Introduction

In this paper we study topological actions of compact Lie groups G on cohomology manifolds, and using the p-version of the geometric weight system introduced by W.Y. Hsiang [5], we try to deduce properties of the orbit structure. Our main interest is the study of fixed point sets and orbit types. As is shown by Hsiang, the weight system defined by a maximal torus of G is a very important invariant. For example, Hsiang shows that the weights completely describe the connected principal isotropy type (G_φ^o), where φ is an action on an acyclic space. Moreover, the connected component G_x^o of the isotropy group G_x can be determined in many cases.

However, this success of the weight system is due to the role played by maximal tori of compact Lie groups, and since maximal p-tori are not even conjugate in general, difficulties arise when one try to use p-weights, imitating the methods used in the case $p = 0$. As far as isotropy groups are concerned, there seems to be a close connection between p-weights and p-torsion of G_x or of G_x/G_x^o (e.g., see § 2 when G is classical).

As is well known, the knowledge of the principal isotropy type (G_φ) is of primary interest in the study of compact transformation groups, and trying to determine G_φ is important even when $\dim G_\varphi = 0$.

In § 1 we are first concerned with the relation between the p-weights and the principal isotropy type. For $p = 0$ the results are essentially due to Hsiang. For example, the case $p = 0$ of theorem (1.10) is a local version of Hsiangs algorithm, which describes a sequence of tori descending to a maximal torus T_φ of G_φ. When G_φ is finite and $p \neq 0$, the algorithm will end up with a p-torus T_φ of G_φ, possibly not maximal. However, under suitable conditions (see Remark (1.11)) we may find the maximal p-tori and hence the p-rank of G_φ.

Let m be the maximum of the p-ranks of the isotropy groups, for some p, and let l be the p-rank of G_φ. Then there are isotropy groups having p-rank q for each q between l and m, as shown in theorem (1.13). More generally, even if there is no well defined principal isotropy type, the "principal p-rank" may still be defined, for example if the space is a $\mathbb{F}_p$-cohomology manifold, see remark (1.14). Theorem (1.13) also applies in this generality.

(1.15) is a fixed point theorem relating the fixed point set of G to the fixed point set of a p-torus T of G, when the nonzero p-weights and roots, with respect to T, are disjoint.

§ 2 is devoted to the study of regular (topological) actions of the classical groups SO(n), SU(n), Sp(n). In the literature regular actions are usually known as (smooth) actions modelled after the representation $k\delta_n + l\theta$, where δ_n is the standard representation and θ is trivial. In this paper we are concerned with actions on (integral) cohomology manifolds, and we say the action is regular if it has exactly the same isotropy groups as the above representation, and k is the order of the regular action.

In the above linear case it is clear how to interpret the number k in terms of orbit structure invariants. First, if k is small $(k < n)$, k is determined by the principal isotropy type, and conversely. Secondly, k defines the multiplicity of the nonzero weights of the representation. Thirdly, the codimension d of the fixed point set F(G) is a linear function of k (In fact, $d = kn$, $2kn$, $4kn$, when $G = SO(n), SU(n), Sp(n)$ resp.)

Now, due to the Borel formula, [2] Ch. XIII, in the topological case it is possible to define a workable substitute for the linear weight system. This is the geometric p-weight systems (p prime or zero) originally introduced by Hsiang. Then the above three describtions of k will also apply for regular actions.

The geometric p-weight system of an action is called <u>regular</u> of order k if it is derived from the representation $k\delta_n$. As shown by Hsiang, [6] for example, if the action has regular 0-weights, the action is almost regular in the sense that the connected isotropy types (G_x^0) are the same as for the linear model $k\delta_n$. This also holds for any p, more precisely, if the classical group acts on an acyclic $\mathbb{F}_p$-cohomology manifold and the p-weights are regular, then the connected isotropy types are as above. In addition, the finite group G_x/G_x^0 has no p-torsion, (2.16)-(2.18).

If the p-weights are regular for two different p, their order k must be the same for both p. Theorem (2.19) says that the action is regular if and only if the p-weights are regular for all p.

Let φ be an action whose p-weights are regular of order k, for some p. Then our problem is to show the regularity of p-weights for all p. This can be shown rather easily when k is small (say, $k < n-2$). However, when k is large, there are some technical difficulties, closely related to the validity of the Borel formula for torus actions on certain $\mathbb{F}_p$-cohomology manifolds. To avoid these we may assume the action on X has some local properties, for example, the fixed point sets of the p-tori (p prime) of G have finitely generated local cohomology groups over the integers, see remarks (0.3) and definition (2.23).

Finally, theorem (2.26) sums up some equivalent formulations of regular actions. For example, letting $p = 2$, we find that the action is regular if the 2-weights are regular. Thus, the property of acting regularly is already determined by restricting the action to the maximal 2-torus of the classical group, i.e. the subgroup of diagonal matrices $\langle e_i \rangle$ with entries $e_i = \pm 1$.

All G-spaces X will be cohomology manifolds over the integers $\mathbb{Z}$, rationals $\mathbb{F}_0$ or the field $\mathbb{F}_p$ of order p, and in § 2 X is acyclic (i.e. X has trivial (Čech) cohomology with coefficients $\mathbb{Z}, \mathbb{F}_0$ or $\mathbb{F}_p$ respectively).

§ 0. Preliminaries

A p-torus of rank r is a direct product $T = (\mathbb{Z}_p)^r$ of the group $\mathbb{Z}_p$ of order p(prime). If $p = 0$, $\mathbb{Z}_p$ is replaced by a circle group and T is an ordinary torus.

Definition (0.1) Let φ be an action of a torus or a p-torus T on a $\mathbb{F}_p$-cohomology manifold X with nonempty fixed point set $F(T)$, and let F^1 be a component of $F(T)$. For each corank 1 sub-p-torus H of T, put

$$m(H) = \dim F^1(H) - \dim F^1 ,$$

where $F^1(H)$ is the component of $F(H)$ containing F^1 and dim is cohomological dimension over $\mathbb{F}_p$ (e.g. see Borel [2]). Then H is a nonzero local weight at F^1 if $m(H) > 0$, and $m(H)$ is its multiplicity. The set of all nonzero weights, counted with multiplicity, is denoted by $\Omega'(\varphi)$ (or $\Omega'_p(\varphi)$, when we want to stress that T is a p-torus). T is the zero weight and it has multiplicity $\dim F^1$. The set of all weights, written $\Omega(\varphi)$, is called the local geometric weight system of φ at F^1.

Borel formula (0.2) ([2] Ch. XIII) If T is a p-torus acting on a $\mathbb{F}_p$-cohomology manifold X, p prime or zero, then the total multiplicity of all local weights at F^1 equals $\dim X$, i.e.

$$\dim X - \dim F^1 = \sum_{H \in \Omega'(\varphi)} m(H)$$

Remarks (0.3) The multiplicity $m(H)$ is even if T is not a 2-torus.

If X is $\mathbb{F}_p$-acyclic, then $F(T)$ is also $\mathbb{F}_p$-acyclic (P.A.Smith's theorem). Then $F(T)$ is nonempty and connected, and the geometric weight system is a global invariant.

If T is a torus acting on a $\mathbb{F}_p$-cohomology manifold X, $p \neq 0$, then we do not know if the Borel formula (0.2) is still valid. However, if the formula fails, some integral local cohomology groups of X must be infinitely generated, in such a way that X is not a rational cohomology manifold.

Let Σ_i, $i = 1,2$, be sets whose elements are sub-p-tori H of T having corank ≤ 1, counted with multiplicity $m_i(H)$. Then we define the <u>sum</u> and <u>difference</u>, $\Sigma = \Sigma_1 \pm \Sigma_2$, to be the set of elements H with multiplicity $m(H) = m_1(H) \pm m_2(H)$, respectively. ($m(H) \leq 0$ simply means that $H \notin \Sigma$).

If T' is a sub-p-torus of T, and Σ is a set of the above type, define the <u>restriction</u>

$$\Sigma|T' = \{H' = H \cap T' \ \ ((H \cap T')^0, \text{ if } p=0) \ ; \ H \in \Sigma\}$$

where the multiplicity of H' is the total multiplicity of all H having same restriction H'. This is consistent with the notion of geometric weight system. In fact, the following standard property is a consequence of the Borel formula.

<u>Proposition (0.4)</u> Let T and $\Omega(\varphi)$ be as in (0.1) and let T' be a sub-p-torus of T. Then

$$\Omega(\varphi|T') = \Omega(\varphi)|T'$$

<u>Definition (0.5)</u> Let φ be an action of a compact Lie group G on a space X ($\mathbb{F}_p$-cohomology manifold) and let T be a fixed p-torus of G. Then the <u>(local) geometric p-weight system</u> of φ, with respect to T, is defined to be the (local) weight system of the restricted action $\varphi|T$, as defined in (0.1). It is also

denoted by $\Omega_p(\varphi)$ (or $\Omega(\varphi)$, when there is no ambiguity).

