Hyperfinite product factors, III.

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1. Introduction. In an earlier paper [3] we introduced the concept of product factors and showed that a factor $\mathcal{R}$ is $*$-isomorph to an ITPFI-factor if and only if $\mathcal{R}$ is a hyperfinite product factor. Subsequently we showed [4] that the countably generated semi-finite product factors are all hyperfinite, and in particular ITPFI-factors. In the present paper we shall improve the above results by showing that a factor $\mathcal{R}$ is $*$-isomorph to an ITPFI-factor if and only if $\mathcal{R}$ is a countably generated product factor. This is then a characterization of ITPFI-factors in terms of their normal states.

We say a normal state $\omega$ on a factor $\mathcal{R}$ is asymptotically a product state if given a finite type I subfactor $M$ of $\mathcal{R}$ and $\epsilon > 0$ there is a finite type I subfactor $N$ of $\mathcal{R}$ containing $M$ such that $\|\omega - \omega|N \otimes \omega|N^C\| < \epsilon$, where we identify $\mathcal{R}$ and $N \otimes N^C$, $N^C = N' \cap \mathcal{R}$. $\mathcal{R}$ is said to be a product factor if every normal state on $\mathcal{R}$ is asymptotically a product state.

$\mathcal{R}$ is said to be an ITPFI-factor if $\mathcal{R} = \pi_w(\mathcal{H})$, where $w = \bigotimes_{i=1}^{\infty} w_i$ is a product state on an infinite $C^*$-algebra tensor product $\mathcal{H} = \bigotimes_{i=1}^{\infty} \mathcal{H}_i$, where $\mathcal{H}_i$ is a type $I_{n_i}$-factor, $2 \leq n_i < \infty$, and $\pi_w$ is the cyclic representation of $\mathcal{H}$ defined by $w$.

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2. The cone $\mathcal{P}^\#$. Let $\mathcal{R}$ be a von Neumann algebra with a separating and cyclic vector $\xi_0$ acting on a Hilbert space $\mathcal{H}$. Let $\Delta$ be the modular operator defined by $\xi_0$ relative to $\mathcal{R}$ [5, §7]. In the notation of [5] $\mathcal{D}^\#$ is the domain of $\Delta^{1/2}$, and $\mathcal{P}^\#$ is the set of all vectors $\xi \in \mathcal{D}^\#$ such that $\omega_{\xi, \xi_0}$ defined by $\omega_{\xi, \xi_0}(S) = (S\xi, \xi_0)$ is a positive linear functional on $\mathcal{R}^\prime$. In the proof of [5, Lemma 15.2] it is shown that if $\xi \in \mathcal{P}^\#$ then $\xi = \lim_n H_n \xi_0$ with $H_n \in \mathcal{R}^+$. Conversely, it is clear that $\mathcal{R}^+ \xi_0 \subseteq \mathcal{P}^\#$. We shall in the present section show that if $\{\xi_n\}$ is a sequence in $\mathcal{P}^\#$ such that $\omega_{\xi_n} \to \omega_{\xi}$ in $\mathcal{R}^\ast$ with $\xi \in \mathcal{P}^\#$ then $\xi_n \to \xi$ in $\mathcal{H}$. For this we need a probably well known lemma on the absolute value of an element in $\mathcal{R}^\ast$.

Recall that if $f \in \mathcal{R}^\ast$ then $f$ has a polar decomposition $f = |f| \cdot U$, where $|f|$ is the unique positive normal linear functional $\varphi$ on $\mathcal{R}$ and $U$ the unique partial isometry in $\mathcal{R}$ such that $f = \varphi \cdot U$ and $\varphi = f \cdot U^*$, where $(\varphi \cdot U)(A) = \varphi(UA)$, see [2, Ch. I, § 4, Thm. 4].

