# Hyperfinite product factors, III.

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1. Introduction. In an earlier paper [3] we introduced the concept of product factors and showed that a factor $\mathcal{R}$ is $*$-isomorphic to an ITPFI-factor if and only if $\mathcal{R}$ is a hyperfinite product factor. Subsequently we showed [4] that the countably generated semi-finite product factors are all hyperfinite, and in particular ITPFI-factors. In the present paper we shall improve the above results by showing that a factor $\mathcal{R}$ is $*$-isomorphic to an ITPFI-factor if and only if $\mathcal{R}$ is a countably generated product factor. This is then a characterization of ITPFI-factors in terms of their normal states.

We say a normal state $\omega$ on a factor $\mathcal{R}$ is asymptotically a product state if given a finite type I subfactor $M$ of $\mathcal{R}$ and $\epsilon > 0$ there is a finite type I subfactor $N$ of $\mathcal{R}$ containing $M$ such that $\|\omega - \omega|N \otimes \omega|N^c\| < \epsilon$ , where we identify $\mathcal{R}$ and $N \otimes N^c$, $N^c = N' \cap \mathcal{R}$. $\mathcal{R}$ is said to be a product factor if every normal state on $\mathcal{R}$ is asymptotically a product state. $\mathcal{R}$ is said to be an ITPFI-factor if $\mathcal{R} = \Pi_\omega(\mathfrak{A})''$ where $\omega = \bigotimes_{i=1}^{\infty} \omega_i$ is a product state on an infinite $C^*$-algebra tensor product $\mathfrak{A} = \bigotimes_{i=1}^{\infty} \mathfrak{A}_i$ , where $\mathfrak{A}_i$ is a type $I_{n_i}$-factor, $2 \leq n_i < \infty$ , and $\Pi_\omega$ is the cyclic representation of $\mathfrak{A}$ defined by $\omega$.

(*) This work is partially supported by an N.S.F. grant.

The author is indebted to M. Takesaki for a remark which initiated this paper, and to all his colleagues at University of Pennsylvania, where this work was done, for their warm hospitality while the author visited Philadelphia.

2. The cone $\mathcal{P}^{\#}$. Let $\mathcal{R}$ be a von Neumann algebra with a separating and cyclic vector $\xi_o$ acting on a Hilbert space $\mathcal{H}$. Let $\Delta$ be the modular operator defined by $\xi_o$ relative to $\mathcal{R}$ [5, §7]. In the notation of [5] $\mathcal{D}^{\#}$ is the domain of $\Delta^{\frac{1}{2}}$, and $\mathcal{P}^{\#}$ is the set of all vectors $\xi \in \mathcal{D}^{\#}$ such that $\omega_{\xi,\xi_o}$ defined by $\omega_{\xi,\xi_o}(S) = (S\xi,\xi_o)$ is a positive linear functional on $\mathcal{R}'$. In the proof of [5, Lemma 15.2] it is shown that if $\xi \in \mathcal{P}^{\#}$ then $\xi = \lim_n H_n\xi_o$ with $H_n \in \mathcal{R}^+$. Conversely, it is clear that $\mathcal{R}^+\xi_o \subset \mathcal{P}^{\#}$. We shall in the present section show that if $\{\xi_n\}$ is a sequence in $\mathcal{P}^{\#}$ such that $\omega_{\xi_n} \to \omega_\xi$ in $\mathcal{R}_*$ with $\xi \in \mathcal{P}^{\#}$ then $\xi_n \to \xi$ in $\mathcal{H}$. For this we need a probably well known lemma on the absolute value of an element in $\mathcal{R}_*$. Recall that if $f \in \mathcal{R}_*$ then $f$ has a polar decomposition $f = |f|\cdot U$, where $|f|$ is the unique positive normal linear functional $\varphi$ on $\mathcal{R}$ and $U$ the unique partial isometry in $\mathcal{R}$ such that $f = \varphi\cdot U$ and $\varphi = f\cdot U^*$, where $(\varphi\cdot U)(A) = \varphi(UA)$, see [2. Ch.I, § 4, Thm.4].

