Introduction.

In this paper we shall investigate the Poincaré series of a finitely generated module $M$ over a local (noetherian) ring $R$, that is the power series

$$P_M^R(t) := \sum_{\ell \geq 0} \dim_k \text{Tor}_p^R(k,M)t^\ell$$

$k$ being the residue field of $R$. $P_k^R(t)$ will be called the Poincaré series of the ring $R$. Although not much is known about $P_M^R(t)$ in general, there is evidence to believe that this power series always represents a rational function. This is of course the case if $R$ is a regular local ring, in which case $P_M^R(t)$ is a polynomial of degree $\text{rot}$ exceeding the global dimension of $R$. Recently it has been shown that $P_M^R(t)$ is a rational function if $R$ is a local complete intersection, Gulliksen [3].

In the present paper we shall establish the rationality of $P_M^R(t)$ also in the case where $R$ is a Golod ring (see section 2 for the definition). It is known that if $R$ is a factor of a regular ring by two relations, or if $R$ has imbedding dimension less than or equal to two, then $R$ is either a complete intersection or a Golod ring. Hence the rationality of $P_M^R(t)$ is established in those cases. This generalizes results of Shamash [5] and Scheja [6] who worked with the case $M = k$.

In theorem 5 in section 3 we give the following reduction formula:
Let \( y \) be a regular element in \( \mathcal{M} \). Let \( \mathfrak{a} \) be an ideal in \( R \) and put \( R' := R/\mathfrak{a} \). If \( \mathfrak{a}M = 0 \) then

\[
(i) \quad P_M^R(t) = P_M^R(t)[1 - t(\alpha(t) - 1)]^{-1}
\]

where \( \alpha(t) = P_R^R(t)/\alpha(t) \).

In section 4 we give applications of theorem 5 and the results of section 3. Several examples are worked out. In particular it is shown that if \( k \) is a field and \( \mathfrak{a} \) is an ideal generated by monomials in the ring \( A = k[[X_1, X_2, X_3]] \), then the Poincaré series of \( A/\mathfrak{a} \) is rational.

In the last section we remark that in order to prove the rationality of \( P_M^R(t) \) for all \( R \) and \( M \), it suffices to prove the rationality of \( P_R^R/\mathcal{M}(t) \) for all local rings \( R \) of dimension zero.

**Notations.**

If \( N = \oplus_{p \geq 0} N_p \) is a graded \( R \)-module where each homogeneous component \( N_p \) is a free \( R \)-module of finite rank, we let \( \chi_R(N) \) or simply \( \chi(N) \) denote the power series

\[
\sum_{p \geq 0} \text{rank}(N_p)t^p
\]

The term "\( R \)-algebra" will be used in the sense of Tate [7]. By an augmented \( R \)-algebra \( F \) we will mean an \( R \)-algebra \( F \) with a surjective augmentation map \( F \to R/\mathcal{M} \) which is a homomorphism of \( R \)-algebras. Recall that the Koszul complex generated over \( R \) by a minimal set of generators for \( \mathcal{M} \) is an \( R \)-algebra which up to (a non-canonical) isomorphism depends only of the ring \( R \). Thus we shall talk about the Koszul complex of \( R \).
1. On Massey operations.

Let $F$ be an augmented $R$-algebra with a trivial Massey operation $\gamma$, and let $S$ be the set of cycles associated with $\gamma$. For the definitions and details the reader is referred to Gulliksen [2]. Recall that $S$ represents a minimal set of generators for the kernel of the map $H(F) \to R/\mathfrak{m}$ induced by the augmentation on $F$. $\gamma$ is a function with values in $F$, defined on the set of finite sequences of elements in $S$ such that $\gamma(z) = z$ for $z \in S$. By means of $F$ and $\gamma$ it is possible to construct an $R$-free resolution of $R/\mathfrak{m}$. We will briefly recall the construction:

To each cycle $z$ in $S$ select a symbol $u$ of degree one more than the degree of $z$. Let $N = \bigoplus q N_q$ be the free graded $R$-module generated by the set of selected symbols $u$. Let $T = T_R(N)$ be the tensor algebra generated over $R$ by $N$. Put

$$X := F \otimes_R T$$

By means of the canonical map $F \to F \otimes T$, sending $f$ to $f \otimes 1$, $F$ will be considered as a submodule of $X$. We will now extend the differential $d$ on $F$ to a differential on $X$ (also denoted by $d$) in the following way: It suffices to define $d$ on a set of generators for the $R$-module $X$. If $f$ is a homogeneous element in $F$ of degree $\deg f$ and if $u_1, \ldots, u_n$ ($n \geq 1$) are selected symbols corresponding to the cycles $z_1, \ldots, z_n$ in $S$ we put

$$d(f \otimes u_1 \otimes \ldots \otimes u_n) = d(f \otimes u_1 \otimes \ldots \otimes u_{n-1}) \otimes u_n + (-1)^{\deg f} f \gamma(z_1, \ldots, z_n)$$

