_k s_k)$ are analytic in the coupling constant λ , for complex λ such that $|\lambda| < \lambda_o$, where $\lambda_o > 0$. [4)] Moreover they satisfy, for complex λ with $|\lambda| < \lambda_o$, the estimates:

$$|\rho^k(x_1 s_1,\dots,x_k s_k)| \leq C^{-k} \, |\lambda| (1-\frac{|\lambda|}{\lambda_o})^{-1} \quad . \tag{2.13}$$

The $\rho^k(x_1 s_1,\dots,x_k s_k)$ are continuous in all the variables and translation invariant in space and time.

Using (2.13) and the exponential decrease, in the τ-variables, of the integrand, we now get from (2.12) that $\eta^k(\alpha_1\vec{\beta}_1,\dots,\alpha_{k-1}\vec{\beta}_{k-1};s_1,\dots,s_k)$ is, for $\alpha_j > 0$, $j = 1,\dots,k-1$, analytic in λ for $|\lambda| < \lambda_o$, as a tempered distribution in $\vec{\beta}_1,\dots,\vec{\beta}_{k-1}$. Moreover we get, from (2.12) and (2.13), that η^k is analytic in the energy variables $\alpha_1,\dots,\alpha_{k-1}$ for $\mathrm{Re}\,\alpha_j > 0$ and λ complex with $|\lambda| < \lambda_o$. By the same argument as above we also obtain that it is analytic in λ for $|\lambda| < \lambda_o$ and

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4) One has $\lambda_o = C^{-1} e^{-2B-1}$, where C is defined in section 4 of [2] and B is defined by (4.10) of [2].

Re $\alpha_j > 0$, $j = 1,\dots,k-1$.

Let now λ be real and $-\lambda_o < \lambda < \lambda_o$. By (2.10) we then have that η^k is analytic in $\alpha_j > 0$, $j = 1,\dots,k-1$, as a tempered distribution in $\vec{\beta}_1,\dots,\vec{\beta}_k$, since by (2.1) $\sigma^k(x_1s_1,\dots,x_ks_k)$, is a uniformly bounded continuous functions of all its variables, when λ is real.

From the linked cluster expansion for ρ^k as given by Lemma 3.1 of [1] and the fact that the measure $d\nu(s)$ satisfies the relation $\overline{d\nu(s)} = d\nu(-s)$, we have, for real λ :

$$\bar{\rho}^k(x_1s_1,\dots,x_ks_k) = \rho^k(x_1,-s_1,\dots,x_k,-s_k) \ . \tag{2.14}$$

This implies, by (2.12):

$$\overline{\eta^k}(\alpha_1\vec{\beta}_1,\dots,\alpha_{k-1}\vec{\beta}_{k-1};s_1\dots s_k) = -\eta^k(\alpha_1,-\vec{\beta}_1,\dots,\alpha_{k-1},-\vec{\beta}_{k-1};-s_1,\dots,-s_k), \tag{2.15}$$

where $\bar{\eta}^k$ is the complex conjugate of η^k .
By the analyticity of η^k in Im $\alpha_j > 0$, $j = 1,\dots,k-1$, we obtain from (2.15) that $\eta^k(\alpha_1\vec{\beta}_1,\dots,\alpha_{k-1}\vec{\beta}_{k-1};s_1\dots s_k)$ is also analytic for Im $\alpha_j < 0$ and by its analyticity for Re $\alpha_j > 0$ we have that it is the continuation of the same function.
Let us now define the integrated spectral density functions

$$\begin{aligned} &\eta^k(\alpha_1f_1,\dots,\alpha_{k-1}f_{k-1};s_1,\dots,s_k) = \\ &= \int\dots\int\eta^k(\alpha_1\vec{\beta}_1,\dots,\alpha_{k-1}\vec{\beta}_{k-1};s_1\dots s_k)f_1(\vec{\beta}_1)\dots f_k(\vec{\beta}_k)d\vec{\beta}_1\dots d\beta_k \ . \end{aligned} \tag{2.16}$$

The following theorem follows from what is said above

Theorem 2.1 Let $f_1,\dots,f_{k-1}$ be in $\mathcal{S}(R^3)$, then the integrated spectral density functions $\eta^k(\alpha_1f_1,\dots,\alpha_{k-1}f_{k-1};s_1,\dots,s_k)$ for λ real and $-\lambda_o < \lambda < \lambda_o$, are analytic functions in the energy variables $\alpha_1,\dots,\alpha_{k-1}$ in the product of the cut planes

$C - [-\infty, 0]$, i.e. the complex planes cut along the negative real axis.

