INTRODUCTION

We will in this paper study the cohomology groups of algebras, due to André [A] and Quillen, for a given graded algebra. We shall see that the groups

$$H^i(S,A,M)$$

have a grading if S is noetherian and graded and if $S \to A$ is finitely generated and where M is a graded A-module. In fact if we let

$${}_\nu H^i(S,A,M)$$

corresponds to the S-derivations of degree ν, we shall prove that there are canonical isomorphisms

$$\coprod_{\nu=-\infty}^{\infty} [{}_\nu H^i(S,A,M)] \cong H^i(S,A,M)$$

for every $i \geq 0$ (chapter 1)

Our main interest will lie in deformation problems. It is well known that the group

$$H^2(S,A,A \underset{S}{\otimes} \ker \Pi)$$

contains an obstruction for deforming A to R where

$$R \xrightarrow{\pi} S$$

is a surjective ringhomomorphism such that $(\ker \pi)^2 = 0$. And $H^1(S,A,A \underset{S}{\otimes} \ker \pi)$ measures the amount of deformations. It is trivial that we have corresponding results in the graded case if we use the groups ${}_oH^2(S,A,A \underset{S}{\otimes} \ker \pi)$ and ${}_oH^1(S,A,A \underset{S}{\otimes} \ker \pi)$. Since the canonical morphism

$${}_oH^2(S,A,A \underset{S}{\otimes} \ker \pi) \to H^2(S,A,A \underset{S}{\otimes} \ker \pi)$$

take the obstruction onto the obstruction, we conclude that A can be lifted to R if and only if there is a graded lifting of A to R. It would be nice to generalize this result to an arbitrary surjection $R \overset{\pi}{\to} S$ of artinian rings where R, S and π are all local. I can not. But if we assume

$$ {}_{\nu}H^{1}(S,A,A) = 0 $$

for $\nu > 0$ or $\nu < 0$, then it is true. This is a consequence of what we do in chapter 2 when S is a field. What we actually state, is that the canonical local Λ-homomorphism

$$ R(A) \to R^{O}(A) $$

has a section. Here, Λ is a noetherian local ring with maximal ideal m_{Λ} and the Λ-algebra $R(A)$ (respectively $R^{O}(A)$) is the hull for the deformation functor (respectively graded deformation functor). These functors are defined on the category of artinian local Λ-algebras with residue field Λ/m_{Λ}. The existence of a section of

$$ R(A) \to R^{O}(A) $$

comes out of a Λ-isomorphism

$$ R(A) \simeq R^{O}(B) $$

where $B = A[T]$ and where T has degree one or minus one. This has much to do with Pinkham's theorem in [P].

In chapter 3 we generalized his theorem to the non-smooth and non-equicaracteristic case. We end this chapter by relating the lifting theory of graded algebras to the corresponding theory for the projective schemes.

I should like to thank O.A. Laudal for his many suggestions.

Chapter 1

COHOMOLOGY GROUPS OF GRADED ALGEBRAS.

We shall consider only commutative rings with one.

The purpose of this chapter is to introduce to the reader the cohomology groups

$$ {}_kH^i(S,A,M) $$

for $i \geq 0$ and every k, when $S \to A$ is a graded (or homogeneous) ringhomomorphism of graded rings and M is a graded A-module. As mentioned in the introduction, we shall prove that there are canonical isomorphisms of groups

$$ \coprod_{k=-\infty}^{\infty} {}_kH^i(S,A,M) \xrightarrow{\cong} H^i(S,A,M) $$

for every $i \geq 0$ if S is noetherian and if $S \to A$ is finitely generated. To prove this, we will use a spectral sequence which we find in [LI] . We also find a proof for this theorem in Illusie [I], using graded simplicial resolutions.

But first, let us recall some definitions and theorems from the non-graded case, and see how it can be carried out in the graded case too.

Let

$$ S \to A $$

be any homomorphism and let M be a A-module. The cohomology groups of algebras

$$ H^i(S,A,M) $$

can be introduced in the following way. Let <u>S-alg</u> be the category of S-algebras and let $\underline{SF}$ be the full subcategory of free S-algebras. We denote by

$$\mathrm{Der}_S(-,M)$$

the functor on $\underline{SF}/A$ with values in $\underline{Ab}$, defined by

$$\mathrm{Der}_S(-,M)\left(\begin{smallmatrix} F \\ \downarrow\varphi \\ A \end{smallmatrix}\right) = \mathrm{Der}_S(F,M)$$

where M is given the structur of a F-module by $\varphi \in \mathrm{ob}\ \underline{SF}/A$. $\underline{Ab}$ is the category of abelian groups.