Lemma 2.1. Let $\mathcal{R}$ be a von Neumann algebra and $\{f_n\}$ a sequence in $\mathcal{R}_*$ converging uniformly to $f$ in $\mathcal{R}_*$. Then $\{|f_n|\}$ converges uniformly to $|f|$.

Proof. We may assume $|f| = \omega_\xi$, where $\xi$ is a unit vector in the Hilbert space $\mathcal{H}$ upon which $\mathcal{R}$ acts. Let $f_n = |f_n|\cdot U_n$ and $f = |f|\cdot U$ be the polar decompositions of $f_n$ and $f$

respectively. Since $f(U_n^*) = |f|(UU_n^*) = (\xi, U_nU^*\xi)$, we have

$$\begin{aligned} |(\xi, U_nU^*\xi) - \|\xi\|^2| &= |f(U_n^*) - \|f\|| \\ &\leq |f(U_n^*) - \|f_n\|| + |\|f_n\| - \|f\|| \\ &= |f(U_n^*) - f_n(U_n^*)| + |\|f_n\| - \|f\|| \\ &\leq \|f - f_n\| + |\|f_n\| - \|f\||, \end{aligned}$$

which converges to $0$ as $n \to \infty$. In particular, since $\|U_nU^*\xi\| \leq 1$, $U_nU^*\xi \to \xi$ in norm. Let $\epsilon > 0$, and choose $n_o$ so large that $\|U_nU^*\xi - \xi\| < \epsilon/2$ and $\|f - f_n\| < \epsilon/2$ for $n \geq n_o$. Then if $A \in \mathcal{R}$ we have

$$\begin{aligned} |\,|f|(A) - |f_n|(A)| &= |\,|f|(A) - f_n(U_n^*A)| \\ &\leq |\,|f|(A) - f(U_n^*A)| + |f(U_n^*A) - f_n(U_n^*A)| \\ &\leq |(A\xi,\xi) - (A\xi, U_nU^*\xi)| + \|f - f_n\|\,\|A\| \\ &\leq \|A\|\,\|\xi - U_nU^*\xi\| + \|f - f_n\|\,\|A\| \\ &< \epsilon\,\|A\|. \end{aligned}$$

Thus $\|\,|f| - |f_n|\,\| < \epsilon$ for $n \geq n_o$. The proof is complete.

<u>Proposition 2.2</u>. Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and having a separating and cyclic vector $\xi_o$. Let $\xi \in \mathcal{P}^\#$ and $\{\xi_n\}$ be a sequence in $\mathcal{P}^\#$ such that $\omega_{\xi_n} \to \omega_\xi$ uniformly in $\mathcal{R}_*$. Then $\|\xi_n - \xi\| \to 0$ as $n \to \infty$.

<u>Proof</u>. By a result of Connes [1, Lem.3.5.5] there are vectors $\psi_n \in \mathcal{H}$ such that $\omega_{\psi_n} = \omega_{\xi_n}$ on $\mathcal{R}$ and $\psi_n \to \xi$. Let $U_n'$ be a partial isometry in $\mathcal{R}'$ such that $U_n'\xi_n = \psi_n$ (define $U_n'$ by $U_n'A\xi_n = A\psi_n$ and $U_n' = 0$ on $[\mathcal{R}\xi_n]^\perp$). Then in particular $\omega_{\xi_o,\psi_n} \to \omega_{\xi_o,\xi}$ in $\mathcal{R}_*'$. Since $\xi_n \in \mathcal{P}^\#$ we have $\omega_{\xi_o,\xi_n} \geq 0$ on $\mathcal{R}'$. Now $|\omega_{\xi_o,\psi_n}| = \omega_{\xi_o,\xi_n}$. Indeed, $U_n'^*\psi_n = U_n'^*U_n'\xi_n = \xi_n$. Thus if $A' \in \mathcal{R}'$ we have $(A'\xi_o,\psi_n) = (U_n'^*A'\xi_o,\xi_n) = (\omega_{\xi_o,\xi_n}\cdot U_n'^*)(A')$, and $(\omega_{\xi_o,\psi_n}\cdot U_n')(A') = (U_n'A'\xi_o,U_n'\xi_n) = (A'\xi_o,\xi_n)$.