One can show that $d^2 = 0$ and that $X$ is an $R$-free resolution of $k$. Cf. [2]. Moreover, if $F$ is minimal in the sense that
dF \subseteq \mathcal{M} F$, and if $\text{Im} \gamma \subseteq \mathcal{M} F$, then $X$ is a minimal resolution.

**DEFINITION.** The resolution $X$ constructed above will be called the Golod extension of the couple $(F, \gamma)$ and will be denoted $X = (F, \gamma, N)$.

**THEOREM 1.** Let $R$ be a local ring with residue field $k$ and let $\mathfrak{a} \mathfrak{L}$ be an ideal in $R$. Let $M$ be a finitely generated $R$-module of finite projective dimension such that $\mathfrak{a} \mathfrak{L} M = 0$. Let $F$ be an augmented $R$-algebra which is an $R$-free resolution of $k$. Put $R' := R/\mathfrak{a} \mathfrak{L}$, $F' := F/\mathfrak{a} \mathfrak{L} F$ and assume that $F'$ has a trivial Massey operation $\gamma$. Then there exists a polynomial $\pi(t)$ with integral coefficients such that

$$\pi(t) = \pi(t)[1 - t(\pi(t) - 1)]^{-1}$$

**PROOF.** Let $X = (F', \gamma, N)$ be the Golod extension of $(F', \gamma)$. Then we have an identity of graded $R$-modules

$$X = F' \otimes X \otimes N$$

(1)

We let $Y$ be the complex whose underlying graded module is $X \otimes N$, and whose differential is $d \otimes 1_N$, $d$ being the differential on $X$. (1) leads to an exact sequence of complexes

$$0 \rightarrow F' \underset{\alpha}{\rightarrow} X \underset{\beta}{\rightarrow} Y \rightarrow 0.$$  

(2)

where $\alpha$ is the canonical injection, and $\beta$ is the canonical projection onto the second factor. Since $\mathfrak{a} \mathfrak{L} M = 0$ and $M$ has finite projective dimension we have

$$H_p^R(M \otimes F') = H_p^R(M \otimes F) = \text{Tor}_{p}^{R}(M, k) = 0$$

for all $p$ sufficiently large. Hence from (2) we obtain
for all \( p \) sufficiently large. Hence we have

\[
\text{Tor}_p^R(M,k) = H_p(M \otimes X) \cong H_p(M \otimes Y) = \bigoplus_q H_{p-q}(M \otimes X) \otimes N_q
\]

for all \( p \) sufficiently large. Thus there exists a polynomial \( \pi(t) \) with integral coefficients such that

\[
\frac{P_{R}}{M}(t) = \chi(H(M \otimes X) \otimes N) + \pi(t) = \frac{P_{R}}{M}(t) \chi(N) + \pi(t)
\]

This yields the desired formula since

\[
\chi(N) = t(\chi(H(F')) - 1) = t(\frac{P_{R}}{M}(t) - 1)
\]


We will first recall the definition of a Golod ring. Let \( R \) be a local ring with maximal ideal \( \mathfrak{m} \), and let \( K \) be the Koszul complex of \( R \). \( R \) is called a Golod ring if the canonically augmented \( R \)-algebra \( K \) has a trivial Massey operation in the sense of Gulliksen [2]. This is equivalent to saying that all Massey operations on \( H(K) \) vanish in the sense of Golod [1].

The following result shows that Golod rings can be characterized entirely in terms of the Poincaré series. The proposition is due to Golod, and the proof can be found in [1].

**PROPOSITION 2.** A local ring \( R, \mathfrak{m} \) is a Golod ring if and only if

\[
\frac{P_{R}}{R}(t) = (1+t)^n[1 - c_1 t^2 - c_2 t^3 - \ldots - c_n t^{n+1}]^{-1}
\]

where \( n = \dim \mathfrak{m}/\mathfrak{m}^2 \) and \( c_i = \dim H_i(K) \) for \( 1 \leq i \leq n \).
Examples of Golod rings:

I. The rings of the form $k[[X_1, X_2]]/\mathfrak{a}$ (where $k$ is a field), which are not complete intersections. That these rings are Golod rings follows from Satz 9 in Scheja [6] and proposition 2 above.