We define

$$H^i(S,A,M) = \varprojlim_{\underline{SF}/A}{}^{(i)} \mathrm{Der}_S(-,M)$$

where $\varprojlim^{(i)}$ is i-th derivative of $\varprojlim$.

If given any surjection $R \xrightarrow{\pi} S$ such that

$$(\ker \pi)^n = 0$$

for some integer n , we shall say that a R-algebra A' is a lifting (or deformation) of A to R if there is a cocartesian diagram

$$\begin{array}{ccc} R & \longrightarrow & A' \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ S & \longrightarrow & A \end{array}$$

such that

$$\mathrm{Tor}_1^R(A',S) = 0$$

Moreover, two liftings $R \to A'$ and $R \to A''$ are equivalent if there are commutative diagrams

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "\pi"'] \arrow[rr, bend left] & A' \arrow[r, "\sim"] \arrow[dr] & A'' \arrow[d] \\
S \arrow[rr] & & A
\end{tikzcd}$$

If we use the word deformation when $R \xrightarrow{\pi} S$ do not satisfy $(\ker \pi)^n = 0$, then we mean a flat R-algebra A' and a cocartesian diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "\pi"'] & A' \arrow[d] \\
S \arrow[r] & A
\end{tikzcd}$$

We will ask whether or not a given S-algebra A can be lifted to R . If we assume

$$(\ker \pi)^2 = 0$$

then $\ker \pi$ is a S-module and the answer is given by

THEOREM 1.1

There is an element

$$o(A) \in H^2(S,A,A \otimes_S \ker \pi)$$

which is zero if and only if A can be lifted to R .

If $o(A) = 0$, then the set of non-equivalent liftings is a principal homogeneous space over $H^1(S,A,A \otimes_S \ker \pi)$.

Again, let us assume that $R \xrightarrow{\pi} S$ satisfies $(\ker \pi)^n = 0$ for some n . Let $\varphi : A \to B$ be a S-algebrahomomorphism. If A' and B' are liftings of A and B respectively to R , we shall say that a R-algebrahomomorphism

$$\varphi' : A' \longrightarrow B'$$

is a lifting of φ to R with respect to A' and B' if

$$\varphi' \otimes_R \mathrm{id}_S \simeq \varphi$$

where id_S is the identity on S .

If we assume

$$(\ker \pi)^2 = 0 \ ,$$

then we can prove

THEOREM 1.2

There is an element

$$o(\varphi;A',B') \in H^1(S,A,B \otimes_S \ker \pi)$$

which is zero if and only if φ can be lifted to R with respect to A' and B' . If $o(\varphi;A',B') = 0$, then the set of liftings is a pricipal homogeneous space over $H^0(S,A,B \otimes_S \ker \pi) = \mathrm{Der}_S(A,B \otimes_S \ker \pi)$.

The element $o(A)$ and $o(\varphi;A',B')$ are called obstructions.

If now S,A are graded rings, M a graded A-module and if the ringhomomorphism

$$S \to A$$

is graded (or homogeneous), then it is possible to define the cohomology groups

$${}_kH^i(S,A,M)$$

by simply repeting what we did above. To be specific, let

<u>Sg-alg</u>

be the category of graded S-algebras and <u>SgF</u> the full subcategory of free S-algebras.

Moreover, we denote by

$$ {}_k\mathrm{Der}_S(-,M) $$

the functor on $\underline{SgF}/A$ with values in $\underline{Ab}$, defined by

$$ {}_k\mathrm{Der}_S(-,M)\left(\begin{array}{c}F\\ \downarrow\varphi\\ A\end{array}\right) = {}_k\mathrm{Der}_S(F,M) = \{D\in \mathrm{Der}_S(F,M) \mid D \text{ is graded of degree } k\} $$

M is a graded F-module by φ .

With these notations, we define

DEFINITION 1.3

We let

$$ {}_kH^i(S,A,M) = \varprojlim_{\underline{SgF}/A}{}^{(i)}[{}_k\mathrm{Der}_S(-,M)] $$

Let $R \xrightarrow{\pi} S$ be a graded surjection of graded rings such that

$$ (\ker \pi)^n = 0 $$

for some n .