Thus by [2, Ch.I, § 4, Thm.4] the assertion follows. By Lemma 2.1 $\omega_{\xi_o,\xi_n} = |\omega_{\xi_o,\psi_n}| \to |\omega_{\xi_o,\xi}| = \omega_{\xi_o,\xi}$ uniformly, since $\xi \in \mathcal{P}^{\#}$, so $\omega_{\xi_o,\xi} \geq 0$ on $\mathcal{R}'$.

Since $\omega_{\xi_n} \to \omega_\xi$ we have in particular that $\|\xi_n\| \to \|\xi\|$, hence the vectors $\{\xi_n - \xi\}$ form a uniformly bounded set. Since the unit ball in $\mathcal{H}$ is weakly compact there is a subnet $\{\xi_{n_\alpha} - \xi\}$ which converges weakly to a vector $\psi \in \mathcal{H}$. Let $\eta \in \mathcal{R}'\xi_o$; then there is a subsequence $\{\xi_{n_j} - \xi\}$ of $\{\xi_{n_\alpha} - \xi\}$ such that $(\xi_{n_j} - \xi, \eta) \to (\psi, \eta)$. But $\omega_{\xi_o,\xi_{n_j}} \to \omega_{\xi_o,\xi}$ in $\mathcal{R}_*'$, hence $(\xi_{n_j} - \xi, \eta) \to 0$. Thus $(\psi, \eta) = 0$. Since $\mathcal{R}'\xi_o$ is dense in $\mathcal{H}$, $\psi = 0$. Therefore $0$ is the only weak limit point of the sequence $\{\xi_n - \xi\}$, hence $\xi_n \to \xi$ weakly. This, together with the fact that $\|\xi_n\| \to \|\xi\|$, shows that $\xi_n \to \xi$ is norm. The proof is complete.

## 3. Product factors.

We prove a slight improvement over [4, Lem.3.1].

Lemma 3.1. Let $\mathcal{R}$ be a factor acting on a Hilbert space $\mathcal{H}$. Then $\mathcal{R}$ is hyperfinite if and only if $\mathcal{R}$ is countably generated and given a finite type I subfactor $M$ of $\mathcal{R}$, $T \in \mathcal{R}^+$, $\epsilon > 0$, and $\xi_1, \dots, \xi_r \in \mathcal{H}$, there is a finite type I factor $N$ with $M \subset N \subset \mathcal{R}$, and $S \in N^+$ such that $\|(S-T)\xi_j\| < \epsilon$ for $j = 1, \dots, r$.

Proof. The only difference between this lemma and [4, Lem.3.1] is that in [4] we require $\|S\| \leq \|T\|$. Thus in order to show the lemma it suffices by [4] to reduce it to the case when $S$ can be chosen with $\|S\| \leq \|T\|$. Let $M, T, \epsilon, \xi_1, \dots, \xi_r$ be as in the lemma. We may assume $0 \leq T \leq I$. We employ the argument used in [2] to prove the Kaplansky Density Theorem. Choose $A \in \mathcal{R}$

with $0 \leq A \leq I$ such that $T = 2A(I+A^2)^{-1}$. By assumption we can find a finite type I factor $N$ with $M \subset N \subset \mathcal{R}$ and $B \in N^+$ such that $\|(B-A)(I+A^2)^{-1}\xi_j\| < \epsilon/4$ and $\|(B-A)T\xi_j\| < \epsilon$. Let $S = 2B(I+B^2)^{-1}$. Then $S \in N^+$ and $0 \leq S \leq I$. From the identity

$$S - T = 2(I+B^2)^{-1}(B-A)(I+A^2)^{-1} + \tfrac{1}{2}S(A-B)T$$

we have

$$\begin{aligned}\|(S-T)\xi_j\| &\leq 2\|(I+B^2)^{-1}\| \ \|B-A)(I+A^2)^{-1}\xi_j\| + \tfrac{1}{2}\|S\| \ \|(A-B)T\xi_j\| \\ &< 2\ \epsilon/4 + \tfrac{1}{2}\epsilon = \epsilon .\end{aligned}$$

The proof is complete.