The p-roots $\Delta_p(G)$ of G, with respect to T, is the p-weights of the adjoint representation Ad_G, i.e.,

$$\Delta_p(G) = \Omega_p(Ad_G) .$$

Usually, T is taken to be maximal in this definition.

As a consequence of the topological slice theorem, each orbit G/G_x has a tubular neighborhood G-equivalent to a twisted product

$$O_x = G \times_{G_x} S_x ,$$

where S_x is a slice at x. O_x is a fiber bundle over G/G_x with fiber S_x. Choose a p-torus T of G_x and let $\Omega_p(G/G_x)$, $\Omega_p(S_x)$ be the p-weights, locally at x, of the action of G_x on the orbit and slice, respectively. Then we have the following transversality equation

$$\begin{aligned} \Omega_p(\varphi) &= \Omega_p(G/G_x) + \Omega_p(S_x) \\ &= \Delta_p(G) - \Delta_p(G_x) + \Omega_p(S_x) . \end{aligned} \tag{0.6}$$

The first equality follows from the fibre bundle structure of the fixed point set $F(H,O_x)$ for each closed subgroup H of G_x. The last equality of (0.6) is due to the fact that the action of G_x on the orbit is smooth and the local weights at x (i.e. eG_x) are derived from the isotropy representation - the representation of G_x on the tangent space of G/G_x at eG_x. This space is identified in the usual way with the subspace $\mathfrak{g}_x^{\perp}$ of the Lie algebra $\mathfrak{g}$ of G.

The existence of a principal isotropy type (G_φ) is well known when G is a compact Lie group acting on a connected integral cohomology manifold X . Notice that G_x is a principal isotropy group if and only if G_x acts trivially on the slice S_x . The union of principal orbits is an open dense subset of X .

If X is a rational cohomology manifold, the connected principal isotropy type (G_φ^0) is well defined. Moreover, if X is a $\mathbb{F}_p$-cohomology manifold, we may still define the notion of principal p-rank and 0-rank, see Remark (1.14). Then, if T is a p-torus of G_x acting trivially on a slice S_x at x , or equivalently $\Omega_p'(S_x) = \emptyset$, T must be contained in an isotropy group of principal p-rank.

§ 1. p-weights and p-rank of isotropy groups.

We say that G is p-regular if all maximal p-tori of G are conjugate. If G is connected, G has no p-torsion if $H_*(G;\mathbb{Z})$ has no p-torsion. This is equivalent to saying that each p-torus of G is contained in a (connected) torus, and this clearly implies p-regularity. (See Borel [3] for the relation between p-tori and p-torsion). The p-rank of G is the largest integer r such that G has a p-torus of rank r. The Weyl group $W(T)$ of a p-torus T in G is the group of automorphisms of T which restrict from inner automorphisms of G. Note that the root system $\Delta(G)$, defined by T, is invariant under the natural action of the Weyl group, as is the weight system when T has connected fixed point set $F(T)$.

Definition (1.1) (Hsiang) Let T be a p-torus (p prime or zero) acting on a $\mathbb{F}_p$-cohomology manifold X. The F-variety at $x \in X$ is the component of the fixed point set $F(T_x)[F(T_x^0)$, if $p = 0]$ containing x. (T_x^0 is the connected component of the isotropy group T_x).

The following is a direct consequence of the Borel formula (0,2).

Lemma (1.2) (Hsiang [5],[6]) Let T and X be as in (1.1) and consider the isotropy group T_x at x. Let Ω be the local weights at a component F^1 of the fixed point set $F(T)$. Then, if the F-variety at x intersects F^1, there exist weights $H_i \in \Omega$, $i = 1,2,\ldots s$, such that

$$T_x = H_1 \cap H_2 \cap \ldots H_s, \quad p \neq 0$$

$$T_x^0 = (H_1 \cap H_2 \cap \ldots H_s)^0, \; p = 0$$

<u>Lemma (1.3)</u> Let φ be an action of a compact Lie group G on a $\mathbb{F}_p$-cohomology manifold X, p prime or zero. Let F^1 be a component of the fixed point set $F(T)$ of a maximal p-torus T and let $\Omega(\varphi)$, $\Delta(G)$ be the local p-weights at F^1 and p-roots, respectively. If $p \neq 0$, assume 1) $F(T)$ is connected and G is p-regular, or 2) G_x is p-regular for some $x \in F^1$. Then if

$$H \in \Omega'(\varphi) - \Delta(G) ,$$

there is a point z in the component $F^1(H) \supset F^1$ of $F(H)$ such that H is a maximal p-torus of G_z.

<u>Proof</u> This lemma is a modification of Lemma 2, p. 373 in Hsiang [6], where $p = 0$ and X is acyclic, and (1.3) can be proved in essentially the same way, by the following two steps:

a) Assume first G has a fixed point $y \in F^1$, choose a small G-invariant neighborhood of y in X and prove (1.3).

b) Choose the isotropy group G_x, $x \in F^1$, and apply the first part a) to the action of G_x on the slice S_x.

<u>Lemma (1.4)</u> Let G be a compact Lie group acting non-trivially on an integral cohomology manifold X and let G_φ be a principal isotropy group. Then the fixed point set $F(G)$ has

$$\operatorname{codim}_{\mathbb{Z}} F(G) > \dim G/G_\varphi$$

If $G = SO(3)$ or $SU(2)$, then $\operatorname{codim} F(G) \geq 3$.

<u>Proof</u> The following equation

$$\dim F(G) + \dim G/G_\varphi \leq \dim X - 1$$

can be deduced from Th. 2.2, Borel [2] p. 118, and the first statement follows readily. If G is simple and has dimension 3, then each proper subgroup has dimension 0 or 1 , and so

$$\operatorname{codim} F(G) \geq 1 + \dim G/G_\varphi \geq 3 .$$

Theorem (1.5) Let φ be a nontrivial action of a compact Lie group G on an integral cohomology manifold X . Let T be a p-torus of G , p prime or zero, and F^1 a component of the fixed point set $F(T)$. Let $\Omega(\varphi)$, $\Delta(G)$ be the local p-weights at F^1 and p-roots, respectively, defined by T . Consider the following two statements

(a) Some principal orbit G/G_φ intersects F^1 , (hence G_φ contains T)

(b) $\Omega'(\varphi) \subset \Delta(G)$.

Then (a) implies (b), and the p-roots of G_φ defined by T are

$$\Delta'(G_\varphi) = \Delta'(G) - \Omega'(\varphi) .$$

Conversely, if G_φ is finite and $p \neq 0$, or if T is a maximal torus $(p = 0)$, then (b) implies (a).

Proof (i) Consider the equation

$$(0.6) \qquad \Omega(\varphi) = \Delta(G) - \Delta(G_x) + \Omega(S_x)$$

where G_x is the isotropy group at $x \in F^1$. If $G_x = G_\varphi$ is principal, then $\Omega'(S_x) = \emptyset$ and the first statement follows directly from this equation.

(ii) Next, assume $p \neq 0$ and G_φ finite, or $p = 0$ and T maximal. We prove the implication from (b) to (a) by induction on Lie group structure.

Suppose (b) is true. We claim that F^1 cannot be fixed by G, and assume this for the moment. (We prove the claim below.)

Choose $x \in F^1 - F(G)$. Then $T \subset G_x \neq G$ and from equation (0.6) we have

$$(b)_x \qquad \Omega'(S_x) \subset \Delta(G_x) .$$

Therefore G_x acts on the slice S_x with local weights (at x) satisfying $(b)_x$, so by induction, there is a principal isotropy group $G_\varphi = G_y$, $y \in S_x \cap F^1$. This implies (a).

(iii) It remains to show the above claim. Since the codimension of F^1 is the total multiplicity of nonzero weights (Borel formula), the assumption of (b) gives

$$\operatorname{codim} F^1 \leq \dim G .$$

If $\dim G_\varphi = 0$ $(p \neq 0)$, we use lemma (1.4),

$$\operatorname{codim} F^1 \leq \dim G < \operatorname{codim} F(G)$$

and F^1 cannot be contained in $F(G)$.

In the case $p = 0$, assuming (b), there is a root α of G with $T_\alpha = \alpha^\perp \in \Omega'(\varphi)$. (If $\Omega'(\varphi) = \emptyset$, then G^0 acts trivially). From standard Lie theory the centralizer Z of T_α has dimension $\dim Z = \dim T + 2$, and Z/T_α is a simple group of dimension 3. Let F^1_α be the component of $F(T_\alpha)$ containing F^1. Then Z, being connected, leaves this component invariant and induces a non-trivial action of Z/T_α on F^1_α with fixed point set

$$F(Z/T_\alpha, F^1_\alpha) = F(Z, F^1_\alpha) \subset F^1 .$$

Applying lemma (1.4) to this action,

$$\dim F^1_\alpha - \dim F(Z/T_\alpha, F^1_\alpha) \geq 3 .$$

Now, by definition of multiplicity, (b) also implies

$$\dim F^1_\alpha - \dim F^1 = 2 .$$

Hence Z and, a fortiori, G does not fix the set F^1 , and the proof is complete.