DEFINITION 1.4

By a graded lifting (or deformation) of A to R we shall mean a graded R-algebra A' such that A' is a lifting of A to R and such that every morphism in the cocartesian diagram

$$ \begin{array}{ccc} R & \longrightarrow & A' \\ \pi\downarrow & & \downarrow \\ S & \longrightarrow & A \end{array} $$

are graded.

Of course, two graded liftings $R \to A'$ and $R \to A''$ are equivalent if everything are graded in the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[rr, bend left] \arrow[d] & A' \arrow[r, "\sim"] \arrow[d] & A'' \arrow[dl] \\ S \arrow[r] & A & \end{tikzcd}$$

It is obvious how we will define graded liftings of graded S-algebrahomomorphisms.

Assuming that

$$(\ker \pi)^2 = 0 \ ,$$

then the lifting problem of a graded S-algebra and the corresponding problem for graded S-algebramorphisms are formally solved by our next two theorems.

THEOREM 1.5

There is an element

$$o_0(A) \in {}_0H^2(S,A,A \otimes_S \ker \pi)$$

which is zero if and only if A can be lifted to a graded R-algebra.

If $o_0(A) = 0$, then the set of non-equivalent liftings is a principal homogeneous space over ${}_0H^1(S,A,A \otimes_S \ker \pi)$

If

$$\varphi : A \longrightarrow B$$

is a graded S-algebrahomomorphism, and if A' and B' are graded liftings of A and B respectively, then

THEOREM 1.6

There is an element

$$o_0(\varphi;A',B') \in {}_0H^1(S,A,B \otimes_S \ker \pi)$$

which is zero if and only if φ can be lifted as a graded morphism to R with respect to A' and B'.

Moreover, if $o_o(\varphi;A',B') = 0$, then the set of graded liftings is a principal homogeneous space over

$${}_oH^o(S,A,B \otimes_S \ker \pi) = {}_o\mathrm{Der}_S(A,B \otimes_S \ker \pi)$$

In [LI] we find proofs for theorem 1.1 and 1.2, and these can be repeted in the graded case too.

If we want to define the (graded) obstructions, we use that the cohomology groups of (graded) algebras can be defined as the cohomology of certain complexes. For instance, in the graded case, we have

$${}_kH^i(S,A,M) \simeq H^i({}_kC^\cdot(M))$$

where

$${}_kC^p(M) = \prod_{\substack{F_o \xrightarrow{\psi_1} F_1 \to \cdots \xrightarrow{\psi_p} F_p \\ \searrow \quad \downarrow \quad \swarrow \\ A}} [{}_k\mathrm{Der}_S(F_o,M)]$$

The index set is every tuple $(\psi_1,\ldots,\psi_p)$ of morphisms from <u>SgF</u>/A where "aim" for ψ_i is "source" for ψ_{i+1} for all i.

The differentials

$$d^p\colon {}_kC^p(M) \to {}_kC^{p+1}(M)$$

are defined by

$$d^p(\xi)(\psi_1,\ldots,\psi_{p+1}) = \psi_1\cdot\xi(\psi_2,\ldots,\psi_{p+1}) + \sum_{i=1}^{p}(-1)^i\xi(\psi_1,\ldots,\psi_i\cdot\psi_{i+1},\ldots,\psi_{p+1})$$
$$+ (-1)^{p+1}\xi(\psi_1,\ldots,\psi_p)$$

where $\xi \in {}_kC^p(M)$

(the composition $\psi_i \psi_{i+1}$ is written in the opposite way).

Define a map

$$\sigma : \mathrm{Mor}\ \underline{SgF} \longrightarrow \mathrm{Mor}\ \underline{RgF}$$

such that if

$$\varphi : F_o \to F_1 \ , \quad \varphi \in \mathrm{Mor}\ \underline{SgF}$$

then

$$\sigma(\varphi) : F_o' \to F_1', \ \sigma(\varphi) \in \mathrm{Mor}\ \underline{RgF}$$

is a graded lifting of φ to R with respect to F_o' and F_1'. And F_o', F_1' are the unique graded liftings of F_o, F_1 respectively. We call σ a graded quasisection for the functor $(-) \otimes_R S : \underline{RgF} \to \underline{SgF}$.