Lemma 3.2. Let $\mathcal{R}$ be a product factor. Let $\rho$ and $\omega$ be normal states of $\mathcal{R}$. Let $M$ be a finite type I subfactor of $\mathcal{R}$ and let $\epsilon > 0$. Then there exist two finite type I subfactors $N$ and $P$ of $\mathcal{R}$ both containing $M$ such that

i) $$\|(\rho-\omega)|N^c\| < \epsilon$$

ii) $\|\rho-\rho|P\otimes\rho|P^c\| < \epsilon$ , $\|\omega-\omega|P\otimes\omega|P^c\| < \epsilon$ .

Proof. We first prove i). If $(\rho-\omega)|M^c = 0$ the assertion is trivial. Otherwise choose a self-adjoint operator $A \in M^c$ such that $\rho(A) \neq \omega(A)$. An easy approximation argument shows that we may assume $A$ belongs to a finite type I subfactor $N_0$ of $M^c$. Let $N_1 = M \otimes N_0$. Then $N_1$ is a finite type I subfactor of $\mathcal{R}$ containing $M$, and there is a self-adjoint operator $A$ in $N_1$ of norm 1 such that $c = |\rho(A) - \omega(A)| \neq 0$. Since $\mathcal{R}$ is a product factor there is a finite type I subfactor $N_2$ of $\mathcal{R}$ containing $N_1$ such that $\|\rho - \rho|N_2 \otimes \rho|N_2{}^c\| < \frac{\epsilon}{6}c$. Then choose a finite type I subfactor $N_3$ of $\mathcal{R}$ containing $N_2$ such that $\|\omega-\omega|N_3\otimes\omega|N_3{}^c\| < \frac{\epsilon}{6}c$. Since $\frac{1}{2}(\rho+\omega)$ is a normal state there is a finite type I subfactor $N = N_4$ of $\mathcal{R}$ containing $N_3$ such that

$$\|\tfrac{1}{2}(\rho+\omega) - \tfrac{1}{2}(\rho+\omega)|N \otimes \tfrac{1}{2}(\rho+\omega)|N^c\| < \frac{\varepsilon}{12}c .$$

Let $B \in N^c$ . Then we have the identity

$$\tfrac{1}{2}(\rho+\omega)(A)\cdot\tfrac{1}{2}(\rho+\omega)(B) - \tfrac{1}{2}(\rho+\omega)(AB) =$$
$$\tfrac{1}{2}(\rho(A)\rho(B) - \rho(AB)) + \tfrac{1}{2}(\omega(A)\omega(B) - \omega(AB))$$
$$- \frac{1}{4}(\rho(A) - \omega(A))(\rho(B) - \omega(B)).$$

Since $A \in N_1$ which is contained in $N_j$ , $j = 2,3,4,$ and $B \in N^c$ which is contained in $N_j^{\ c}$ , $j = 1,2,3,$ we have when $\|B\| \leq 1$ ,

$$\frac{c}{4}|\rho(B) - \omega(B)| < \frac{\varepsilon}{12}c + \tfrac{1}{2}\frac{\varepsilon}{6}c + \tfrac{1}{2}\frac{\varepsilon}{6}c = \frac{\varepsilon}{4}c .$$

Thus $\|(\rho-\omega)|N^c\| < \varepsilon$ , and i) is proved.