<u>Corollary (1.6)</u> Assume X is $\mathbb{F}_p$-acyclic, $p \neq 0$. If the p-torus T of G has finite centralizer and the p-roots $\Delta'(G)$, defined by T , consist of a single orbit of the action of the Weyl group $W(T)$, then

$$T \subset G_\varphi \quad \text{if and only if}$$

$$G_\varphi \text{ is finite and } \Omega'(\varphi) = \Delta'(G) .$$

[Example: $G = SO(n)$, T = maximal 2-torus]

<u>Corollary (1.7)</u> If T is a maximal torus of G and T has a nonempty fixed point set $F(T) = \bigcup_i F^i$. Then the following statements are equivalent:

(1) The principal isotropy group G_φ has maximal rank.

(2) For some i , $\Omega_i' \subset \Delta(G)$, where Ω_i is the local weight system at the component F^i .

(3) For all i , $\Omega_i' \subset \Delta(G)$.

Using the fact that a compact connected Lie group is a torus if and only if it has no nonzero root (i.e. 0-root), and using the formula for the roots of G_φ given by theorem (1.5), the following modification of (1.7) is obvious.

Corollary (1.8) In the situation of (1.7), the following statements are equivalent:

(1) The connected principal isotropy type is $(G_\varphi^o) = (T)$.

(2) For some i , $\Omega_i' = \Delta'(G)$.

(3) For all i , $\Omega_i' = \Delta'(G)$.

Remark (1.9) The p-version of (1.7) is wrong in general. A simple counterexample is the adjoint action of $SO(n)$ with $p = 2$.

(1.8) is the local version of Hsiang [6], Th. 4, p. 357, where the space is acyclic. Now, if G_φ has not maximal rank, the Hsiang algorithm, [6] p. 367, computes a maximal torus of G_φ from the weights. We will describe a local version of this algorithm, together with a partial p-version of it when G_φ is finite. By "partial" we mean that we cannot ensure the maximality of the p-tori in the isotropy groups involved, since a p-version of the crucial lemma (1.3) is not known except, of course, when these groups are p-regular. However, see Remark (1.11).

Theorem (1.10) (The local Hsiang algorithm)

Let G be a compact Lie group acting on an integral cohomology manifold X with principal isotropy type (G_φ) . Choose an isotropy group G_{x_o} and let T_o be a p-torus or a maximal torus $(p = 0)$ of G_{x_o} . If $p \neq 0$, assume G_φ is finite. Let Ω, $\Delta(G)$ be the weight system at x_o and p-roots, respectively, defined by T_o , and put

$$\Sigma_o = \Omega' - \Delta(G) .$$

If $\Sigma_o = \emptyset$, then for some principal isotropy group G_φ ,

(1) $$T_o \subset G_\varphi \subset G_{x_o} .$$

If $\Sigma_o \neq \emptyset$, then there are isotropy groups G_{x_i} and sub-p-tori T_i of T_o such that

$$(2) \qquad \begin{array}{ccccccccc} G_{x_o} & \underset{\neq}{\supset} & G_{x_1} & \underset{\neq}{\supset} & G_{x_2} & \underset{\neq}{\supset} & \cdots & G_{x_q} & = G_\varphi \\ \cup & & \cup & & \cup & & & \cup & \\ T_o & \underset{\neq}{\supset} & T_1 & \underset{\neq}{\supset} & T_2 & \underset{\neq}{\supset} & \cdots & T_q & = T_\varphi , \end{array}$$

where rank T_i = rank $T_{i-1} - 1$, $T_{i-1} \not\subset G_{x_i}$, and T_i is maximal in G_{x_i} if $p = 0$. T_i is given by intersection of weights $H_j \in \Sigma_o$ as follows: [If $p=0$, replace $T_i \cap H_j$ by $(T_i \cap H_j)^o$]

$$\begin{array}{ll} T_1 \in \Sigma_o & T_1 = H_1 \\ T_2 \in \Sigma_1 = \{\Omega | T_1 - \Delta(G) | T_1\}' \neq \emptyset , & T_2 = T_1 \cap H_2 \\ - \; - \; - \; - \; - \; - & - \; - \; - \; - \; - \\ (3) \quad T_i \in \Sigma_{i-1} = \{\Omega | T_{i-1} - \Delta(G) | T_{i-1}\}' \neq \emptyset , & T_i = T_{i-1} \cap H_i \\ - \; - \; - \; - \; - \; - & - \; - \; - \; - \; - \\ T_q \in \Sigma_{q-1} = \{\Omega | T_{q-1} - \Delta(G) | T_{q-1}\}' \neq \emptyset , & T_q = T_{q-1} \cap H_q \\ \qquad \Sigma_q = = \{\Omega | T_q - \Delta(G) | T_q\}' = \emptyset . & \end{array}$$

Conversely, for each sequence of weights $H_i \in \Sigma_o$ satisfying (3) there is a corresponding chain (2) of isotropy groups.

<u>Proof</u> The theorem is proved by induction, using theorem (1.5), and lemma (1.3) if $p = 0$. Assume the theorem true for all proper closed subgroups of G and actions satisfying the hypothesis of the theorem.

Using equation (0.6), we may put

$$(\Sigma_o)_{x_o} = \Sigma_o = \Omega' - \Delta(G) = \Omega'(S_{x_o}) - \Delta(G_{x_o}) .$$

If $\Sigma_o = \emptyset$, apply (1.5) to the action of G_{x_o} on the slice S_{x_o}, and (1) follows readily.

If $\Sigma_o \neq \emptyset$, choose $T_1 = H_1 \in \Sigma_o$ and a point x_1 in the component of $F(H_1) \cap S_{x_o}$ at x_o such that

$$T_o \neq H_1 = (T_o)_{x_1} [(T_o)^o_{x_1}, \text{ if } p = 0]$$

Then $T_1 \subset G_{x_1} \subsetneq G_{x_o}$ and if $p = 0$, we may also assume T_1 is maximal in G_{x_1}, by lemma (1.3).

Define the next set of weights, Σ_1, with respect to T_1, by

$$\Sigma_1 = (\Sigma_o)_{x_1} = \{\Omega | T_1 - \Delta(G)|T_1\}' = \Omega'(S_{x_1}) - \Delta(G_{x_1}) .$$

By induction hypothesis, the algorithm is true for the action of G_{x_1} on the slice S_{x_1}. Having the corresponding (2) and (3) for this action, starting with G_{x_1} and T_2, respectively, we get (2) and (3) for the original action of G as well, by the usual slice argument.

Remark (1.11) Assume G is p-regular and the space X is $\mathbb{F}_p$-acyclic. Then we may assume T_o is a maximal p-torus of G, i.e. G_{x_o} has maximal p-rank. Observe that the number q of (3) depends on the choice of weights H_i, and let q_o be the smallest q ever possible in (3). Then each sequence of T_i in (3) for which $q = q_o$, leads down to a maximal rank p-torus T_φ of a principal isotropy group G_φ. In fact, from lemma (1.2) we may assume $T_\varphi = T_o \cap G_\varphi$ is some intersection of weights H_i, and theorem (1.5) and the minimality of q_o imply that T_φ has maximal p-rank in G_φ.

Problem (1.12) In the case $p \neq 0$, is it possible to construct the algorithm such that the p-tori T_i are maximal in G_{x_i}? How do we construct the algorithm when $\dim G_\varphi > 0$?

<u>Theorem (1.13)</u> Let φ be an action of a compact Lie group G on a connected integral cohomology manifold X. Put

$$m_p = \max\{p\text{-rank } G_x;\ x \in X\}\ ,\ p \text{ prime or zero}\ ,$$

and let l_p be the p-rank of a principal isotropy group G_φ. Then the p-ranks of the isotropy groups form a string of numbers k, $l_p \leq k \leq m_p$,
and hence there are at least $m_p - l_p + 1$ different isotropy types.

<u>Proof</u> Using induction on Lie group structure, assume the theorem is true for actions of all proper closed subgroups of G.

Choose x such that $K = G_x$ has a p-torus T of rank m_p. If $m_p = l_p$, there is nothing to prove, so assume $m_p > l_p$.

As usual, let $\Omega'(S_x)$ be the nonzero weights at x, defined by T acting on the slice S_x. T cannot act trivially on S_x, otherwise T would be contained in a principal isotropy group, contradicting $m_p > l_p$. Thus there is a nonzero weight $H \in \Omega'(S_x)$ and a point $y \in S_x$ such that $H = T_y (T_y^0$, if $p = 0)$. Then $H \subset K_y = G' \neq G$, and

$$p\text{-rank } G' \geq \text{rank } H = m_p - 1\ .$$

By assumption, the theorem is true for the action φ' of G' on the slice S_y. By the usual slice argument, the isotropy groups of φ' are isotropy groups of φ, $G'_{\varphi'} \sim G_\varphi$, and so the theorem is true for φ.