If

$$\begin{tikzcd} F_o \arrow[r, "\psi_1"] \arrow[dr, "\varphi_o"'] & F_1 \arrow[r, "\psi_2"] \arrow[d, "\varphi_1"] & F_2 \arrow[dl, "\varphi_2"] \\ & A & \end{tikzcd}$$

is an index for ${}_oC^2(-)$, then let

$$o_o(\sigma)_{(\psi_1,\psi_2)} = [\sigma(\psi_1\psi_2) - \sigma(\psi_1)\sigma(\psi_2)]\cdot(\varphi_2 \otimes \mathrm{id}_{\ker\pi})$$

where $\mathrm{id}_{\ker\pi}$ is the identity on $\ker\pi$. $o_o(\sigma)$ is a 2-cocycle in ${}_oC^2(A \otimes_S \ker\pi)$ defining the obstruction $o_o(A) \in {}_oH^2(S,A,A \otimes_S \ker\pi)$ (which is independent of σ) .

Correspondingly we have

$$H^i(S,A,M) \cong H^i(C^\cdot(M))$$

where $C^\cdot(M)$ is defined in a similar way. The proofs in [LI] work with this complex. And it should be remarked that the definition by Andre in [A] uses this complex too. For more details, see [LI].

The main problem in relating the groups ${}_kH^i(S,A,M)$ to the groups $H^i(S,A,M)$ is that they are defined as $\varprojlim^{(i)}$ on different categories. However, I claim that the forgetful functor

$$j : \underline{SgF}/A \longrightarrow \underline{SF}/A$$

induce isomorphisms

$$\varprojlim_{\underline{SF}/A}{}^{(i)} \mathrm{Der}_S(-,M) \xrightarrow{\sim} \varprojlim_{\underline{SgF}/A}{}^{(i)}[\mathrm{Der}_S(-,M)\cdot j]$$

for every $i \geq 0$.

To prove this, we shall use a spectral sequence which is theorem 2.1.3 in [LI] .

Let

$$F \rightarrow A$$

be a graded S-algebrasurjection and let

$$F_i = F \times_A F \times_A \dots \times_A F \qquad (i+1)\text{-times} .$$

All projection morphisms

$$F_i \rightarrow F_{i-1}$$

are graded.

If

$$D = \mathrm{Der}_S(-,M) \circ j : \underline{SgF}/A \rightarrow \underline{Ab}$$

is the composed functor

$$\underline{SgF}/A \xrightarrow[j]{} \underline{SF}/A \xrightarrow[\mathrm{Der}_S(-,M)]{} \underline{Ab}$$

then look at the complex

$$\lim_{\underset{\underline{SgF}/F}{\longleftarrow}}{}^{(q)} D \to \lim_{\underset{\underline{SgF}/F_1}{\longleftarrow}}{}^{(q)} D \to \quad \to \lim_{\underset{\underline{SgF}/F_i}{\longleftarrow}}{}^{(q)} D \to$$

where the differentials are the alternating sum of group-morphisms

$$\lim_{\underset{\underline{SgF}/F_{i-1}}{\longleftarrow}}{}^{(q)} D \longrightarrow \lim_{\underset{\underline{SgF}/F_i}{\longleftarrow}}{}^{(q)} D$$

induced by the projections $F_i \to F_{i-1}$. In this situation, there is a spectral sequence

$$E_2^{p,q} = H^p(\lim_{\underset{\underline{SgF}/F_*}{\longleftarrow}}{}^{(q)} D)$$

which is the homology of the complex above, converging to

$$\lim_{\underset{\underline{SgF}/A}{\longleftarrow}}{}^{(\cdot)} D \, .$$

Correspondingly, there is a spectral sequence

$$E_2^{p,q} = H^p(\lim_{\underset{\underline{SF}/F_*}{\longleftarrow}}{}^{(q)} \mathrm{Der}_S(-,M))$$

converging to

$$\lim_{\underset{\underline{SF}/A}{\longleftarrow}}{}^{(\cdot)} \mathrm{Der}_S(-,M) \, .$$