Lemma 2.1. Let $\mathcal{R}$ be a von Neumann algebra and $\{f_n\}$ a sequence in $\mathcal{R}^\ast$ converging uniformly to $f$ in $\mathcal{R}^\ast$. Then $\{|f_n|\}$ converges uniformly to $|f|$.

Proof. We may assume $|f| = \omega_{\xi}$, where $\xi$ is a unit vector in the Hilbert space $\mathcal{H}$ upon which $\mathcal{R}$ acts. Let $f_n = |f_n| \cdot U_n$ and $f = |f| \cdot U$ be the polar decompositions of $f_n$ and $f$. 

respectively. Since $f(U^*_n) = |f|(UU^*_n) = (\xi, U_nU^*_n\xi)$, we have

$$|(\xi, U_nU^*_n\xi) - \|\xi\|^2| = |f(U^*_n)| - \|f\| \leq |f(U^*_n)| - \|f\| + \|f_n\| - \|f\| = |f(U^*_n) - f(U^*_n)| + \|f_n\| - \|f\| \leq \|f - f_n\| + \|f_n\| - \|f\|,$$

which converges to 0 as $n \to \infty$. In particular, since

$$\|U_nU^*_n\xi\| \leq 1, \ U_nU^*_n\xi \to \xi$$

in norm. Let $\epsilon > 0$, and choose $n_0$ so large that $\|U_nU^*_n\xi - \xi\| < \epsilon/2$ and $\|f - f_n\| < \epsilon/2$ for $n \geq n_0$. Then if $A \in \mathcal{A}$ we have

$$|\|f\| - \|f_n\|| = |\|f\| - f(U^*_n A)| \leq |\|f\| - f(U^*_n A)| + |f(U^*_n A) - f(U^*_n A)| \leq |(A\xi, \xi) - (A\xi, U_nU^*_n\xi)| + \|f - f_n\| \|A\| \leq \|A\| \|\xi - U_nU^*_n\xi\| + \|f - f_n\| \|A\| < \epsilon \|A\|.$$

Thus $\|\|f\| - \|f_n\|| < \epsilon$ for $n \geq n_0$. The proof is complete.

**Proposition 2.2.** Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and having a separating and cyclic vector $\xi_0$. Let $\xi \in \mathcal{R}^\#$ and $\{\xi_n\}$ be a sequence in $\mathcal{R}^\#$ such that $w_{\xi_n} \to w_{\xi}$ uniformly in $\mathcal{R}^\ast$. Then $\|\xi_n - \xi\| \to 0$ as $n \to \infty$.

**Proof.** By a result of Connes [1, there are vectors $\psi_n \in \mathcal{H}$ such that $w_{\psi_n} = w_{\psi_n}$ on $\mathcal{R}$ and $\psi_n \to \xi$. Let $U_n'$ be a partial isometry in $\mathcal{R}'$ such that $U_n'\xi_n = \psi_n$ (define $U_n'$ by $U_n'A\xi_n = A\psi_n$ and $U_n' = 0$ on $[\mathcal{R}\xi_n]$). Then in particular $w_{\xi_n} \psi_n \to w_{\xi_n} \xi = \xi_n$ in $\mathcal{R}^\ast$. Since $\xi_n \in \mathcal{R}^\#$ we have $w_{\xi_0, \xi} \geq 0$ on $\mathcal{R}'$. Now $|w_{\xi_0, \psi_n}| = w_{\xi_0, \xi_n}$. Indeed, $U_n'^*\psi_n = U_n'^*U_n'\xi_n = \xi_n$. Thus if $A' \in \mathcal{R}'$ we have $(A'\xi_0, \psi_n) = (U_n'^*A'\xi_0, \xi_n) = (w_{\xi_0, \xi_n}U_n'^*)(A')$, and $(w_{\xi_0, \psi_n}U_n')(A') = (U_n'^*A'\xi_0, U_n'\xi_n) = (A'\xi_0, \xi_n)$.
Thus by [2, Ch.I, § 4, Thm.4] the assertion follows. By Lemma 2.1
\( w_{\xi_0}, \xi_n = |w_{\xi_0}, \psi_n| - |w_{\xi_0}, \xi| = w_{\xi_0}, \xi \) uniformly, since \( \xi \in \mathcal{R} \),
so \( w_{\xi_0}, \xi \geq 0 \) on \( \mathcal{R} \).