Moreover for complex λ , the integrated spectral density functions are analytic in the coupling constant λ and the energyvariables $\alpha_1, \ldots, \alpha_{k-1}$ in the product of the disk $|\lambda| < \lambda_o$ and the right hand half planes $\mathrm{Re}\ \alpha_i > 0$, $i = 1, \ldots, k-1$. ∎

By formula (6.4) of [1] we have that for $t_1 \leq \ldots \leq t_k$ and $-\lambda_o < \lambda < \lambda_o$:

$$\rho^k(t_1\vec{x}_1 s_1, \ldots, t_k\vec{x}_k s_k) = $$
$$= (-\lambda)^k(\Omega, e^{is_1\varphi_\varepsilon(\vec{x}_1)} e^{-(t_2-t_1)H} \ldots e^{-(t_k-t_{k-1})H} e^{is\varphi_\varepsilon(\vec{x}_k)} \Omega) \ . \tag{2.17}$$

Introducing now $\vec{P} = \{P_1, P_2, \ldots, P_{n-1}\}$ as the self adjoint infinitesimal generator for the unitary group of space translations, we see that (2.17) may be written

$$\rho^k(t_1\vec{x}_1 s_1, \ldots, t_k\vec{x}_k s_k) = $$
$$= (-\lambda)^k(\Omega, e^{is_1\varphi_\varepsilon(o)} e^{-i\vec{\xi}_1\vec{P}} e^{-\tau_1 H} e^{is_2\varphi_\varepsilon(o)} \ldots e^{-i\vec{\xi}_{k-1}\cdot\vec{P}} e^{-\tau_{k-1}H} e^{is_k\varphi_\varepsilon(o)} \Omega). \tag{2.18}$$

Inserting this expression for ρ^k into (2.12) we get the following ezpression for the spectral density function for $\alpha_j > 0$, $j = 1, \ldots, k-1$.

$$\eta^k(\alpha_1\vec{\beta}_1, \ldots, \alpha_{k-1}\vec{\beta}_{k-1}; s_1, \ldots, s_k) = $$
$$= \frac{((2\Pi)^{n-1}\lambda)^k}{(2\Pi)^{n-1} i}(\Omega, e^{is_1\varphi_\varepsilon(o)} \frac{\delta(\vec{P}-\beta_1)}{H+\alpha_1} e^{is_2\varphi_\varepsilon(o)} \ldots \frac{\delta(\vec{P}-\beta_{k-1})}{H+\alpha_{k-1}} e^{is_k\varphi_\varepsilon(o)} \Omega), \tag{2.19}$$

where we have used that H and $\vec{P}$ commute, and the identity (2.19) is in the sense of tempered distribution.

<u>Theorem 2.2</u> For $-\lambda_o < \lambda < \lambda_o$ and $\alpha_1, \ldots, \alpha_{k-1}$ complex and outside the negative real half axis, the spectral density func-

tions are given by

$$\eta^k(\alpha_1\vec{\beta}_1,\dots,\alpha_{k-1}\vec{\beta}_{k-1},s_1,\dots,s_k) =$$

$$= \frac{((2\Pi)^{n-1}\lambda)^k}{2\Pi)^{n-1}i}(\Omega, e^{is_1\varphi_\varepsilon(o)}\frac{\delta(\vec{P}-\vec{\beta}_1)}{H+\alpha_1}e^{is_2\varphi_\varepsilon(o)}\dots\frac{\delta(\vec{P}-\vec{\beta}_{k-1})}{H+\alpha_{k-1}}e^{is_k\varphi_\varepsilon(o)}\Omega).$$