We shall prove that the canonical morphism

$$\lim_{\underset{\underline{SF}/A}{\longleftarrow}}{}^{(i)} \mathrm{Der}_S(-,M) \to \lim_{\underset{\underline{SgF}/A}{\longleftarrow}}{}^{(i)} D$$

is an isomorphism by induction on $i \geq 0$. For $i = 0$, the isomorphism is trivial since there is a commutative diagram

$$\begin{array}{ccc} \varprojlim_{\underline{SF}/A} \mathrm{Der}_S(-,M) & \longrightarrow & \varprojlim_{\underline{SgF}/A} D \\ \parallel & & \parallel \\ \mathrm{Der}_S(\varinjlim_{\underline{SF}/A} f, M) & \longrightarrow & \mathrm{Der}_S(\varinjlim_{\underline{SgF}/A} f \circ j, M) \end{array}$$

where $f : \underline{SF}/A \to \underline{Ab}$ is the functor

$$f(F \xrightarrow[\varphi]{} A) = F .$$

And moreover

$$A \simeq \varinjlim_{\underline{SgF}/A} f \circ j \xrightarrow{\sim} \varinjlim_{\underline{SF}/A} f .$$

Assuming the isomorphism for $i \leq n$ and for every object A in $\underline{Sg\text{-}alg}$, we conclude that the morphism

$$'E_2^{p,q} \longrightarrow E_2^{p,q}$$

is an isomorphism for $q \leq n$ and every p .

Since by definition

$$\varprojlim_{\underline{SF}/F}{}^{(q)} \mathrm{Der}_S(-,M) = H^q(S,F,M)$$

then $'E_2^{o,q} = 0$ for every q .

Moreover

$$\varprojlim_{\underline{SgF}/F}{}^{(q)} D = 0$$

since $F \in \mathrm{ob}\ \underline{SgF}$, proving that $E_2^{o,q} = 0$ for all q .

By theory for spectral sequences, we know that there are morphisms

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

such that

$$E_{r+1}^{p,q} = \ker d_r^{p,q} / \mathrm{im}\ d_r^{p-r,q+r-1} .$$

Furthermore for given p and q , then

$$E_\infty^{p,q} = E_r^{p,q}$$

if r is big enough .

With this in mind, we conclude that the morphisms

$${}'E_\infty^{p,q} \longrightarrow E_\infty^{p,q}$$

are isomorphisms for every p and q such that

$$p+q \leq n+1 .$$

Hence we have proved

LEMMA 1.7

The forgetful functor

$$j : \underline{SgF}/A \to \underline{SF}/A$$

induce isomorphisms

$$H^i(S,A,M) \xrightarrow{\sim} \varprojlim_{\underline{SgF}/A}{}^{(i)} \mathrm{Der}_S(-,M)$$

for every $i \geq 0$.

Since there will be no confusion, we simply write $\mathrm{Der}_S(-,M)$ instead of $\mathrm{Der}_S(-,M)\circ j$.

Assume that S is noetherian and that

$$S \to A$$

is finitely generated.

Let

$$(\underline{SgF}/A)_{fg} \subseteq \underline{SgF}/A$$

be the full subcategory whose objects $\varphi : F \to A$ have the property that F is a finitely generated S-algebra.

I can prove

LEMMA 1.8

If S is noetherian and A a finitely generated S-algebra, then the canonical morphisms

$$_kH^i(S,A,M) \longrightarrow \varprojlim_{(\underline{SgF}/A)_{fg}}{}^{(i)} \; [_k\mathrm{Der}_S(-,M)]$$

and

$$H_i(S,A,M) \longrightarrow \varprojlim_{(\underline{SgF}/A)_{fg}}{}^{(i)} \; \mathrm{Der}_S(-,M)$$

are isomorphisms for $i \geq 0$.

Again it is clear that the derivation functors are composed with the obvious forgetful functors.

<u>Proof</u>

Let us prove the isomorphism

$$H^i(S,A,M) \longrightarrow \varprojlim_{(\underline{SgF}/A)_{fg}}{}^{(i)} \; \mathrm{Der}_S(-,M)$$

Choose a graded S-algebrasurjection

$$F \to A$$

such that F is a finitely generated S-algebra. Then, with the same notations as in the proof of lemma 1.7, there is a spectral sequence

$$''E_2^{p,q} = H^p(\varprojlim_{(\underline{SgF}/F_{\bullet})_{fg}}{}^{(q)} \ \mathrm{Der}_S(-,M))$$

converging to

$$\varprojlim_{(\underline{SgF}/A)_{fg}}{}^{(\cdot)} \ \mathrm{Der}_S(-,M) \ .$$

Since F is finitely generated, then $''E_2^{o,q} = 0$ for every q. Since F_i is finitely generated too, the induction argument from lemma 1.7 goes through. Q.E.D.