II. The rings $A/y\mathfrak{a}$ where $A$ is a regular local ring and $y$ is a non-unit in $A$. Cf. Schamash [5].

III. $A/(a_1, a_2)$ where $A$ is regular, and $a_1$ and $a_2$ do not form a regular sequence. This is a special case of example II.

IV. $k[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^p$. ($k$ is a field). Cf. Golod [1].

PROPOSITION 3. Let $R$, $M$ be a local ring and let $y \in M - M^2$ be a regular element. Then $R$ is a Golod ring if and only if $R/yR$ is a Golod ring.

PROOF. Let $K$ be the Koszul complex of $R$. Put $k = R/M$. Since the $k$-algebra $H(K)$ and the Poincaré series $P(t)_R$ are invariant under $M$-adic completion, Proposition 2 shows that there is no loss of generality assuming that $R$ is complete.

Hence by the Cohen structure theorem we may assume that $R = A/\mathfrak{a}$ where $A$ is a regular ring and $\mathfrak{a}$ is an ideal contained in the square of the maximal ideal of $A$. Let $y'$ be an element of $A$ that represents $y$ in $R$. We have an isomorphism of $k$-vector-spaces.

$$H_i(K) \cong \text{Tor}_i^A(R, k)$$ (1)

Similarly the homology of the Koszul complex of $R/yR$ is isomorphic to $\text{Tor}_i^A/(y')^A(R/yR, k)$. Since $y'$ is regular on $A$ and $R$, and since $y'k = 0$ we have a canonical isomorphism

$$\text{Tor}_i^A/(y')^A(R/yR, k) \cong \text{Tor}_i^A(R, k)$$ (2)
It follows from Satz 1 in Scheja [6] that
\[ P_k^R(t) = (1 + t) P_k^{R/yR}(t) \]  
(3)

Let \( \mathfrak{m} \) denote the maximal in \( R/yR \). We have
\[ \dim \mathfrak{m}/\mathfrak{m}^2 = \dim \mathfrak{m}/\mathfrak{m}^2 + 1 \]  
(4)

The proposition now follows from prop. 2 using (1), (2), (3) and (4).

**THEOREM 4.** Let \( R, \mathfrak{m} \) be a local Golod ring and let \( M \) be a finitely generated \( R \)-module. Then \( P_M^R(t) \) is a rational function.

**PROOF.** Let \( \hat{R} \) and \( \hat{M} \) be the \( \mathfrak{m} \)-adic completions of \( R \) and \( M \). It can easily be shown that \( P_M^R(t) = P_M^R(t) \), hence as in the proof of proposition 3 we may assume that \( R = A/\mathfrak{a} \) where \( A \) is a regular local ring and \( \mathfrak{a} \) is an ideal which is contained in the square of the maximal ideal of \( A \). Let \( K \) be the Koszul complex of \( A \). Then \( K' := K/\mathfrak{a}K \) is the Koszul complex of \( R \). Since \( R \) is a Golod ring, \( K' \) has a trivial Massey operation. Since \( M \) has finite projective dimension over the regular ring \( A \), theorem 2 gives that \( P_M^R(t) \) is a rational function.

3. Reduction formulas for the Poincaré series.

**THEOREM 5.** Let \( R, \mathfrak{m} \) be a local ring, let \( y \) be a regular element in \( \mathfrak{m} \), and let \( \mathfrak{a} \) be an ideal in \( R \). Put \( R' = R/y\mathfrak{a} \) and \( \alpha(t) = P_{R/\mathfrak{a}}^R(t) \). Then

(i) If \( M \) is an \( R \)-module such that \( \mathfrak{a}M = 0 \), then
\[ P_M^{R'}(t) = P_M^R(t)[1 - t(\alpha(t) - 1)]^{-1}. \]
(ii) If \( \mathcal{O} \neq R \) and if \( M \) is an \( R \)-module of finite projective dimension such that \( \gamma \mathcal{O} M = 0 \), then

\[
I_M^R(t) = \pi(t)[1 - t(\alpha(t) - 1)]^{-1}
\]

where \( \pi(t) \) is a polynomial with integral coefficients.