<u>Remark (1.14)</u> If X is a connected $\mathbb{F}_p$-cohomology manifold, put

$$l_p = \min\{p - \text{rank } G_x;\ x \in X\}$$

$$l_0 = \min\{0 - \text{rank } G_x;\ x \in X\}\ .$$

Then theorem (1.13) and its proof are still valid. In fact, we claim the <u>principal p-rank</u> l_p (or 0-rank l_o) is well defined in the sense that the subset

$$X_{l_p} = \{x \in X;\ \text{p-rank } G_x = l_p\}\ ,\quad p \text{ prime or zero}$$

is open and dense. This is proved by induction on Lie group structure, and letting $p \neq 2$, it can be shown as follows:

Assume the above claim is true for all proper closed subgroups of G. Let T_p be a p-torus of maximal rank in G. Put $Y = X - F(G)$. Considering cohomological dimension mod $\mathbb{F}_p$, if codim $F(G) < 2$, then also codim $F(T_p) < 2$ and hence $(p \neq 2)$

$$\dim F(T_p) = \dim X\ ,\quad F(T_p) = X\ .$$

In this case T_p acts trivially, so we may assume $\dim F(G) \leq \dim X-2$, and consequently Y is connected.

$G_y \neq G$ for all $y \in Y$, and G_y acts on the (connected) slice S_y with principal p-rank l_y, by assumption. In the tubular neighborhood (§ 0)

$$O_y = G \times_{G_y} S_y\ ,$$

the action of G has principal p-rank l_y, by the slice theorem. Clearly, if $O_y \cap O_z \neq \emptyset$, then $l_y = l_z$. Therefore, by the connectedness of Y, there is a well defined principal p-rank in Y, say l, and we must have $l = l_p$ (defined above). The set Y_l is open and dense in Y, and so the set X_{l_p} is open and dense in X. (It is obviously open, by the slice theorem).

According to a theorem of Hsiang [5], if G is a compact connected Lie group acting on an acyclic rational cohomology manifold such that no nonzero weight is a root, with respect to a maximal torus T, then G and T have the same fixed point set, $F(T) = F(G)$. We extend this in the following way.

Theorem (1.15) Let G be a compact connected Lie group acting on a $\mathbb{F}_p$-cohomology manifold X, p prime or zero. Let T be a torus or a p-torus of G having nonempty fixed point set $F(T) = \bigcup_i F^i(T)$, and let $\Omega, \Delta(G)$ be the local p-weights at $F^1(T)$ and p-roots, respectively, defined by T. Define N to be the largest connected normal subgroup of G which centralizes T. Assume the nonzero weights and roots are disjoint, i.e.

$$\Omega' \cap \Delta(G) = \emptyset .$$

Then the fixed point set $F(G)$ of G is related to $F^1(T)$ by

$$F(G) \cap F^1(T) = F(N) \cap F^1(T) .$$

Hence, if $F(T)$ is connected (e.g., X is $\mathbb{F}_p$-acyclic), then

$$F(G) = F(T) \cap F(N) .$$

Proof Choose $x \in F(N) \cap F^1(T)$, and we must show $x \in F(G)$, or equivalently, $G_x = G$. Now, G_x contains N and T, and from the equation

$$(0.6) \qquad \Omega' = \Delta'(G) - \Delta'(G_x) + \Omega'(S_x)$$

we get

$$\Omega'(G/G_x) = \Delta'(G) - \Delta'(G_x) = \emptyset .$$

This says that T acts on the orbit G/G_x with fixed point set of codimension zero, and since the orbit is connected, T must act trivially on G/G_x, i.e.

$$gTg^{-1} \subset G_x , \quad \text{for all } g \in G .$$

Put

$$L = \bigcap_{g \in G} g G_x g^{-1} , \quad T \subset L \subset G_x .$$

Then L and its connected component L^o are normal subgroups of G , and from the structure theory of compact connected Lie groups we can find a connected normal subgroup K of G such that

$$G = K \cdot L^o , \quad K \cap L^o \text{ finite} .$$

K and L^o commute, and $K \cap L^o$, being normal in G , is in fact central in G . Therefore K and L commute, and since L contains T , K commutes with T and so $K \subset N$. Thus, G_x contains both K and L , and consequently $G_x = G$.

<u>Corollary (1.16)</u> Let G be a compact connected simple Lie group acting on an acyclic $\mathbb{F}_p$-cohomology manifold, p prime or zero. Let T be a torus of G or a p-torus not contained in the center of G . Assume the nonzero p-weights and p-roots, defined by T , are disjoint, i.e.,

$$\Omega' \cap \Delta(G) = \emptyset .$$

Then G and T have the same fixed point set,

$$F(G) = F(T) .$$

§ 2. Regular actions of classical groups.

In order to treat the classical compact Lie groups in a unified manner, we use the following notation and terminology. $O(n)$, $U(n)$, $Sp(n)$ are the linear groups leaving invariant the standard inner product on the n-space Λ^n, where $\Lambda = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (quaternions) respectively. Standard inclusions are

$$(2.0)\qquad \begin{array}{ccccc} O(n) & \subset & U(n) & \subset & Sp(n) \\ \cup & & \cup & & \\ SO(n) & \subset & SU(n) , & & \end{array}$$

where $SO(n)$, $SU(n)$ are defined by requiring the determinant to be 1 . Then $SO(n)$, $SU(n)$, $Sp(n)$ are connected and simple ($SO(4)$ is semi-simple) and these are denoted by <u>$G(n)$</u> in the sequel.

The subgroup $G(V) \cong G(q)$, fixing a $(n-q)$-dimensional linear subspace $V^{\perp} \subset \Lambda^n$, is called a <u>regular</u> subgroup. The standard (orthogonal) decomposition

$$\Lambda^n = \Lambda^q \oplus \Lambda^{n-q}$$

defines the subgroup $G(q) = G(\Lambda^q)$ and its <u>complementary</u> group $G'(n-q) = G(\Lambda^{n-q})$.

Define $G(q)$ to be the trivial group if q is an integer < 1 . To simplify notation we often write $G = SO$, SU or Sp instead of $G(n) = SO(n)$, $SU(n)$ or $Sp(n)$, when the omission of n makes no confusion.

Let T_o be the standard (usual) maximal torus of $G(n)$. The subgroup $T_p \subset T_o$ of elements of order p is a maximal p-torus of $G(n)$, except when $G = SO$ and $p = 2$. In the latter case, the group of diagonal matrices is a maximal 2-torus T_2 . Note that all groups (2.0) are p-regular for all p , i.e.

maximal p-tori are conjugate. In § 2 all p-weights and p-roots are taken with respect to T_p, even if it is not explicitly stated.

Remark (2.1) A regular subgroup $G(V) \subset G(n)$ is uniquely determined by any of its maximal p-tori [p prime or zero, but $\dim V = 2r$ (even) if $G = SO$ and $p \neq 2$]. In fact, if the p-torus T is maximal in both $G(V)$ and $G(W)$, then

$$T \subset G(V) \cap G(W) = G(V \cap W)$$

and hence

$$\text{rank } T = \text{p-rank } G(V) = \text{p-rank } G(W) = \text{p-rank } G(V \cap W) .$$

Now,

$$\text{p-rank } G(V) = \begin{cases} \dim V , & G = Sp \\ \dim V-1, & G = SU, \text{ or } G = SO \text{ and } p = 2 \\ [\dim V/2] , & G = SO,\ p \neq 2 \end{cases}$$

and consequently, $\dim(V \cap W) = \dim V = \dim W$, i.e. $V = W$.

Let $NG(V)$ be the normalizer of $G(V)$ in $G(n)$. From the definition of $G(V)$,

$$gG(V)g^{-1} = G(gV) ,$$

where $g \in G(n)$ acts on Λ^n by the standard representation δ_n. From this it is easy to calculate $NG(V)$. Assuming $G(q) \neq 1$,

$$(2.2) \qquad NG(q) = [Sp(q) \times Sp'(n-q)] \cap G(n) ,$$

using the standard inclusions (2.0).

The following property of the classical group $G(n)$ will be used in the sequel.

Proposition (2.3) (Hsiang [4] § 2) Let K be a closed connected subgroup of $G(n)$ and assume K contains a regular subgroup $G(V)$, where $\dim_\Lambda V \geq 1,3$ when $G = Sp$, $G = SO$ or SU respectively. Then there is a linear subspace $W \subset \Lambda^n$ such that

$$G(V) \subset G(W) \subset K \subset NG(W) \subset G(n) .$$

The regular model. The group $G(n)$ acts naturally on euclidean space Λ^{nk}, given by k copies, $k\delta_n$, of the standard representation δ_n. Let G_v be the isotropy group at $v = (v_1, v_2, \ldots, v_k) \in \Lambda^{nk}$. Then

$$G_v = G_{v_1} \cap G_{v_2} \cap \ldots G_{v_k}, \quad G_{v_i} = G(v_i^\perp) \sim G(n-1) \quad \text{if} \quad v_i \neq 0 .$$

Put $V = [v_1, v_2, \ldots v_k]^\perp$, $\dim_\Lambda V = q$. Clearly

$$G_v = G(V) \sim G(q) ,$$

and so all isotropy groups are regular subgroups, more precisely,

a) The isotropy types are the types of all $G(q)$, $n-k \leq q \leq n$.

b) The principal isotropy type is the type of $G(n-k)$.