Since \( w_{\xi_n} \to w_\xi \) we have in particular that \( \|\xi_n\| \to \|\xi\| \),
hence the vectors \( \{\xi_n - \xi\} \) form a uniformly bounded set. Since
the unit ball in \( \mathcal{H} \) is weakly compact there is a subnet \( \{\xi_{n_\alpha} - \xi\} \)
which converges weakly to a vector \( \psi \in \mathcal{H} \). Let \( \eta \in \mathcal{R}', \xi_0; \)
then there is a subsequence \( \{\xi_{n_j} - \xi\} \) of \( \{\xi_{n_\alpha} - \xi\} \) such that
\( (\xi_{n_j} - \xi, \eta) \to (\psi, \eta) \). But \( w_{\xi_0}, \xi_{n_j} \to w_{\xi_0}, \xi \) in \( \mathcal{R} \),
hence \( (\xi_{n_j} - \xi, \eta) \to 0 \).

Thus \( (\psi, \eta) = 0 \). Since \( \mathcal{R}', \xi_0 \) is dense in \( \mathcal{H} \), \( \psi = 0 \).
Therefore \( 0 \) is the only weak limit point of the sequence \( \{\xi_n - \xi\} \),
hence \( \xi_n \to \xi \) weakly. Thus, together with the fact that
\( \|\xi_n\| \to \|\xi\| \), shows that \( \xi_n \to \xi \) is norm. The proof is complete.

3. Product factors.

We prove a slight improvement over [4, Lem.3.1].

Lemma 3.1. Let \( \mathcal{R} \) be a factor acting on a Hilbert space \( \mathcal{H} \).
Then \( \mathcal{R} \) is hyperfinite if and only if \( \mathcal{R} \) is countably generated
and given a finite type I subfactor \( M \) of \( \mathcal{R} \), \( T \in \mathcal{R}^+ \), \( \epsilon > 0 \),
and \( \xi_1, \ldots, \xi_r \in \mathcal{H} \), there is a finite type I factor \( N \) with
\( M \subset N \subset \mathcal{R} \), and \( S \in \mathcal{N}^+ \) such that \( \|(S-T)\xi_j\| < \epsilon \) for \( j = 1, \ldots, r \).

Proof. The only difference between this lemma and [4, Lem.3.1]
is that in [4] we require \( \|S\| \leq \|T\| \). Thus in order to show the
lemma it suffices by [4] to reduce it to the case when \( S \) can
be chosen with \( \|S\| \leq \|T\| \). Let \( M, T, \epsilon, \xi_1, \ldots, \xi_r \) be as in the
lemma. We may assume \( 0 \leq T \leq I \). We employ the argument used
in [2] to prove the Kaplansky Density Theorem. Choose \( A \in \mathcal{R} \)
with $0 \leq A \leq I$ such that $T = 2A(I + A^2)^{-1}$. By assumption we can find a finite type I factor $N$ with $M \subseteq N \subseteq \mathcal{R}$ and $B \in N^+$ such that $\|(B-A)(I+A^2)^{-1}\xi_j\| < \varepsilon/4$ and $\|(B-A)T\xi_j\| < \varepsilon$. Let $S = 2B(I+B^2)^{-1}$. Then $S \in N^+$ and $0 \leq S \leq I$. From the identity $S - T = 2(I+B^2)^{-1}(B-A)(I+A^2)^{-1} + \frac{1}{2}S(A-B)T$

we have

$$\|(S-T)\xi_j\| \leq 2\|(I+B^2)^{-1}\| |B-A)(I+A^2)^{-1}\xi_j\| + \frac{1}{2}\|S\| \|(A-B)T\xi_j\|$$

$$< 2 \varepsilon/4 + \frac{1}{2} \varepsilon = \varepsilon .$$

The proof is complete.