The jumps of the spectral density functions across the cuts along the negative real axis for α_j is obtained from the formula above by substituting $2\pi i\delta(H+\alpha_j)$ for $\frac{1}{H+\alpha_j}$. The equality above is to be understood in the sense of tempered distributions. Hence for the integrated spectral density functions we have

$$\eta^k(\alpha_1 f_1,\dots,\alpha_{k-1}f_{k-1};s_1,\dots,s_k) =$$

$$= \frac{((2\Pi)^{n-1}\lambda)^k}{(2\Pi)^{n-1}i}(\Omega, e^{is_1\varphi_\varepsilon(o)}\frac{f_1(\vec{P})}{H+\alpha_1}e^{is_2\varphi_\varepsilon(o)}\dots\frac{f_{k-1}(\vec{P})}{H+\alpha_{k-1}}e^{is_k\varphi_\varepsilon(o)}\Omega).$$

Proof: Integrating (2.19) with respect to $f_1(\vec{\beta}_1)\dots f_{k-1}(\vec{\beta}_{k-1})$ and utilizing that both sides are then analytic functions for $\mathrm{Re}\,\alpha_i > 0$, $i = 1,\dots,k-1$, because of $H \geq 0$, we get the corresponding identity for all $\alpha_1,\dots,\alpha_{k-1}$ in the cut planes, and from this formula it also follows that η^k is joint analytic for all $\alpha_1,\dots,\alpha_{k-1}$ in the cut planes, since the spectrum of H is contained in the positive real axis. ∎

<u>Theorem 2.3</u> Consider now the scattering functions in momentum space $\mathcal{S}^k(p_1 s_1,\dots,p_k s_k)$ as functions on the hyperplane $\sum_{i=1}^k p_i = 0$. Then for real λ, $-\lambda_o < \lambda < \lambda_o$, $\mathcal{S}^k(p_1 s_1,\dots,p_k s_k)$ is complex analytic in the energyvariables $p_1^o,\dots,p_k^o$ in the complex $k-1$ dimensional space $\sum_{i=1}^k p_i^o = 0$ outside the union of the real

hyperplanes of the form

$$\operatorname{Im} \sum_{i \in I} p_i^0 = 0 ,$$

where I is any subset of $\{1,\dots,k\}$.

Proof: The Theorem is proven by expressing $\mathcal{S}^k$ in terms of η^{k-1} by the formulae (2.6), (2.7) and (2.11) and using theorem 2.1, which gives the analyticity of η^{k-1} in $\alpha_1,\dots,\alpha_{k-1}$. ∎

3. The infinite volume off shell scattering matrix.

3.1 Analyticity in the energy variables for the off shell scattering matrix.

We now introduce the truncated spectral density functions

$$\eta_T^k(\alpha_1\vec{\beta}_1,\ldots,\alpha_{k-1}\vec{\beta}_{k-1};s_1,\ldots,s_k)$$

$$= \frac{((2\Pi)^{n-1}\lambda)^k}{(2\Pi)^{n-1}i}\Big(\Omega, e^{is_1\varphi_\varepsilon(o)} \frac{\delta(\vec{P}-\vec{\beta}_1)}{H+\alpha_1}F\, e^{is_2\varphi_\varepsilon(o)} \frac{\delta(\vec{P}-\vec{\beta}_2)}{H+\alpha_2}F \tag{3.1}$$

$$\ldots \frac{\delta(\vec{P}-\vec{\beta}_{k-1})}{H+\alpha_{k-1}}F\, e^{is_k\varphi_\varepsilon(o)}\Omega\Big) ,$$

where F is the projection on the orthogonal complement of Ω in the physical Hilbert space $\mathcal{H}$. Let F_o be the projection onto Ω, then by utilicing that $F+F_o = 1$, we see that if we define the truncated time ordered scattering functions in momentum space $\mathcal{S}_T^k(p_1s_1,\ldots,p_ks_k)$ by the formulae (2.6) and (2.11) with η_T^k instead of η^k, then $\mathcal{S}_T^k(p_1s_1,\ldots,p_ks_k)$ is actually the Fourier transform of the time ordered truncated scattering functions $\check{\sigma}_T^k(x_1s_1,\ldots,x_ks_k)$, where $\check{\sigma}_T^k$ is the functions obtained by truncating the time ordered scattering functions (2.3) in the sense of (3.13) of [1].