Putting this together, we get

THEOREM 1.9

If $S \to A$ is any graded (or homogeneous) morphism and M is a graded A-module, then there is a canonical injection

$$\coprod_{k=-\infty}^{\infty} {}_kH^i(S,A,M) \longrightarrow H^i(S,A,M) \ .$$

If S is noetherian and A is a finitely generated S-algebra, then the injection above is an isomorphism for every $i \geq 0$.

Proof

There is a canonical morphism of functors

$$\coprod_{k=-\infty}^{\infty} [_k \ \mathrm{Der}_S(-,M)] \longrightarrow \mathrm{Der}_S(-,M)$$

on the category $\underline{SgF}/A$ which is an isomorphism if we restrict to the category $(\underline{SgF}/A)_{fg}$. Hence the lemmaes complete the proof.

Q.E.D.

Let $R \xrightarrow{\pi} S$ be a graded surjection such that

$$(\ker \pi)^2 = 0 \ .$$

It is easy to see that the injection

$$_{o}H^2(S,A,A \otimes_S \ker \pi) \longrightarrow H^2(S,A,A \otimes_S \ker \pi)$$

maps the obstruction $o_o(A)$ onto $o(A)$, which proves

COROLLARY 1.10

Let $R \xrightarrow{\pi} S$ be a graded surjection such that $(\ker \pi)^2 = 0$. If A is a graded S-algebra, then A can be lifted to R if and only if A can be lifted to a graded R-algebra.

Correspondingly we prove

COROLLARY 1.11

Let $R \xrightarrow{\pi} S$ satisfy $(\ker \pi)^2 = 0$. If $\varphi : A \to B$ is a graded (or homogeneous) S-algebrahomomorphism and A' and B' are graded liftings of A and B respectively, then φ can be lifted to R with respect to A' and B' if and only if φ can be lifted to a graded R-algebrahomomorphism from A' to B' .

REMARK

1. Corollary 1.11 can be generalized in the following way. Let $R \xrightarrow{\pi} S$ satisfy $(\ker \pi)^2 = 0$ and let

$$\varphi : A \to B$$

be any graded S-algebramorphism. Assume that there are liftings A'' and B'' , not necessarily graded, of A and B respectively such that φ can be lifted to R with respect to A'' and B'' . Then there are graded liftings A' and B' of A and B such that φ can be lifted to a graded R-algebrahomomorphism with respect to A' and B' . We express this by saying that φ admits a graded lifting to R if and only if φ admits a lifting.

We omit the proof.

2. Similar results are true for graded S-modules and for graded morphisms of S-modules.

Chapter 2

(GRADED) DEFORMATION FUNCTORS AND HULLS.

In this chapter we will study the relationship between hulls for the graded and non-graded deformation functors. We will deform or lift only noetherian algebras, but the hulls need not be noetherian.

To be more precise, let Λ be a noetherian ring with maximal ideal m_Λ and residue field $k = \Lambda/m_\Lambda$. Let $\underline{C}$ be the category whose objects are artinian local Λ-algebras with residue fields k and the morphisms are local Λ-homomorphisms. Moreover let $\underline{C}_n$ be the full subcategory of $\underline{C}$ whose objects R satisfy $m_R^n = 0$ where m_R is the maximal ideal of R and n an integer. We get to pro-$\underline{C}$ objects by taking projective limits of objects from $\underline{C}$.

Let A be a graded k-algebra.

If $R \in \text{ob}\, \underline{C}$, we let

$$\mathrm{Def}^{\mathrm{o}}(A/k,R) = \left\{ \begin{array}{ccc} R & \to & A' \\ \downarrow & \circ & \downarrow \\ k & \to & A \end{array} \middle|\ A' \text{ is a graded lifting} \right\} \Big/ \sim$$

where $\sim$ is an equivalence relation, given by saying that two deformations $R \to A'$ and $R \to A''$ is equivalent if they are isomorphic in the following sense

$$\begin{tikzcd} & & A' \arrow[dd, bend left] \\ R \arrow[r] \arrow[urr] \arrow[d] & A' \arrow[ur, "\simeq"'] \arrow[d] & \\ k \arrow[r] & A & \end{tikzcd}$$

where all diagrams commutes. It is easy to see that $\mathrm{Def}^{\mathrm{o}}(A/k,-)$

is a covariant functor on $\underline{C}$ with values in <u>Setz</u> and we call it <u>the graded deformation functor for $k \to A$</u>. Correspondingly, if A is an arbritrary k-algebra, we let $\mathrm{Def}(A/k,-)$ be its deformation functor. Since we work with noetherian k-algebras, these functors have always hulls. Schlessinger's general theorem applies when $H^1(k,A,A)$ is a finite k-vectorspace [S] and Laudal proves it in general in [L2] .