**Lemma 3.2.** Let $\mathcal{R}$ be a product factor. Let $\rho$ and $\omega$ be normal states of $\mathcal{R}$. Let $M$ be a finite type I subfactor of $\mathcal{R}$ and let $\varepsilon > 0$. Then there exist two finite type I subfactors $N$ and $P$ of $\mathcal{R}$ both containing $M$ such that

i) $\|(\rho - \omega)|N^0\| < \varepsilon$

ii) $\|\rho - \rho|P \otimes \rho|P^0\| < \varepsilon$, $\|\omega - \omega|P \otimes \omega|P^0\| < \varepsilon$.

**Proof.** We first prove i). If $(\rho - \omega)|M^0 = 0$ the assertion is trivial. Otherwise choose a self-adjoint operator $A \in M^0$ such that $\rho(A) \neq \omega(A)$. An easy approximation argument shows that we may assume $A$ belongs to a finite type I subfactor $N_0$ of $M^0$. Let $N_1 = M \otimes N_0$. Then $N_1$ is a finite type I subfactor of $\mathcal{R}$ containing $M$, and there is a self-adjoint operator $A$ in $N_1$ of norm 1 such that $c = |\rho(A) - \omega(A)| \neq 0$. Since $\mathcal{R}$ is a product factor there is a finite type I subfactor $N_2$ of $\mathcal{R}$ containing $N_1$ such that $\|\rho - \rho|N_2 \otimes \rho|N_2^0\| < \frac{\varepsilon}{6} c$. Then choose a finite type I subfactor $N_3$ of $\mathcal{R}$ containing $N_2$ such that $\|\omega - \omega|N_2 \otimes \omega|N_3^0\| < \frac{\varepsilon}{6} c$. Since $\frac{1}{2}(\rho + \omega)$ is a normal state there is a finite type I subfactor $N = N_4$ of $\mathcal{R}$ containing $N_3$ such
that
\[ \| \frac{1}{2}(p + w) - \frac{1}{2}(p + w) \| \| N \otimes \frac{1}{2}(p + w) \| N^c \| < \frac{e}{12} c . \]

Let \( B \in N^c \). Then we have the identity
\[
\begin{align*}
\frac{1}{2}(p + w)(A) \cdot \frac{1}{2}(p + w)(B) - \frac{1}{2}(p + w)(AB) &= \\
\frac{1}{2}(p(A)p(B) - p(AB)) + \frac{1}{2}(w(A)w(B) - w(AB)) + \frac{1}{4}(p(A) - w(A))(p(B) - w(B)).
\end{align*}
\]

Since \( A \in N_1 \) which is contained in \( N_j \), \( j = 2, 3, 4 \), and \( B \in N^c \) which is contained in \( N_j^c \), \( j = 1, 2, 3 \), we have when \( \| B \| \leq 1 \),
\[
\frac{C}{4} |p(B) - w(B)| < \frac{e}{12} c + \frac{1}{2} \frac{e}{6} c + \frac{1}{2} \frac{e}{6} c = \frac{e}{4} c .
\]

Thus \( \| (p - w) \| N^c \| < \varepsilon \), and i) is proved.