We define the off shell finite volume truncated scattering amplitudes $S_{n,m}^{l,T}(p_1,\ldots,p_n;q_1,\ldots,q_m)$ by the formula in theorem 6.4 of [1], without the restrictions $p_i^o = \mu(\vec{p}_i)$, $q_j^o = \mu(\vec{q}_j)$.
Introduce the notation

$$S_{n,m}^{l,T}(A,B) = A_{n,m}^{l,T}(p_1,\ldots,p_n;q_1,\ldots,q_m) \tag{3.2}$$

with $A = \{p_1,\ldots,p_n\}$ and $B = \{q_1,\ldots,q_m\}$.
With this notation the formula in theorem 6.4 of [1] takes the

form

$$S^{l,T}(A,B) = \sum_{r=1}^{|A|+|B|} \frac{1}{r!} \sum_{\substack{A=A_1\cup\ldots\cup A_r \\ B=B_1\cup\ldots\cup B_r}} \int\ldots\int (is_1)^{|A_1|+|B_1|}\ldots(is_r)^{|A_r|+|B_r|} \tag{3.3}$$

$$|\tilde{\chi}_\varepsilon(A)|^2|\tilde{\chi}_\varepsilon(B)|^2\tilde{\sigma}^r_{l,T}(\sum_{i\in A_1} p_i - \sum_{j\in B_1} q_j,\ldots,\sum_{i\in A_i} p_i - \sum_{i\in A_r} q_j, s_1,\ldots,s_r) \prod_{j=1}^{r} d\mu(s_j)$$

$$+ \delta_{|A|,1}\,\delta_{|B|,1}\,\delta(\vec{p}_1-\vec{q}_1)\ ,$$

where the sum runs over all disjoint partitions of A and B into r subsets, and $|A|$ stands for the number of points in the set A . $\tilde{\chi}_\varepsilon(A) = \prod_{j\in A}\tilde{\chi}_\varepsilon(\vec{p}_j)$ and $\delta_{|A|,1} = 1$ if $|A| = 1$ and zero if not.

By theorem 6.1 of [1] we know that the finite volume scattering functions $\sigma^k_l(x_1s_1,\ldots,x_ks_k)$ converge pointwise as uniformly bounded functions to the infinite volume scattering functions $\sigma^k_l(x_1s_1,\ldots,x_ks_k)$ for λ real and $-\lambda_o< \lambda < \lambda_o$. Hence the corresponding time ordered scattering functions also converge pointwise as uniformly bounded functions and of course also the corresponding truncated functions converge pointwise as uniformly bounded functions. Therefore their Fourier transforms converge as tempered distributions, and their limits are given by the truncated time ordered scattering functions in momentum space $\mathscr{S}^k_T(p_1s_1,\ldots,p_ks_k)$. We formulate this result in the following theorem.

<u>Theorem 3.1</u> The finite volume truncated off shell scattering amplitudes $S^{l,T}(A,B)$ given in (3.3) converge in the sense of tempered distribytions to the infinite volume truncated off shell

scattering amplitudes $S^T(A,B)$ given by the following formula for λ real and $-\lambda_o < \lambda < \lambda_o$:

$$S^T(A,B) = \sum_{r=1}^{|A|+|B|} \frac{1}{r!} \sum_{\substack{A=A_1\cup\ldots\cup A_r \\ B=B_1\cup\ldots\cup B_r}} \int\ldots\int (is_1)^{|A_1|+|B_1|} \ldots (is_r)^{|A_r|+|B_r|}$$