We next show ii). By i) there is a finite type I subfactor $P_1$ of $\mathcal{R}$ containing M such that $\|(\omega-\rho)|P_1^{\ c}\| < \varepsilon/6$ . Since $\mathcal{R}$ is a product factor there are finite type I factors $P_2$ and $P_3$ such that $P_1 \subset P_2 \subset P_3 \subset \mathcal{R}$ and $\|\omega - \omega|P_2 \otimes \omega|P_2^{\ c}\| < \varepsilon/6$ and $\|\rho - \rho|P_3 \otimes \rho|P_3^{\ c}\| < \varepsilon/6$ . Let $P = P_3$ . Then the three inqualities above imply

$$\begin{aligned}
&\|\omega - \omega|P \otimes \omega|P^c\| \\
&< \|\omega - \omega|P \otimes \rho|P^c\| + \varepsilon/6 \\
&< \|\omega - \omega|P_2 \otimes \omega|P_2^{\ c} \cap P \otimes \rho|P^c\| + 2\ \varepsilon/6 \\
&< \|\omega - \omega|P_2 \otimes \rho|P_2^{\ c} \cap P \otimes \rho|P^c\| + 3\ \varepsilon/6 \\
&< \|\omega - \omega|P_2 \otimes \rho|P_2^{\ c}\| + 4\ \varepsilon/6 \\
&< \|\omega - \omega|P_2 \otimes \omega|P_2^{\ c}\| + 5\ \varepsilon/6 \\
&< 6\ \varepsilon/6 = \varepsilon .
\end{aligned}$$

The proof is complete.

<u>Theorem 3.3</u>. Let $\mathcal{R}$ be a factor which is not finite of type I. Then $\mathcal{R}$ is *-isomorphic to an ITPFI-factor if and only if $\mathcal{R}$ is a countably generated product factor.

<u>Proof</u>. The necessity follows from [3]. In order to show the converse it suffices by [3] to show that a countably generated product factor is hyperfinite. In order to do this it suffices by Lemma 3.1 to prove that if $M$ is a finite type I subfactor of $\mathcal{R}$ $H \in \mathcal{R}^+$, $\epsilon > 0$, and $\xi_1, \dots, \xi_r \in \mathcal{H}$, the underlying Hilbert space, then there is a finite type I subfactor $N$ of $\mathcal{R}$ containing $M$ and $S \in N^+$ such that $\|(S-H)\xi_j\| < \epsilon$.

Since $\mathcal{R}$ is countably generated we may assume $\mathcal{R}$ has a separating and cyclic vector $\xi_o$ and that $\|H\xi_o\| = 1$. By [5, Thm.10.1] there is a Tomita algebra (called modular Hilbert algebra in [5]) $\mathcal{R}_o$ which is strongly dense in $\mathcal{R}$. (more correctly we should consider $\mathcal{R}_o\xi_o$). Thus there is $K \in \mathcal{R}_o^+$ such that $\|(K-H)\xi_j\| < \epsilon/2$ for $j = 0,1,\dots,r$ and $\|K\xi_o\| = 1$. By [5, §3] it follows that $\mathcal{R}_o\xi_o \subset \mathcal{R}'\xi_o$. Hence there is $K' \in \mathcal{R}'$ such that $K'\xi_o = K\xi_o$. Let $k = \|K'\|^2$. Then $\omega_{K\xi_o} = \omega_{K'\xi_o} \leq k\omega_{\xi_o}$ on $\mathcal{R}$.

We assert that given $\delta > 0$ there is a finite type I subfactor $N$ of $\mathcal{R}$ containing $M$ and $T \in N^+$ such that $\|T\| \leq k$ and $\|\omega_{K\xi_o} - \omega_{T\xi_o}\| < \delta$. Indeed, by Lemma 3.2 there is a finite type I subfactor $N_1$ of $\mathcal{R}$ containing $M$ such that

1) $\quad \|(\omega_{K\xi_o} - \omega_{\xi_o})|N_1^c\| < \delta/4$.

Again by Lemma 3.2 there is a finite type I subfactor $N$ of $\mathcal{R}$ containing $N_1$ such that

2) $\quad \|\omega_{K\xi_o} - \omega_{K\xi_o}|N \otimes \omega_{K\xi_o}|N^c\| < \delta/4$

and

3) $\qquad \|\omega_{\xi_o} - \omega_{\xi_o}|N \otimes \omega_{\xi_o}|N^c\| < \delta/2k^2$ .