We next show ii). By i) there is a finite type I subfactor \( P_1 \) of \( \mathcal{R} \) containing \( M \) such that \( \| (w - p) \| P_1^C \| < \varepsilon/6 \). Since \( \mathcal{R} \) is a product factor there are finite type I factors \( P_2 \) and \( P_3 \) such that \( P_1 \subset P_2 \subset P_3 \subset \mathcal{R} \) and \( \| w - w \| P_2 \otimes \mathcal{W} \| P_2^C \| < \varepsilon/6 \) and \( \| p - p \| P_3 \otimes \mathcal{W} \| P_3^C \| < \varepsilon/6 \). Let \( P = P_3 \). Then the three inequalities above imply
\[
\begin{align*}
\| w - w \| P \otimes \mathcal{W} \| P^C \| &< \| w - w \| P \otimes \mathcal{W} \| P^C \| + \varepsilon/6 \\
&< \| w - w \| P \otimes \mathcal{W} \| P_2^C \otimes P \otimes \mathcal{W} \| P^C \| + 2 \varepsilon/6 \\
&< \| w - w \| P_2 \otimes \mathcal{W} \| P_2^C \otimes P \otimes \mathcal{W} \| P^C \| + 3 \varepsilon/6 \\
&< \| w - w \| P_2 \otimes \mathcal{W} \| P_2^C \| + 4 \varepsilon/6 \\
&< \| w - w \| P_2 \otimes \mathcal{W} \| P_2^C \| + 5 \varepsilon/6 \\
&< 6 \varepsilon/6 = \varepsilon .
\end{align*}
\]

The proof is complete.
**Theorem 3.3.** Let $\mathcal{R}$ be a factor which is not finite of type I. Then $\mathcal{R}$ is *-isomorphic to an ITPFI-factor if and only if $\mathcal{R}$ is a countably generated product factor.

**Proof.** The necessity follows from [3]. In order to show the converse it suffices by [3] to show that a countably generated product factor is hyperfinite. In order to do this it suffices by Lemma 3.1 to prove that if $M$ is a finite type I subfactor of $\mathcal{R} H \in \mathcal{R}^+$, $\epsilon > 0$, and $\xi_1, \ldots, \xi_r \in \mathcal{H}$, the underlying Hilbert space, then there is a finite type I subfactor $N$ of $\mathcal{R}$ containing $M$ and $S \in N^+$ such that $\|(S-H)\xi_j\| < \epsilon$.

Since $\mathcal{R}$ is countably generated we may assume $\mathcal{R}$ has a separating and cyclic vector $\varphi_0$ and that $\|H\varphi_0\| = 1$. By [5, Thm.10.1] there is a Tomita algebra (called modular Hilbert algebra in [5]) $\mathcal{R}_0$ which is strongly dense in $\mathcal{R}$. (more correctly we should consider $(\mathcal{R}_0 \varphi_0)$). Thus there is $K \in \mathcal{R}_0^+$ such that $\|(K-H)\xi_j\| < \epsilon/2$ for $j = 0, 1, \ldots, r$ and $\|K\varphi_0\| = 1$. By [5, §3] it follows that $\mathcal{R}_0 \varphi_0 \subset \mathcal{R} \varphi_0$. Hence there is $K' \in \mathcal{R}'$ such that $K'\varphi_0 = K\varphi_0$. Let $k = \|K'\|^2$. Then $w_{K\varphi_0} = w_{K'\varphi_0} \leq kw_{\varphi_0}$ on $\mathcal{R}$.

We assert that given $\delta > 0$ there is a finite type I subfactor $N$ of $\mathcal{R}$ containing $M$ and $T \in N^+$ such that $\|T\| \leq k$ and $\|w_{K\varphi_0} - w_{T\varphi_0}\| < \delta$. Indeed, by Lemma 3.2 there is a finite type I subfactor $N_1$ of $\mathcal{R}$ containing $M$ such that

1) $\|(w_{K\varphi_0} - w_{\varphi_0})|N_1^C\| < \delta/4$ .

Again by Lemma 3.2 there is a finite type I subfactor $N$ of $\mathcal{R}$ containing $N_1$ such that

2) $\|w_{K\varphi_0} - w_{K\varphi_0}|N \otimes w_{K\varphi_0}|N^C\| < \delta/4$.
3) \[ \|w_{\xi_0} - w_{\xi_0} | N \otimes w_{\xi_0} | N^C \| < \delta/2k^2. \]

Since in particular \( w_{K_{\xi_0}} | N \leq k w_{\xi_0} | N \) it follows from Sakai's Radon Nikodym Theorem [2, Ch.I, § 4. Thm.5] applied to \( N \) that there is \( T \in N^+ \) such that \( w_{K_{\xi_0}} | N = w_{T_{\xi_0}} | N \), and \( \|T\| \leq k \).