PROOF. Let \( F \rightarrow k \) be an augmented \( R \)-algebra which is an \( R \)-free resolution of \( k = R/\mathcal{M} \). Put \( F' := F/\gamma \mathcal{O} F \). We are going to construct a trivial Massey operation on \( F' \). First we remark that there exists an element \( a' \) of degree one in \( F' \) such that every cycle \( z' \) in \( F' \) is homologue with a cycle of the form \( a'x' \) where \( x' \) is in \( \mathcal{O} F' \). In fact let \( z' \in F \) represent a given cycle \( z' \) in \( F' \). Then we may write \( dz = yx \) where \( x \in \mathcal{O} F \). Since

\[
0 = d^2 z = ydx
\]

and since \( y \) is regular in \( R \) we have \( dx = 0 \). Let \( a \in F_1 \) be such that \( da = y \). We have

\[
d(z - ax) = 0.
\]

Since \( F \) is acyclic, \( z - ax \) is a boundary. Hence \( z' - a'x' \) is a boundary in \( F' \), where \( a'x' \) is the image of \( ax \) in \( F' \).

Now choose a basis for the \( k \)-vectorspace \( \text{Ker}(H(F') \rightarrow k) \). Let \( S \) be a set of cycles in \( F' \) representing that basis, and choose \( S \) such that the cycles \( z' \) in \( S \) have the form \( z' = a'x' \) where \( x' \in \mathcal{O} F' \). Since \( a' \) has degree 1, we have \( (a')^2 = 0 \). Thus we can construct a trivial Massey operation \( \gamma \) on \( F' \) by putting

\[
\gamma(z') = z' \quad \text{for} \quad z' \in S
\]

\[
\gamma(z_1', \ldots, z_n') = 0 \quad \text{for} \quad n \geq 2 \quad ; \quad z_1', \ldots, z_n' \in S.
\]

Now (ii) follows from theorem 2 since we have
We are now going to prove (i). Let \( X = (F', \gamma, N) \) be the Golod extension of the couple \((F', \gamma)\). Assume that \( \mathcal{A} M = 0 \). Since \( \text{Im} \gamma \subseteq \mathcal{A} F' \) the following diagram is commutative

\[
\begin{array}{ccc}
M \otimes X & \to & M \otimes F' \\
\downarrow 1_M \otimes \delta & & \downarrow 1_M \otimes 1_M \otimes \delta \otimes N \\
M \otimes X & \to & M \otimes F' \otimes (M \otimes X) \otimes N
\end{array}
\]

Here the horizontal isomorphisms are induced by the identity (1) in the proof of theorem 2. The diagram yields

\[
H(M \otimes X) \cong H(M \otimes F') \otimes H(M \otimes X) \otimes N
\]

We have \( M \otimes F' \cong M \otimes F \). Hence if \( \mathcal{A} \neq R \) the desired formula in (i) follows from (1) and (2). If \( \mathcal{A} = R \), then \( M = 0 \) and in this case the formula in (i) is trivial.

**COROLLARY 6.** Let \( \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_r \) be a chain of ideals in a local ring \( R \), \( M (r \geq 1) \). Let \( y_1, \ldots, y_r \) be a sequence of elements in \( M \) such that \( y_1 \) is regular in \( R \), and for every \( i \) \((1 \leq i \leq r-1)\) \( y_{i+1} \) is regular on \( R^1 := R/ \bigoplus_{k=1}^{i} y_k \mathcal{A}_n \).

Let \( M \) be an \( R \)-module such that \( \mathcal{A}_r M = 0 \). Then we have

\[
\mathcal{P}_M^R(t) = \mathcal{P}_M^R[(1+t)^R - t \sum_{0 \leq p < r} \mathcal{A}_{r-p}(t)]^{-1}
\]

where \( \mathcal{P}_q(t) = \mathcal{P}_R / \mathcal{A}_q(t) \) for \( 1 \leq q \leq r \).

In particular, if \( R \) is a local complete intersection or a Golod ring, then \( \mathcal{P}_M^R(t) \) represents a rational function.
PROOF. The formula will be proved by induction on \( r \). For \( r = 1 \) the formula is valid by theorem 5. Now let \( i \) be an integer such that \( 1 \leq i < r \). Put

\[
\beta_i(t) = (1 + t)^i - t \sum_{p=0}^{i-1} (1 + t)^p \alpha_{i-p}(t)
\]