Definition (2.4) The action of a classical group $G(n)$ on a topological space is called regular of order k if the action has the same isotropy groups as the representation $k\delta_n$ for some $k > 0$. [To define k uniquely when $G(n-k) = 1$, we require the ($\mathbb{F}_2$-cohomological) codimension of the fixed point set to be the same as in the representation space of $k\delta_n$.]

Regular weights Let $(\theta_1, \theta_2, \ldots)$ be the coordinates of the Cartan algebra defined by the maximal torus T_0 of $G(n)$. If $\omega = \Sigma n_i \theta_i$ is an integral functional, $\omega^\perp$ is the corank 1 subtorus given by $\omega = 0$. The nonzero p-weights of the regular model, defined by the maximal p-torus T_p , are as follows:

(2.5) $\Omega_0'(k\delta_n) = \{(\theta_i)^{\perp} ;\ 1 \le i \le n\ ([n/2],$ if $G = SO)$,

mult. = 4k, 2k, G = Sp, G = SO or SU resp.}

If $G \neq SO$ or $p = 2$, $T_p \subset T_0$ and the p-weights are simply the restriction of 0-weights,

(2.6) $\Omega_p'(k\delta_n) = \{S_i = (\theta_i)^{\perp} \cap T_p$; same mult. as in (2.5)} .

If $G = SO$ and $p = 2$,

(2.7) $\Omega_2'(k\delta_n) = \{S_i ;\ 1 \le i \le n ,\ \text{mult.} = k\}$

where S_i consists of the diagonal matrices $\langle e_j \rangle$ with entry $e_i = 1$ (This is also the describtion of S_i in (2.6) when $p = 2$).

Definition (2.8) A p-subtorus $H \subset T_p$ of corank 1 is called a regular weight if it is a weight of the standard representation δ_d , i.e. $H \in \Omega_p'(\delta_n)$. A system Σ of regular weights, counted with multiplicity, is called regular of order k if it is the nonzero p-weights of $k\delta_n$, as described in (2.5)-(2.7).

Remark. When $G = Sp$, the weights of (2.5), (2.6) have multiplicity 4k. Therefore, we also permit k to be half integral in the above definition. However, if Sp(n) acts with regular weights having order $k < n$, then we know k must be integral, see (2.16), (2.18), (2.26).

Now we characterize regular weights.

Lemma (2.9) Let T_p be a maximal p-torus of the classical group G(n) , p prime or zero, and let H be a p-torus of corank 1 in T_p . Then H is a regular weight if and only if H is a maximal p-torus of a regular subgroup $G(V) \sim G(q)$.

Proof If H is a regular weight, then clearly H is a maximal p-torus of a group $G(V) \sim G(q)$ [$q = n-1$, or $q = n-2$ if $G(n) = SO(2r+1)$ and $p \neq 2$].

Conversely, assume H is a maximal p-torus of $G(V)$. Assume first $G \neq SO$ or $p = 2$. As usual we may choose T_p to be standard, i.e. it consists of diagonal matrices $\langle d_i \rangle$. Now, $G(V) \sim G(n-1)$, so we can write $G(V) = G_v$, where $0 \neq v \perp V$ and G_v is the isotropy group at v by the standard representation δ_n. $H \subset G_v$ implies

$$\langle d_i \rangle v = (d_1 v_1, d_2 v_2, \dots d_n v_n) = (v_1, v_2, \dots v_n) = v$$

for all $\langle d_i \rangle \in H$. At least one component of v , say v_i , is $\neq 0$. Then $d_i = 1$ for all $\langle d_i \rangle \in H$. However, the relation $d_i = 1$ is just the definition of a regular weight, so H is regular.

In the case $G = SO$ and $p \neq 2$, we may assume $\dim V = n-2$, and $G(V)$ fixes a 2-dimensional space $V^{\perp} \subset \mathbb{R}^n$. T_p splits in 2-dimensional rotations

$$t_i \in SO(2) , \quad t = (t_1, t_2, \dots t_r) \in T_p , \quad r = [n/2] .$$

Choose $v \in V^{\perp}$ with at least two nonzero components. H fixes v and it is easily seen that for some i , $t_i = 1$ for all $t \in H$, i.e. H is regular.

Lemma (2.10) Let K be a closed subgroup of $G(n)$ whose connected component K^o is a regular subgroup $G(V) \sim G(q)$, and assume the finite group K/K^o has p-torsion (p prime). Then

$$\text{p-rank } K > \text{p-rank } G(q) ,$$

more precisely, a maximal p-torus of K^0 is not maximal in K.

<u>Proof</u> We may assume $K^0 = G(q) \neq 1$. By assumption, K has an element z such that

$$z^{p-1} \notin G(q) , \quad z^p \in G(q) .$$

We must find an element $z' \in K - G(q)$ of order p, commuting with the maximal p-torus T_p^q of $G(q)$. Now, K normalizes $G(q)$, consequently

$$1 \neq G(q) \subset K \subset NG(q)$$

$$(1) \qquad = [Sp(q) \times Sp'(n-q)] \cap G(n) , \qquad (2.2)$$

In the decomposition (1), $z = (a,b) = ab$ and $b^p = 1$. First, if $G = SO$ and $p > 2$, or if $G = Sp$, we clearly have $a \in G(q)$. Then $z' = b \in K - G(q)$ and z' commutes with $G(q)$.

In the other cases, T_p^q (standard) consists of diagonal matrices. Let $d \in G(n)$ be the diagonal matrix with first entry $d_1 = \det(a)$ and $d_i = 1$ when $i \neq 1$. Then, since $da^{-1} \in G(q)$,

$$z' = db = (da^{-1})(ab) = da^{-1})z \in K - G(q) .$$

Now, $\det(a^p) = 1$ implies $d^p = 1$, and d commutes with b, so z' has order p. Moreover, d, and hence $db = z'$, commutes with T_p^q, and the proof is complete.

<u>The p-roots of G(n)</u>

The (ordinary) roots of the classical groups are as follows (see Adams [1]).

$$(2.11) \qquad \begin{aligned} \Delta'(Sp(n)) &= \{(\theta_i \pm \theta_j),\ i < j \leq n;\ (2\theta_i),\ i \leq n\} \\ \Delta'(SU(n)) &= \{(\theta_i - \theta_j),\ i < j \leq n\} \\ \Delta'(SO(2r)) &= \{(\theta_i \pm \theta_j),\ i < j \leq r\} \\ \Delta'(SO(2r+1)) &= \Delta'(SO(2r)) + \{(\theta_i),\ i \leq r\} \end{aligned}$$

These are integral functionals on the Cartan algebra of the maximal torus T_o. In all cases, except when $G = SO$ and $p = 2$, $T_p \subset T_o$ and then the p-roots $\Delta_p'(G(n))$ are calculated by restricting the roots (2.11). The roots $(2\theta_i)$ of $Sp(n)$ vanish on T_2, i.e., $(2\theta_i)|T_2 = 0$, but all other roots restrict to nonzero p-roots α,

$$T_p \neq \alpha^{\perp} = (\theta_i \pm \theta_j)^{\perp} \cap T_p \quad \text{or} \quad (\theta_i)^{\perp} \cap T_p .$$

In the special case $p = 2$, the elements of T_2 are diagonal matrices $\langle e_i \rangle$ with entries $e_i = \pm 1$, for all $G(n)$. Let S_{ij} be the corank 1 subgroup of T_2 defined by $e_i \cdot e_j = 1$. Clearly $S_{ij} = (\theta_i \pm \theta_j)^{\perp} \cap T_2$ when $G = SU$ or Sp.

The Lie algebra $\mathfrak{g}$ of $SO(n)$ splits in 1-dimensional root spaces $\mathfrak{g}_{ij}$, $1 \leq i < j \leq n$, in which T_2 acts by

$$v \to tvt^{-1} = (e_i \cdot e_j) \cdot v , \quad t = \langle e_i \rangle \in T_2 .$$

Therefore the 2-roots have a common expression for all $G(n)$, namely

$$(2.12) \qquad \Delta_2'(G(n)) = \{S_{ij},\ 1 \leq i < j \leq n;\ \text{mult.} = 1,2,4 \text{ when } G = SO, SU, Sp \text{ respectively}\} .$$

To demonstrate the power of p-weights of classical groups, we first note the following corollary of (1.15), (1.16).

<u>Proposition (2.13)</u> Let φ be an action of $G(n)$ on an acyclic $\mathbb{F}_p$-cohomology manifold, and let T_p be a maximal p-torus of $G(n)$. Assume the set of nonzero p-weights and the set of p-roots are disjoint, i.e.

$$\Omega_p'(\varphi) \cap \Delta_p(G(n)) = \emptyset .$$

Then $G(n)$ and T_p have the same fixed point set

$$F(G(n)) = F(T_p) .$$

<u>Theorem (2.14)</u> Let φ be an action of the classical group $G(n)$ on an acyclic $\mathbb{F}_p$-cohomology manifold, p prime or zero. Let T_p^q be a maximal p-torus of the regular subgroup $G(q) \subset G(n)$, and assume q is even if $G = SO$ and $p \neq 2$. Then, if the p-weights of φ are regular, $G(q)$ and T_p^q have the same fixed point set

$$(1) \qquad F(G(q)) = F(T_p^q) ,$$

except possibly when 1) $G(q) = SO(2), SU(2), Sp(1)$, or 2) $p = 2$, $G(q) = SO(3), SU(3)$.