$$|\tilde{\chi}_\varepsilon(A)\chi_\varepsilon(B)|^2 \mathcal{S}^k_T(\Sigma A_1 - \Sigma B_1, s_1, \ldots, \Sigma A_r - \Sigma B_r, s_r) \prod_{j=1}^{r} d\mu(s_j)$$

$$\delta_{|A|,1}\ \delta_{|B|,1}\ \delta(\vec{p}_1 - \vec{q}_1) ,$$

where $A = \{p_1,\ldots,p_n\}$ and $B = \{q_1,\ldots,q_m\}$ and $|A|$ is the number of points in the set A . The sum runs over all disjoint partitions $A = A_1\cup\ldots\cup A_r$ and $B = B_1\cup\ldots\cup B_r$, such that $A_i \cup B_i \neq \emptyset$ for $i = 1,\ldots,r$. $\tilde{\chi}_\varepsilon(A) = \prod_{p\in A} \tilde{\chi}_\varepsilon(p)$, and $\Sigma A_j = \sum_{p\in A_j} p$ and $\delta_{|A|,1} = 1$ of $|A| = 1$ and zero if not.

The truncated scattering function $\mathcal{S}^k_T(p_1 s_1,\ldots,p_k s_k)$ is given in term of the truncated spectral density functions $\eta^k_T(\alpha\vec{\beta}_1,\ldots,\alpha\vec{\beta}_{k-1}; s_1\ldots s_k)$ by

$$\mathcal{S}^k_T(p_1 s_1,\ldots,p_k s_k) = \sum_\pi \eta^k_T(p_{\pi(1)}, p_{\pi(1)}+p_{\pi(2)},\ldots$$

$$, p_{\pi(1)}+\ldots+p_{\pi(k-1)}; s_{\pi(1)},\ldots, s_{\pi(k)})\,\delta(p_1+\ldots+p_k),$$

where the sum runs over all permutations π of $\{1,\ldots,k\}$, and η^k_T is given by (3.1).

We remark that the truncated scattering amplitudes $S^T_{n,m}(p_1,\ldots,p_n; q_1,\ldots,q_m)$ are symmetric in $p_1,\ldots,p_n$ and in $q_1,\ldots,q_m$.

<u>Theorem 3.2</u> For λ real and $-\lambda_o < \lambda < \lambda_o$ the truncated off shell scattering amplitudes $S^T_{n,m}(p_1,\ldots,p_n; q_1,\ldots,q_m)$ considered as functions on the hyperplane $\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} q_j = 0$, are analytic

functions in the complex energy variables $p_1^o, \ldots, p_n^o; q_1^o, \ldots, q_m^o$ in the $n+m-1$ dimensional complex space which is the hyperplane $\sum_{i=1}^{n} p_i^o - \sum_{i=1}^{m} q_i^o = 0$, in a domain which is the complement to the union of the real hyperplanes of the form

$$\mathrm{Im}(\sum_{i\in I} p_i^o - \sum_{j\in J} q_j^o) = 0 ,$$

where I is any non empty subset of $\{1,\ldots,n\}$ and J is any non empty subset of $\{1,\ldots,m\}$.

Proof. This theorem follows from the analyticity of the scattering functions in momentum space theorem 2.3, and the immediate observation that the truncated scattering functions in momentum space have the same domain of analyticity, as follows from the fast that η^k and η_T^k have the same domain of analyticity in the energy variabeles, as seen from the definition (3.1) of the truncated spactral density functions η_T^k .

Remark: The structure of the regions where the truncated off-shell scattering emplitudes are not analytic could be more closely spacified if one would have more detailed information on the spactrum of H in a fixed total momentum subspace. This is clearly seen from the formula (3.1) for the truncated spactral density functions, which shows how the maximal domain of analyticity depends on the spactrum of H .

3.2 The crossing symmetry for the off shell scattering amplitudes.

The form of the off shell scattering matrix given in theorem 3.1 together with the analyticity in the energy variables for the off shell scattering matrix as given in theorem 3.2 are sufficient to prove the correspondent in this model of what is usually known as the crossing symmetry for the off shell scattering amplitudes[5]. Namely the property that the scattering amplitudes for different scattering channels are related to one another in the sense that they are boundery values of one and the same analytic function taken at different branch cuts.