By induction we may assume that we have

\[
F_Q^R(t) = F_Q(t) \beta_i(t)^{-1}
\]

for all \( R \)-modules \( Q \) such that \( \sigma_i Q = 0 \). Now let \( L \) be an \( R \)-module such that \( \sigma_{i+1} L = 0 \). We are going to show that (1) remains valid if \( i \) is replaced by \( i+1 \), and \( Q \) is replaced by \( L \). Since \( \sigma_i \leq \sigma_{i+1} \), we have \( \sigma_i (R/\sigma_{i+1}) = 0 = \sigma_i L \), so (1) yields

\[
F_{R/\sigma_{i+1}}^R(t) = \alpha_{i+1}(t) \beta_i(t)^{-1}
\]

and

\[
F_L^R(t) = F_L(t) \beta_i(t)^{-1}
\]

Since \( y_{i+1} \) is regular in \( R^i \), theorem 5 gives

\[
P_{L/\sigma_{i+1}}^R(t) = P_L^R(t) [1 - t( P_R^{R/\sigma_{i+1}}(t) - 1 )]^{-1}
\]

Substituting (2) and (3) in (4) and using the identity

\[
\beta_{i+1}(t) = (1 + t) \beta_i(t) - t \alpha_{i+1}(t)
\]

we obtain the desired result.

We shall now give a lemma which gives conditions implying the hypothesis in the previous corollary. With the notation of that corollary we have
LEMMA 7. Let $\mathcal{I}_1 \subseteq \ldots \subseteq \mathcal{I}_r$ be a sequence of ideals in $R$. Let $y_1, \ldots, y_r$ be a regular sequence contained in the maximal ideal and assume that $y_1, \ldots, y_{i+1}$ is a regular sequence for $R/\mathcal{I}_i$ for all $i \ (1 \leq i \leq r-1)$. Then $y_{i+1}$ is $R^i$-regular for all $i$.

PROOF. We will prove the proposition by induction on $r$, the number of ideals. For $r = 1$ there is nothing to prove. Now let $r > 2$ and $1 \leq i \leq r-1$. Let $\lambda$ be an element of $R$ such that

$$\lambda y_{i+1} \in \sum_{h=1}^{i-1} y_h \mathcal{I}_h \ (1)$$

It suffices to show that $\lambda \in \sum_{h=1}^{i-1} y_h \mathcal{I}_h$. Reading (1) modulo $y_i R$, and using the induction hypothesis one obtains

$$\lambda \in \sum_{h=1}^{i-1} y_h \mathcal{I}_h + y_i R$$

Hence we may write

$$\lambda = \sum_{h=1}^{i-1} y_h a_h + y_i a \ (2)$$

where $a_h \in \mathcal{I}_h$ and $a \in R$. From (1) we obtain $\lambda y_{i+1} \in \mathcal{I}_i$. Hence we have $\lambda \in \mathcal{I}_i$. Now (2) yields

$$y_i a \in \mathcal{I}_i$$

hence $a \in \mathcal{I}_i$, so $\lambda \in \sum_{h=1}^{i-1} y_h \mathcal{I}_h$.

We will end this section by giving an example where lemma 7 can be applied.

Let $S$ be a local ring, let $r \geq 1$ be an integer and let $\mathcal{O}_1 \subseteq \ldots \subseteq \mathcal{O}_r$ be ideals in $S$. Put $R := S[[y_1, \ldots, y_r]]$ and put $\mathcal{O}_i = \mathcal{O}_i^1 R$ for $1 \leq i \leq r$. Then the sequences $y_1, \ldots, y_r$ and $\mathcal{O}_1 \subseteq \ldots \subseteq \mathcal{O}_r$ satisfy the hypothesis in corollary 6.
4. Examples.

I. Let $R$ be a local ring and let $\mathfrak{a}$ be an ideal in $R$.
Let $M$ be an $R$-module such that $\mathfrak{a}M = 0$. Let $y_1, \ldots, y_r$ be a regular sequence in $R$ which is contained in $M$, and assume that $y_1, \ldots, y_r$ is also a regular sequence on $R/\mathfrak{a}$.
Put $R' := R/(y_1, \ldots, y_r)\mathfrak{a}$. Then lemma 7 and corollary 6 yields the formula

$$P_{R'}^R(t) = P_M^R(t)\left(\alpha(t) - \alpha(t)\beta(t) + \beta(t)\right)^{-1}$$

where $\alpha(t) = P_{R/\mathfrak{a}}^R(t)$ and $\beta(t) = P_{R/(y_1, \ldots, y_r)}^R(t) = (1+t)^r$.

In particular if $R$ is a local complete intersection or a Golod ring, then $P_{R'}^R(t)$ represents a rational function.