<u>Proof</u> Observe that if $\Omega_p'(\varphi) = \Omega_p'(k\delta_n)$, then

$$(2.15) \qquad \Omega_p'(\varphi|G(q)) = \Omega_p'(k\delta_n|G(q)) = \Omega_p'(k\delta_q) , \text{ all } q ,$$

hence $G(q)$ acts with regular p-weights of order k. Therefore it is enough to prove (1) for $q = n$. However, apart from the exceptional cases, and $G = Sp$ with $p \neq 2$, no nontrivial p-weights are p-roots, and then (1) follows from (2.13).

Finally, assume $G(n) = Sp(n)$ and $p \neq 2$. Let $x \in F(T_p)$, $T_p = T_p^n$. Since

$$\Omega_p'(\varphi) \cap \Delta_p(Sp(n)) = \{(\theta_i)^{\perp} \cap T_p , i \leq n\} ,$$

we have from equation (0.6) and (2.11)

$$\Delta_p'(G_x) \supset \{(\theta_i \pm \theta_j)^{\perp} \cap T_p , i < j \leq n\} .$$

These p-roots are all different and so it is clear that the Lie algebra $\mathfrak{g}_x$ contains all root spaces of the roots $(\theta_i \pm \theta_j)$. From standard Lie theory $\mathfrak{g}_x$ must contain the root spaces of the roots $(2\theta_i)$ as well, and consequently $G_x = Sp(n)$. This proves (1). [Notice that the root space method can be used to prove (1) in all cases of $G(n)$.]

Corollary (2.16) Let φ be an action of $G(n)$ on an acyclic $\mathbb{F}_2$-cohomology manifold X with regular 2-weights of order k , i.e. $\Omega_2'(\varphi) = \Omega_2'(k\delta_n)$. Then

(a) Each isotropy group G_x has regular connected component $G_x^o \sim G(q)$ and G_x/G_x^o has odd order, except possibly if 2-rank $G_x \leq 2$ (2-rank $G_x = 1$, when $G = Sp$)

(b) Let $m_o = 2,4$ when $G = Sp$, $G = SO$ or SU respectively. The types of $G(q)$, where $\max\{m_o, n-k\} \leq q \leq n$, occur as connected isotropy types. If $k \leq n-m_o$, these are all types.

(c) Assume X also is a rational cohomology manifold, and assume rank $G(n) > 2$ [To simplify the proof, let $G(n) \neq SU(4)$] . Then the connected isotropy types are the types of all $G(q)$, $n-k \leq q \leq n$, and each quotient G_x/G_x^o has odd order.

Proof

(i) Let G_x be an isotropy group with a 2-torus T such that

$$\text{rank } T = \text{2-rank } G_x > 2 \ (> 1, \text{ if } G = Sp) .$$

By conjugation we may assume T is in the maximal 2-torus T_2^n of $G(n)$, and so

$$T = T_2^n \cap G_x = (T_2^n)_x .$$

Then, from (1.2), T must be the intersection of some regular 2-weights $S_i \in \Omega_2'(k\delta_n)$. By (Weyl) conjugation we may therefore assume

$$T = S_n \cap S_{n-1} \cap \dots = T_2^q$$

is the (standard) maximal 2-torus of $G(q)$ for some q, and clearly $q \geq m_0$, see (b). Now, from the above theorem

$$F(T) = F(T_2^q) = F(G(q)) .$$

Put $K = G_x^0$. Then we have $G(q) \subset G_x^0$, and applying prop. (2.3) and the fact

(1) $$\text{2-rank } G_x = \text{2-rank } K = \text{2-rank } G(q) ,$$

we must have

(2) $$G(q) = K \subset G_x \subset NG(q) .$$

Moreover, (2) and (2.10) imply that G_x/K has odd order. This proves (a)

(ii) The extra assumption of (c) is only necessary when treating the isotropy groups with 2-rank 1 or 2. These occur only when $k > n-m_0$. Now, introducing the 0-weights, it will be seen later, (2.20), that they are also regular. Then we refer to the proof of (2.19), part (ii), to show that

$$F(T_2^q) = F(G(q)) , \quad \text{for all } G(q) .$$

Starting with an isotropy group G_x with 2-rank > 0, we still have (1), and (2) is valid except possibly when $G(q) = SO(2)$ or $SU(2)$. However, in these two cases, 2-rank $K = 1$, and if $K \neq G(q)$, the only possibility is $G(q) = SO(2)$ and $K \cong SU(2)$. In the latter case, K would have nontrivial center and would have the same root as $SO(3)$. This is impossible, so (2) must hold in all cases.

(iii) To show which of the $G(q)$ occur as connected isotropy type, we may use the following inductive argument.

First, $F(G(n)) = F(T_2^n) \neq \emptyset$, so $q = n$ occur. Assume we have shown that for some x, $G_x^o = G(q)$, where $q > \max\{m_o, n-k\}$. Since $T_2^q \subset G_x$, we consider weights with respect to T_2^q and equation (0.6) reads

$$(3) \qquad \Omega_2(\varphi) = \Delta_2(G(n))|T_2^q - \Delta_2(G(q)) + \Omega_2(S_x) ,$$

where the 2-roots $\Delta_2'(G(n))$, with respect to T_2^n, are given by (2.12). The left side of (3) is the weights of $\varphi|T_2^q$ [or $\varphi|G(q)$], and so the nonzero weights of (3) are $\Omega_2'(\varphi) = \Omega_2'(k\delta_q)$, as follows from equation (2.15). Using (2.12) we calculate the difference

$$\Delta_2(G(n))|T_2^q - \Delta_2(G(q))$$
$$\equiv \{S_i \cap T_2^q,\ i \leq q;\ \text{mult.} = (n-q)d\} = \Omega_2'((n-q)\delta_q)$$

modulo zero weight, $d = 1,2,4$ when $G = SO, SU, Sp$ respectively. Therefore the action of G_x on the slice S_x has the weights

$$(4) \qquad \Omega_2'(S_x) = \Omega_2'((k-n+q)\delta_q) \qquad (\neq \emptyset \text{ iff } q > n-k) .$$

Choose the weight $S_q \cap T_2^q = T_2^{q-1}$ of (4), and choose a point $y \in S_x$ such that $(T_2^q)_y = T_2^{q-1}$. Then

$$T_2^{q-1} \subset G_y , \quad G_y^o \subsetneq G(q) , \quad q-1 \geq m_o ,$$

and the method og (i) applies to show $G_y^o = G(q-1)$. Thus, inductively we get all connected isotropy types of $G(q)$, $q \geq \max\{m_o, n-k\}$. Similarly, in the case of (c) we get all types of $G(q)$, $q \geq n-k$.

Finally, it follows from (4) that it is impossible to have $G_x^o = G(q)$ when $q < n-k$.

<u>Remark (2.17)</u> A p-version of the above corollary is proved in a similar way for all p. As in the case $p = 2$, there are some

technical subtleties when isotropy groups having p-rank 1 occur (k large).

The case $p = 0$ of (2.18) is a theorem of Hsiang [6]. The proof of [6] does not exclude the possibility $G_x^o \simeq SO(2)$ when $G = SU$ or Sp (k large). However, this is settled when we combine 0-weights and 2-weights, see the proof of (2.19). We state the following simpler p-version of (2.16).

<u>Corollary (2.18)</u> Let φ be an action of $G(n)$ on an acyclic $\mathbb{F}_p$-cohomology manifold X, $p \neq 2$, and assume the p-weights are regular,

$$\Omega_p'(\varphi) = \Omega_p'(k\delta_n) .$$

Then each isotropy group G_x has regular connected component $G_x^o \sim G(q)$, $q \geq n-k$, and the quotient G_x/G_x^o has no p-torsion $(p \neq 0)$, at least if p-rank $G_x > 1$. Isotropy groups of p-rank 1 occur only when $k \geq n-2$ and $G = SO$ or SU, or $k \geq n-1$ and $G = Sp$.

The following is a characterization of regular actions, (2.4), by means of p-weights, and then, in (2.20) we characterize regular p-weights using isotropy groups.

<u>Theorem (2.19)</u> Let X be an acyclic integral cohomology manifold with an action of the classical group $G(n)$, rank $G(n) > 1$, $G(n) \neq SU(3)$. Then the following two statements are equivalent:

(a) The action is regular, i.e. for some $k > 0$, the isotropy types are given by a string

$$G(n), G(n-1), G(n-2), \dots G(n-k) .$$

(b) The nonzero p-weights are regular for all p (prime or zero).

Moreover, if (b) is true, the p-weights have the same order k for all p, and k determines the principal isotropy type $G(n-k)$ of (a). Hence $G(n-k)$ is nontrivial if and only if $k < n$ when $G = Sp$, $k < n-1$ when $G = SO$ or SU.