Since in particular $\omega_{K\xi_o}|N \leq k\omega_{\xi_o}|N$ it follows from Sakai's Radon Nikodym Theorem [2, Ch.I, § 4. Thm.5] applied to $N$ that there is $T \in N^+$ such that $\omega_{K\xi_o}|N = \omega_{T\xi_o}|N$ , and $\|T\| \leq k$ . By 1) and 2) we have

$$\|\omega_{K\xi_o} - \omega_{T\xi_o}|N \otimes \omega_{\xi_o}|N^c\| = \|\omega_{K\xi_o} - \omega_{K\xi_o}|N \otimes \omega_{\xi_o}|N^c\|$$

4) $$< \|\omega_{K\xi_o} - \omega_{K\xi_o}|N \otimes \omega_{K\xi_o}|N^c\| + \delta/4$$

$$< \delta/4 + \delta/4 = \delta/2 .$$

Let $S \in \mathcal{R} = N \otimes N^c$ . Then by 3), identifying $T$ and $T \otimes I$ ,

$$|(\omega_{T\xi_o}|N \otimes \omega_{\xi_o}|N^c)(S) - \omega_{T\xi_o}(S)|$$

$$= |(\omega_{\xi_o}|N \otimes \omega_{\xi_o}|N^c)((T\otimes I)S(T\otimes I)) - \omega_{\xi_o}((T\otimes I)S(T\otimes I))|$$

$$< (\delta/2k^2)\ \|(T\otimes I)S(T\otimes I)\|$$

$$\leq (\delta/2k^2)\ \|T\|^2\|S\| \leq (\delta/2)\ \|S\| .$$

Thus $\|\omega_{T\xi_o}|N \otimes \omega_{\xi_o}|N^c - \omega_{T\xi_o}\| < \delta/2$ . Hence by 4)

$$\|\omega_{K\xi_o} - \omega_{T\xi_o}\| \leq \|\omega_{K\xi_o} - \omega_{T\xi_o}|N \otimes \omega_{\xi_o}|N^c\| + \|\omega_{T\xi_o}|N \otimes \omega_{\xi_o}|N^c - \omega_{T\xi_o}\|$$

$$< \delta/2 + \delta/2 = \delta ,$$

and our assertion is proved.

We can therefore find a sequence $\{N_j\}$ of finite type I subfactors of $\mathcal{R}$ and $T_j \in N_j^+$ with $\|T_j\| \leq k$ such that $\|\omega_{K\xi_o} - \omega_{T_j\xi_o}\| \to 0$. Since $K$ and $T_j$ are all positive, it follows from Proposition 2.2 that $\|T_j\xi_o - K\xi_o\| \to 0$ . Since the $T_j$ are uniformly bounded and $\xi_o$ is separating for $\mathcal{R}$, $T_j \to K$ strongly [2, Ch.I. § 4, Prop.4]. We can thus find $n$ such that $\|(T_n - K)\xi_j\| < \epsilon/2$ for $j = 1,\dots,r$. Let $N = N_n$ and $S = T_n$ .

Then $\|(S-H)\xi_j\| \leq \|(S-K)\xi_j\| + \|(K-H)\xi_j\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ . This completes the proof of the theorem.

Remark. It was shown in [4] that if $\mathcal{R}$ is a countably generated product factor acting on a Hilbert space $\mathcal{H}$ then $E\mathcal{R}E$ is a countably generated product factor for each non-zero projection $E$ in $\mathcal{R}$ , and if $\mathcal{H}$ is separable then $\mathcal{R}'$ is a countably generated product factor. Since by the above theorem it is immediate that the tensor product of two countably generated product factors is itself a countably generated product factor, it follows that on separable Hilbert spaces the product factors are closed under the so-called elementary operations.

## References.

1. A. Connes, Thèse, To appear.

2. J. Dixmier, Les algèbres d'opérateurs dans l'espace Hilbertien, Gauthier-Villars Paris, 1969.

3. E. Størmer, Hyperfinite product factors, Arkiv Math., 9(1971), 165-170.

4. E. Størmer, Hyperfinite product factors II, J. Functional Analysis, 10(1972), 471-481.

5. M. Takesaki, Tomita's theory of modular Hilbert algebras and its applications, Lecture notes in mathematics, 128, Springer-Verlag, 1970.