By 1) and 2) we have

\[ \|w_{K_{\xi_0}} - w_{T_{\xi_0}} | N \otimes w_{\xi_0} | N^C \| = \|w_{K_{\xi_0}} - w_{K_{\xi_0}} | N \otimes w_{\xi_0} | N^C \| \]

4) \[ < \|w_{K_{\xi_0}} - w_{K_{\xi_0}} | N \otimes w_{\xi_0} | N^C \| + \delta/4 \]

\[ < \delta/4 + \delta/4 = \delta/2. \]

Let \( S \in \mathcal{R} = N \otimes N^C \). Then by 3), identifying \( T \) and \( T \otimes I \),

\[ |(w_{T_{\xi_0}} | N \otimes w_{\xi_0} | N^C)(S) - w_{T_{\xi_0}}(S)| \]

\[ = |(w_{\xi_0} | N \otimes w_{\xi_0} | N^C)(T \otimes I)S(T \otimes I)) - w_{\xi_0}((T \otimes I)S(T \otimes I))| \]

\[ < (\delta/2k^2) \| (T \otimes I)S(T \otimes I) \| \]

\[ \leq (\delta/2k^2) \| T \| \| S \| \leq (\delta/2) \| S \|. \]

Thus \( \|w_{T_{\xi_0}} | N \otimes w_{\xi_0} | N^C - w_{T_{\xi_0}} \| < \delta/2 \). Hence by 4)

\[ \|w_{K_{\xi_0}} - w_{T_{\xi_0}} \| \leq \|w_{K_{\xi_0}} - w_{T_{\xi_0}} | N \otimes w_{\xi_0} | N^C \| + \|w_{T_{\xi_0}} | N \otimes w_{\xi_0} | N^C - w_{T_{\xi_0}} \| \]

\[ < \delta/2 + \delta/2 = \delta, \]

and our assertion is proved.

We can therefore find a sequence \( \{N_j\} \) of finite type I subfactors of \( \mathcal{R} \) and \( T_j \in N_j^+ \) with \( \|T_j\| \leq k \) such that \( \|w_{K_{\xi_0}} - w_{T_j_{\xi_0}} \| \to 0 \). Since \( K \) and \( T_j \) are all positive, it follows from Proposition 2.2 that \( \|T_j_{\xi_0} - K_{\xi_0}\| \to 0 \). Since the \( T_j \) are uniformly bounded and \( \xi_0 \) is separating for \( \mathcal{R}, T_j \to K \) strongly [2, Ch.I, § 4, Prop.4]. We can thus find \( n \) such that

\[ \|(T_n - K)_{\xi_j}\| < \varepsilon/2 \text{ for } j = 1, \ldots, r. \]

Let \( N = N_n \) and \( S = T_n \).
Then \( \| (S - H) \xi_j \| \leq \| (S - K) \xi_j \| + \| (K - H) \xi_j \| < \epsilon/2 + \epsilon/2 = \epsilon \).

This completes the proof of the theorem.

**Remark.** It was shown in [4] that if \( \mathcal{R} \) is a countably generated product factor acting on a Hilbert space \( \mathcal{H} \) then \( E \mathcal{R} E \) is a countably generated product factor for each non-zero projection \( E \) in \( \mathcal{R} \), and if \( \mathcal{H} \) is separable then \( \mathcal{R}' \) is a countably generated product factor. Since by the above theorem it is immediate that the tensor product of two countably generated product factors is itself a countably generated product factor, it follows that on separable Hilbert spaces the product factors are closed under the so-called elementary operations.

**References.**