<u>Proof</u> (i) The implication from (a) to (b) is closely related to the p-rank properties of $G(n)$, see remark (2.1).

First, the case $G(n) = SO(2r+1)$ and $p \neq 2$ follows from the even case $n = 2r$, since the restricted action of $SO(2r)$ is also regular and has the same p-weights. Assume therefore $n = 2r$ if $G = SO$ and $p \neq 2$.

Now, notice that $p\text{-rank } G(n-1) < p\text{-rank } G(n)$ for all p. Hence, if H is a p-weight of the action, H must be contained in a proper isotropy group $G_x \sim G(q)$, $q < n$, and H is regular by lemma (2.9).

(ii) To show the reverse implication, we assume (b), i.e.

$$(1) \qquad \Omega_p'(\varphi) = \Omega_p'(k_p\delta_n), \quad k_p > 0 .$$

Let T_p^n be the usual maximal p-torus of $G(n)$. From the Borel formula (0.2) we have the identity (valid for all n)

$$\dim X - \dim F(T_p^n) = \text{mult. } \Omega_p'(k_p\delta_n)$$

$$(2) \qquad = \begin{cases} 2k_p\cdot n,\ 4k_p\cdot n, & G = SU, Sp \text{ resp.} \\ k_p\cdot n, & G = SO,\ p = 2 \\ 2k_p\cdot[n/2], & G = SO,\ p \neq 2 . \end{cases}$$

According to theorem (2.14), $F(T_p^n) = F(G(n))$, except when $G(n) = SO(2r+1)$ and $p \neq 2$. In the latter case, however, $T_p^n = T_p^{n-1}$ and $F(T_p^n) = F(SO(2r)) = F(T_2^{n-1})$. Then it is clear from (2) that all k_p are equal, say, $k_p = k$, and (1) reads

$$(1)' \qquad \Omega_p'(\varphi) = \Omega_p'(k\delta_n) \text{ , all } p \text{ , } k > 0 \text{ .}$$

From the corollaries (2.16), (2.18) of theorem (2.14) we know that all isotropy groups G_x are regular, at least if p-rank $G_x > 1$ (> 2, if $p = 2$) , and so far we only need (1). However, to settle the remaining cases, when p-rank $G_x = 1$ or 2, we apply (1)'. In fact, it is only necessary to show that equation (1) of (2.14)

$$F(T_p^q) = F(G(q))$$

is valid for all p without restriction on rank $G(q)$. Then the same method as in the proof of (2.16), part (i) and (ii), applies to show that all G_x are regular.

It remains to show the above equation of (2.14) in the cases

1) $G(q) = SO(2), SU(2)$ or $Sp(1)$, all p

2) $G(q) = SO(3)$ or $SU(3)$, $p = 2$.

Let $G(q_o)$ denote the groups of 1). Using (2.15) and (1)' , the restricted action of $G(q_o)$ has the weights

$$(3) \qquad \Omega_p'(\varphi | G(q_o)) = \Omega_p'(k\delta_{q_o}) \text{ , all } p \text{ .}$$

Let $\mathbb{Z}_p$ be a cyclic group of order p ($p \neq 0$) or a circle group ($p = 0$) contained in $G(q_o)$ ($\mathbb{Z}_p$ is its maximal p-torus). Then the identities (2) and (3) show that the groups $\mathbb{Z}_p$ have the same fixed point set for all p . If $G(q_o) = SU(2)$ or $Sp(1)$, note that $\mathbb{Z}_2$ is its center, and since $\mathbb{Z}_2$ and $\mathbb{Z}_o$ have the same fixed point set $F(\mathbb{Z}_p)$, this is also the fixed point set of $G(q_o)$. Consequently,

$$(4) \qquad F(G(q_o)) = F(\mathbb{Z}_p) \text{ , all } p$$

and this solves the cases 1).

Finally, we solve the cases 2). Let $p = 2$, $G(3) = SO(3)$ or $SU(3)$, and consider the two subgroups $G(2) = G(q_0)$ and $G'(2)$. Let

$$\mathbb{Z}_2 = T_2^3 \cap G(2) \ , \quad \mathbb{Z}_2' = T_2^3 \cap G'(2) \ .$$

In addition to (4) we clearly have

$$(4)' \qquad F(G'(2)) = F(\mathbb{Z}_2') \ .$$

Using the fact that the closed subgroup generated by $G(2)$ and $G'(2)$ is $G(3)$ itself, (4) and (4)' imply

$$F(T_2^3) = F(\mathbb{Z}_2) \cap F(\mathbb{Z}_2') = F(G(2)) \cap F(G'(2))$$
$$= F(G(2) \cdot G'(2)) = F(G(3)) \ .$$

(iii) The subgroups $G(q)$ of $G(n)$ are distinguished by their 2-rank. Therefore we may use theorem (1.13) to show that the isotropy types are given by a string

$$G(n),\ G(n-1),\ \ldots\ G_\varphi = G(n-k)$$

leading down to a principal isotropy group $G_\varphi = G(l)$, and $l = n-k$ follows, for example, by counting weights in equation (0.6), with $G_x = G(l)$ and $\Omega_p'(S_x) = \emptyset$. Alternatively, the above string is a direct consequence of corollary (2.16). This completes the proof of (2.19).

Proposition (2.20) Let X be an acyclic $\mathbb{F}_p$-cohomology manifold, p prime or zero, and let φ be an action of the classical group $G(n)$ on X. Assume rank $G(n) > 2$, and $G(n) \neq SU(4)$ if $p = 2$. Then the following two statements are equivalent:

(a) There are p-corank 1 isotropy groups. These groups have regular connected component, $G_x^o \sim G(q)$, and the corresponding quotient G_x/G_x^o has no p-torsion.

(b) The p-weights are regular, i.e. $\Omega_p'(\varphi) = \Omega_p'(k\delta_n)$ for some $k > 0$, and $k > 1$ if $G(n) = SO(2r+1)$ and $p \neq 2$.

Proof We already know (a) is a consequence of (b), by (2.16)-(2.18) and lemma (1.3). Conversely, to prove the reverse implication, assume (a) is true and we must show (b).

(i) Consider first the case $G(n) = SO(2r+1)$, $p \neq 2$. In the proof below, the statement that T_p and $G(n)$ have the same fixed point set, is replaced by the statement $F(T_p) = F(SO(2r))$. Then the proof is similar to the $Sp(n)$ case, showing the regularity of the p-weights. Moreover, the weights must have order $k > 1$, otherwise all isotropy groups would have maximal p-rank, (2.18).

(ii) In the following we assume $n = 2r$ if $p \neq 2$ and $G = SO$. First, if

$$H \in \Omega_p'(\varphi) - \Delta_p(G(n)),$$

it follows from lemma (1.3) that H is a maximal p-torus of some isotropy group G_x. By assumption (a), $H \subset G_x^o \sim G(q)$ and so H is a regular weight, lemma (2.9). Secondly, by the definition of weight, (a) and lemma (1.3) imply that the p-weight system must contain the regular weights. Therefore, if the p-weight system contains <u>non-regular</u> weights, the only possibilities are

(1) $\Omega_p'(\varphi) = \Omega_p'(k\delta_n) + \Delta_p'(G(n))$, $k > 0$
or
(2) $\Omega_p'(\varphi) = \Delta_p'(Sp(n))$, $G = Sp$ and $p \neq 0,2$.

(2) is due to the fact that the set of p-roots of $Sp(n)$ contains the regular weights if $p \neq 2$, but $p = 0$ is excluded in (2) by (1.7), since the principal 0-rank cannot be maximal.

We claim that, as a consequence of both (1) and (2), $G(n)$ and its (usual) maximal p-torus T_p must have the same fixed point set. Assuming this for the moment, we choose a p-root H in (1) [or (2)] which is not a regular weight. Then there is a point $z \in X$,

$$H \subset T_p \cap G_z \neq T_p \ , \quad G_z \neq G(n) \ ,$$

and H is a maximal p-torus of G_z since $F(T_p) = F(G(n))$. Consequently, by assumption (a), $G(q) \sim G_z^o \supset H$ and this contradicts lemma (2.9). Thus it is impossible to have (1) or (2) and this proves the proposition.

(iii) Henceforth, we show the above claim that both (1) and (2) imply $F(T_p) = F(G(n))$.

Suppose $F(T_p) \neq F(G(n))$ and let G_x be an isotropy group such that $T_p \subset G_x \neq G(n)$. Since, by (a), the principal p-rank is not maximal, the usual slice argument shows there is a point y in the slice S_x whose isotropy group G_y has p-corank 1. Then $G(q) \sim G_y^o \subset G_x$, and so we have $G(W) \subset G_x^o \subset NG(W)$ for some $W \subset \Lambda^n$, using (2.3). By suitable conjugation we may assume

(3) $G(q) \subset G_x^o \subset NG(q)$, $q = n-1$ (or $n-2$, if $G = SO$) .

From our knowledge (2.2) of $NG(q)$ we find the possible choices of G_x^o, and its normalizer is contained in $NG(q)$, in particular, $G_x \subset NG(q)$. Moreover, G_x must be p-regular, i.e. its maximal p-tori are conjugate.