Recalling that for any physical scattering process the energy variables are of course all positive, we shall see that the crossing symmetry actually follows from the fact that by theorem 3.2 we may continue analytically the energy variables from the cut along the positive real axis to the cut along the negative real axis. In fact we have the following theorem:

Theorem 3.3 For λ real and $-\lambda_o < \lambda < \lambda_o$, the truncated off shell scattering amplitudes $S^T_{n,m}(p_1,\ldots,p_n;q_1,\ldots,q_m)$ are symmetric in $p_1,\ldots,p_n$ and in $q_1,\ldots,q_m$. Moreover they are symmetric with respect to interchanges of p's with q's in the sense that

$$S^T_{n,m}(p_1,\ldots,p_n;q_1,\ldots,q_m) = S^T_{n-1,m+1}(p_1,\ldots,p_{n-1};q_1,\ldots,q_m,-p_n) .$$

Furthermore $S^T_{n,m}(p_1,\ldots,p_n;q_1,\ldots,q_m)$ is analytic in the upper half planes for the energy variables $p_1^o,\ldots,p_n^o;q_1^o,\ldots,q_m^o)$, so

5) See e.g. Ref. [5].

that $S^T_{n,m}(p_1,\ldots,p_n;q_1,\ldots,q_m)$ may be analytically continued in the upper p_n^o half plane from the positive real p_n^o axis to the negative real p_n^o axis. Hence all the truncated off shell scattering amplitudes $S^T_{n,m}(p_1,\ldots,p_n;q_1,\ldots,q_m)$ with $n+m=N$ are boundary values of one and the same analytic function taken at different branch cuts.

Proof: The identity in the theorem follows immediately from the formula for the truncated off shell scattering amplitudes given in theorem 3.1. The analyticity in the energy variables follows from the analyticity in the energy variables given in theorem 3.2. This proves the theorem. ∎

3.3 Remarks on the scattering matrix.

The scattering matrix would be obtained from the off shell scattering matrix by restricting the energy variables p_i^o, $i = 1,\ldots,n$ and q_j^o, $j = 1,\ldots,m$ to the physical mass shell, in a similar way as for the space cut-off scattering matrix in theorem 6.4 of [1]. Of course the physical mass shell in these models would not be a hyperboloid , since the models have a momentum cut-off in the interaction, but it should be given by the eigenvalues $m(\vec{p})$ of H in the subspace of fixed momentum $\vec{P} = \vec{p}$. The corresponding eigenvectors would be the asymptotic or physical one particle states. By (3.1) we see that the eigenvalues $m(\vec{p})$ of H, if they exist, would correspond to poles at the negative α_i-axis at $m(\vec{p}_i)$ for the truncated spectral density functions. From (2.12) and the fact that $\rho^k(x_1s_1,\ldots,x_ks_k)$ is analytic in λ for $|\lambda| < \lambda_o$ and has a zero of order k at $\lambda = 0$, we find from (3.1) that

$$(\Omega, e^{is\varphi_\epsilon(o)} \frac{\delta(\vec{P}-\vec{\beta}_1)}{H+\alpha_1} F\, e^{is_2\varphi_\epsilon(o)} \cdots \frac{\delta(\vec{P}-\vec{\beta}_{k-1})}{H+\alpha_{k-1}} F\, e^{is_k\varphi_\epsilon(o)} \Omega) \qquad (3.4)$$

is an analytic function in λ for $|\lambda| < \lambda_o$, and Re $\alpha_i > 0$ $i = 1,\ldots,k-1$.

For $\lambda = 0$ we know that (3.4) has poles only at $\alpha_i = -\mu(\vec{\beta}_i)$ and that the cuts along the negative real α-axis actually start only at $-2\mu(\vec{\beta}_i)$, so that the poles are isolated from the cuts for $\lambda = 0$. This together with the analyticity in λ of (3.4) seems promising, but we have not yet been able to prove that the spectral density functions have isolated poles.

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