NOTATION

If F is a functor from $\underline{C}$ to <u>Setz</u>, we let

$$t_F = F(k[\varepsilon])$$

and call it the tangent space to F. $k[\varepsilon] \in \mathrm{ob}\ \underline{C}_2$ is the dual numbers.

For the general situation in [L2], let us recall

DEFINITION 2.1

Let A be a k-algebra. A pro-$\underline{C}$ object $R(A)$ is called a hull for $\mathrm{Def}(A/k,-)$ if there is a smooth morphism of functors

$$\mathrm{Hom}_{\Lambda}^{\mathrm{cont}}(R(A),-) \longrightarrow \mathrm{Def}(A/k,-)$$

on $\underline{C}$ which is an isomorphism on its tangent spaces.

Recall that a morphism of functors

$$F \longrightarrow G$$

on $\underline{C}$ is smooth iff the map

$$F(c) \longrightarrow F(d) \times_{G(d)} G(c)$$

is surjective when $c \to d$ is surjective. $h_{R(A)} = \mathrm{Hom}_{\Lambda}^{\mathrm{cont}}(R(A),-)$ denotes continuous local Λ-homomorphisms.

Hulls defined as above, are unique by non-canonical isomorphisms.

With A graded, we define the hull $R^o(A)$ for the functor $\mathrm{Def}^o(A/k,-)$ correspondingly.

If we let

$$R_n(A) = R(A)/m_{R(A)}^n$$

where $m_{R(A)} \subseteq R(A)$ is the maximal ideal, then $R_n(A)$ will be a hull for $\mathrm{Def}(A/k,-)$ restricted to $\underline{C}_n$. In general $R_n(A)$ is not an object of $\underline{C}_n$. However, if the k-vectorspace $H^1(k,A,A)$ is finite dimensional, it is, and we can forget everything about continuity in the definition. For further details, see [L2] .

Let A be a graded k-algebra. Consider the canonical morphism of functors

$$\mathrm{Def}^o(A/k,-) \longrightarrow \mathrm{Def}(A/k,-) .$$

When does it split? To avoid difficulties, we will ask for conditions which guarantees the existence of a section of

$$R(A) \longrightarrow R^o(A) .$$

In fact we will show that a certain kind of rigidity will do .

DEFINITION 2.2

We shall say that $k \to A$ has negative grading (respectively positive grading) if

$${}_{\nu}H^1(k,A,A) = 0 \quad \text{for} \quad \nu > 0$$

$$(\text{respectively} \quad {}_{\nu}H^1(k,A,A) = 0 \quad \text{for} \quad \nu < 0) .$$

If A has negative or positive grading, then

$$R(A) \longrightarrow R^o(A)$$

admits a section. This will follow from the existence of an isomorphism

$$R(A) \simeq R^o(B)$$

where $B = A[T]$ is a polynomial ring in one variable over A, considered as a graded ring by choosing a suitable degree of T.

To begin with, let us prove this isomorphism rather formally.

Let A and B be k-algebras, not necessarily graded, and

$$\varphi : B \to A$$

a k-algebrahomomorphism. φ induces maps

$$\varphi^* : H^i(k,A,A) \longrightarrow H^i(k,B,A)$$

for every $i \geq 0$.

Let $R \xrightarrow[\pi]{} S$ be a small surjection from $\underline{C}$, it is such that

$$m_R \cdot \ker \pi = 0$$

where m_R is the maximal ideal in R.

Consider the commutative diagram

$$(*) \qquad \begin{array}{ccccc} R & \longrightarrow & B' & & \\ \downarrow & & \downarrow & & \\ S & \longrightarrow & B_1 & \xrightarrow{\varphi_1} & A_1 \\ \downarrow & & \downarrow & \circ & \downarrow \\ k & \longrightarrow & B & \xrightarrow[\varphi]{} & A \end{array}$$

where A_1, B_1 lifts A and B respectively and B' lifts B_1.