Choose $x \in F(T_p) - F(G(n))$ such that G_x is minimal, and consequently

$$F(T_p, S_x) = F(G_x, S_x) ,$$

otherwise there is a point $y \in S_x$ for which $T_p \subset G_y \subsetneq G_x$, contradicting the minimality of G_x.

Now, use equation (0.6)

$$\Omega_p'(\varphi) = \Delta_p'(G(n)) - \Delta_p'(G_x) + \Omega_p'(S_x) .$$

Assuming (1) or (2), the above equation gives

$$(1)' \quad \Omega_p'(S_x) = \Omega_p'(k\delta_n) + \Delta_p'(G_x) ,$$

or

$$(2)' \quad \Omega_p'(S_x) = \Delta_p'(G_x) ,$$

respectively. Choose a p-root $H \in \Delta_p'(G_x)$ which is not a regular weight [This is possible because of (3)]. Then using the same argument as in the last part of (ii), with $G(n)$, X replaced by G_x, S_x, respectively, we obtain a contradiction, and the claim is proved.

The exceptional case $G(n) = SO(2r+1)$, $k = 1$, of 2.20 (b) is interesting in the sense that it gives the only example of a regular action having no isotropy groups of p-corank 1 for some p. We state it as follows

<u>Theorem (2.21)</u> Let $G = SO(2r+1)$, $r > 1$, act on an acyclic integral cohomology manifold X. Then the following are equivalent:

(a) The action is regular of order 1, i.e. the isotropy types are the types of $SO(2r+1)$, $SO(2r)$.

(b) For some p (prime or zero) the p-weights are regular of order 1, i.e. $\Omega_p'(\varphi) = \Omega_p'(\delta_{2r+1})$.

(c) For all p the p-weights are regular of order 1 .

<u>Proof</u> (a) and (c) are equivalent by theorem (2.19), and if $p=2$, then (b) and (a) are equivalent, by (2.16).

It remains to prove (a), assuming (b) when $p \neq 2$. Let $T_p (p \neq 2)$ be the maximal p-torus of $SO(2r)$. Then $T_p \subset T_o$ and

$$F(T_p) = F(T_o) = F(SO(2r)), \quad \text{theorem } (2.14) ,$$

and this set has ($\mathbb{Z}$-cohomological) dimension

$$2r = \text{mult. } \Omega_p'(\delta_{2r})$$

by the Borel formula (0.2). If also $F(T_p) = F(G)$, a principal isotropy group would have dimension

$$\dim G_\varphi \geq \dim G - 2r + 1 > \dim SO(2r), \quad \text{lemma } (1.4) ,$$

which is clearly impossible. Now, if $x \in F(T_p) - F(G)$, then $SO(2r) \subset G_x \neq G$, and so $G_x^o = SO(2r)$, (2.3) . Moreover, since

$$\begin{aligned}\Omega_p'(\varphi) = \Omega_p'(\delta_{2r}) &= \Delta_p'(G) - \Delta_p'(SO(2r)) \\ &= \Delta_p'(G) - \Delta_p'(G_x) ,\end{aligned}$$

we have $\Omega_p'(S_x) = \emptyset$ from (0.6), i.e. T_p acts trivially on the slice S_x . Therefore $T_p \subset G_\varphi$, $G_\varphi^o = SO(2r)$. G_φ cannot have maximal 2-rank, by (1.6), consequently $G_\varphi = SO(2r)$.

Now we note that all 2-corank 1 isotropy groups are regular, in fact principal, and hence the 2-weights are regular, by (2.20)

[This part of (2.20) is also valid for $G(n) = SO(5)$] . Thus there is no exceptional isotropy group, $G_x \sim NSO(2r)$, see (2.16), and this proves (a).

Remark (2.22) If $G = SO(3)$, (a) and (c) are equivalent, and equivalent to (b) when $p = 0$. However, (b) does not imply (a) when $p = 2$ (Consider the 5-dim. irreducible representation).

To end our investigations of regular actions we shall discuss the problem of extending the result of theorem (2.21) to all classical groups $G(n)$ and all orders k .

Definition (2.23) Let T be a torus acting on an (acyclic) integral cohomology manifold X . We say T acts rationally if the fixed point set of each p-torus in T is a rational cohomology manifold, for all prime p. If G is a compact connected Lie group acting on X , G acts rationally if its maximal torus T acts rationally.

Lemma (2.24) Let T be a torus acting rationally on X , let T_p (p prime) be the maximal p-torus of T , and let Ω_o', Ω_p' be the set of nonzero weights of the action of T and T_p respectively. If $S \in \Omega_p'$, then for some $H \in \Omega_o'$,

$$S = T_p \cap H .$$

Proof Denote the fixed point set of a group K by X^K , let $\dim X^K$ be dimension mod the rationals $\mathbf{F}_o$, and denote the elements of Ω_o' by H_i .

Suppose $S \subset T_p$ is a corank 1 p-torus which is not contained in any H_i . Then $S \cdot H_i = T_p \cdot H_i$ and using the Borel formula for the action of T on X^S , we have

$$\dim X^S - \dim X^T = \sum_i [\dim (X^S)^{H_i} - \dim X^T] = \sum_i [\dim X^{S\cdot H_i} - \dim X^T]$$
$$= \sum_i [\dim (X^{T_p})^{H_i} - \dim (X^{T_p})^T] \leq \dim X^{T_p} - \dim X^T .$$

Therefore $\dim X^S = \dim X^{T_p}$ and S is not a weight.

Corollary (2.25) Assume the classical group $G(n)$ acts rationally on X with regular 0-weights. Then the p-weights are also regular for all prime p, $p \neq 2$ if $G = SO$. [If we know that T_p and T_o have the same fixed point set, the "rational" assumption is unnecessary].

Proof Since the maximal p-torus T_p of $G(n)$ is contained in T_o, the p-weights are the intersection of T_p and regular 0-weights, by (2.24), and these are regular. [If $F(T_o) = F(T_p)$, it is clear from the Borel formula that the p-weights are the restriction of 0-weights].

Theorem (2.26) Let φ be an action of the classical group $G(n)$ on an acyclic integral cohomology manifold X. Let rank $G(n) > 2$ and $G(n) \neq SU(4)$ (for simplicity). Consider the following statements:

(a) The action is regular of order k, hence the isotropy types are the types of $G(n), G(n-1), \dots G(n-k)$.

(b) For some p (prime or zero) the p-weights are regular of order k, $\Omega_p'(\varphi) = \Omega_p'(k\delta_n)$.

(c) For all p the p-weights are regular of order k.

Then (a) and (c) are equivalent. If $G(n)$ acts rationally, then all three statements are equivalent, with $p = 2$ in (b) if $G = SO$.

Assume $k < n-1, n-2, n-3$ when $G = Sp, SU, SO$ respectively. Then the three statements are equivalent.

Proof The equivalence of (a) and (c) follows from (2.19). If the p-weights are regular for some prime p, then the 0-weights are regular, by (2.16)-(2.21), and if we also assume $G(n)$ acts rationally, the p-weights are regular for all p, (2.25). Hence, (b) implies (c) if $G(n)$ acts rationally.

To show that (b) implies (c) when k is small, we first note that (b) implies $G_\varphi^0 \sim G(n-k)$, (2.16)-(2.18). Now, let p be a prime and $T_p' \subset T_p$, maximal p-tori of $G(n-k) \subset G(n)$. From (0.6),

$$(1) \qquad \Omega_p(\varphi | T_p') \equiv \Delta_p(G(n)) | T_p' - \Delta_p(G(n-k))$$

modulo zero weight T_p'. $\Omega_p'(\varphi)$ is Weyl invariant, i.e. its elements are permuted by the action of the Weyl group $W(T_p)$. The regular weights constitute the shortest orbit under this action. If $\Omega_p'(\varphi)$ contains an orbit of non-regular weights, we can show, by counting weights, that (1) is violated. Details will be omitted.

Problems (2.27) If $G(n) = SO(n)$ acts with regular 0-weights of order $k \geq n-3$, how do we show the regularity of 2-weights?

In (2.26) we have assumed $G(n)$ acts rationally (k large). This seems to be only a technical assumption, and we conjecture that it is superfluous.

References

[1] J.F. Adams, Lectures on Lie groups. Benjamin, New York, 1969.

[2] A. Borel, Seminar on Transformation Groups. Ann. of Math. Studies 46, Princeton Univ. Press, 1960.

[3] A. Borel, Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes. Tohoku Math. J. 13 (1961), 216-240.

[4] W.Y. Hsiang, On the principal orbit type and P.A. Smith theory of SU(p) actions. Topology 6 (1967), 125-135.

[5] W.Y. Hsiang, On the geometric weight system of topological actions I. Preprint, Univ. of Calif., Berkeley.

[6] W.Y. Hsiang, On the splitting principle and the geometric weight system of topological transformation groups I. Proc. of the Second Conf. on Compact Transformation Groups, Part I, 334-402. Springer-Verlag, 1972.