II. Let $A$ be a regular local ring of dimension $n$. Let $r$ and $s$ be integers such that $0 < s < r < n$. Let $y_1, \ldots, y_r$ be a regular sequence in $M^2$ and let $u$ be an element in $M$ such that $y_1, \ldots, y_s$, $u$ is a regular sequence. Put

$$\mathfrak{a} = (y_1, \ldots, y_s, uy_{s+1}, \ldots, uy_r).$$

Then considering $A/\mathfrak{a}$ as a factor ring of the complete intersection $A/(y_1, \ldots, y_s)$, one easily deduces the following formula from theorem 5:

$$P_{A/\mathfrak{a}}^A(t) = (1+t)^{n-s-1}[1-(1-t)^s(1-t(1+t)^{-r-s-1})]^{-1}.$$

III. Let $k$ be a field and consider the following ring of formal powerseries $A = k[[X_1, \ldots, X_n, Y_1, \ldots, Y_r]]$. Let $\mathfrak{a}_1' \subseteq \ldots \subseteq \mathfrak{a}_r'$ be a chain of ideals in $A' = k[[X_1, \ldots, X_n]]$. 
Put $\mathcal{O}_1 = \mathcal{O}_1^i A$ for $1 \leq i \leq r$, and put

$$\mathcal{O} = \sum_{i=1}^{r} y_i \mathcal{O}_1^i.$$ 

Then for each $A$-module $M$ such that $\mathcal{O} M = 0$ we have

$$P_{M}^{A/\mathcal{O}}(t) = P_{M}^{A}(t) [(1+t)^{r} (1 - t \sum_{0 \leq p < r} (1+t)^{p} \alpha_{r-p}^{i}(t))]^{-1}$$

where $\alpha_q^{i}(t) = P_{A}^{A'/\mathcal{O}_q^i}(t)$.

Clearly $P_{M}^{A/\mathcal{O}}(t)$ is a rational function. In particular, if $k'$ denotes the residue field of $A/\mathcal{O}$ we get

$$P_{k'}^{R/\mathcal{O}}(t) = (1+t)^{n} [1 - t \sum_{0 \leq p < r} (1+t)^{p} \alpha_{r-p}^{i}(t)]^{-1}.$$ 

IV. Let $\mathcal{O}$ be an ideal generated by monomials in the ring $A = k[[X_1, X_2, X_3]]$, where $k$ is a field. We shall also let $k'$ denote the residue field of $A/\mathcal{O}$. We will show that $P_{k'}^{A/\mathcal{O}}(t)$ is rational.

We may write

$$\mathcal{O} = \mathcal{O}_1 X_1 + \mathcal{O}_2$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ are ideals in $A$ and $k[[X_2, X_3]]$ respectively. Put $R = k[[X_2, X_3]]/\mathcal{O}_2$. Then $R$ is either a complete intersection or a Golod ring. See example I in section 2. By proposition 3 we see that the same holds for $R[[X_1]]$. Hence $P_{M}^{R[[X_1]]}(t)$ is a rational function for every $R[[X_1]]$-module $M$. Since

$$A/\mathcal{O} \cong (k[[X_2, X_3]]/\mathcal{O}_2)[[X_1]]/\mathcal{O}_1 X_1 = R[[X_1]]/\mathcal{O}_1 X_1$$

it follows from theorem 5 that $P_{M}^{A/\mathcal{O}}(t)$ is rational for every module $M$ such that $\mathcal{O}_1 M = 0$. In particular $P_{k'}^{A/\mathcal{O}}(t)$ is rational.
Using theorem 5 it is also possible to prove that the ring \( k[[X_1, \ldots, X_n]]/(m_1, m_2, m_3) \) has rational Poincaré series, \( m_1, m_2, m_3 \) being monomials.

5. Reduction to the case of dimension zero.

PROPOSITION 8. The following statements are equivalent:

(i) \( P_{R/M}(t) \) is rational for every local ring \( R, M \) of dimension zero.

(ii) \( P_M(t) \) is rational for every local ring \( R \) and every finitely generated \( R \)-module \( M \).

PROOF. It suffices to prove \((i) \implies (ii)\). Suppose that \( P_R(t) \) is rational for every local ring of dimension zero. From theorem 3.17 in Levin [4] one deduces that \( P_R/(R/M) \) is rational for every local ring \( R, M \). By theorem 2 in [2] it then follows that \( P_M(t) \) is rational for all \( R \) and all \( M \).
References.


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