LEMMA 2.3

If φ^* is bijective for $i = 1$ and injective for $i = 2$, then a given diagram $(*)$ can be completed to a commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & B' & \xrightarrow{\varphi'} & A' \\ \pi\downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & B_1 & \xrightarrow[\varphi_1]{} & A_1 \end{array} .$$

If $\varphi' : B' \to A'$ and $\varphi'' : B' \to A''$ both complete (*) , then A' and A'' are equivalent liftings of A_1 to R .

Proof

Consider the diagram

$$\begin{array}{ccc} & & H^2(k,B,B) \otimes_k I \\ & & \downarrow \varphi_* \otimes id_I \\ H^2(k,A,A) \otimes_k I & \xrightarrow[\varphi^* \otimes id_I]{} & H^2(k,B,A) \otimes_k I \end{array}$$

where id_I is the identity on $I = \ker \pi$. Due to [LI] theorem 2.2.5 which says that the obstructions for deforming A_1 and B_1 to R are mapped on the same element in $H^2(k,B,A) \otimes_k I$, we conclude by the injectivity of φ^* that A_1 can be lifted to R .

Moreover

$$H^1(k,A,A) \otimes_k I \xrightarrow[\varphi^* \otimes id_I]{} H^1(k,B,A) \otimes_k I$$

is an isomorphism.

Due to [LI] theorem 3.1.6 (see remark 2), surjectivity gives the existence of a diagram

$$\begin{array}{ccccc} R & \longrightarrow & B' & \xrightarrow{\varphi'} & A' \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & B_1 & \xrightarrow[\varphi_1]{} & A_1 \end{array}$$

and the injectivity gives uniqueness of the lifting A' .

Q.E.D.

REMARK 1

The conditions of lemma 2.3 is fulfilled if

$$H^2(B,A,A) = 0$$

and if there is a k-algebrahomomorphism

$$j : A \to B$$

such that $\varphi \cdot j = id_A$, the identity on A . In fact the existence of $j : A \to B$ implies that

$$H^i(k,A,A) \xrightarrow[\varphi^*]{} H^i(k,B,A)$$

is injective for all $i \geq 0$.

By the exact sequence

$$\to H^{i-1}(k,B,A) \to H^i(B,A,A) \to H^i(k,A,A) \xrightarrow[\varphi^*]{} H^i(k,B,A) \to$$

the remark is proved.

REMARK 2

Theorem 3.1.6 in [LI] says that if B' lifts B_1 and A' and A'' lifts A_1 to R then

$$(\varphi^* \otimes id_I)(\lambda) = o(\varphi_1;B',A') - o(\varphi_1;B',A'')$$

when

$\lambda \in H^1(k,A,A) \otimes_k I$ corresponds to the difference $A''-A'$ and $o(\varphi_1;B',A') \in H^1(k,B,A) \otimes_k I$ is the obstruction for lifting φ_1 to R with respect to B' and A' . Correspondingly if we keep a lifting A' fixed, and let B' and B'' be two liftings of B_1 to R , then

$$(\varphi_* \otimes id_I)(\mu) = o(\varphi_1;B',A') - o(\varphi_1;B'',A')$$

when μ is given by the difference $B'-B''$, $\mu \in H^1(k,B,B) \otimes_k I$

(see [LI], theorem 3.1.3). With this in mind and assuming the conditions of lemma 2.3, we have: Given two commutative diagrams

$$\begin{array}{ccccc} k[\epsilon] & \longrightarrow & B_o & \xrightarrow{\varphi_o} & A_o \\ \downarrow & & \downarrow & & \downarrow \\ k & \longrightarrow & B & \longrightarrow & A \end{array} \qquad \begin{array}{ccccc} k[\epsilon] & \longrightarrow & B' & \xrightarrow{\varphi'} & A' \\ \downarrow & & \downarrow & \circ & \downarrow \\ k & \longrightarrow & B & \xrightarrow[\varphi]{} & A \end{array}$$

then the conposed map

$$H^1(k,B,B) \xrightarrow[(\varphi^*)^{-1}\circ\varphi_*]{} H^1(k,A,A)$$

maps

$$\mu = B'-B_o \quad \text{onto} \quad \lambda = A'-A_o \ .$$

This proves

COROLLARY 2.4

With conditions from lemma 2.3, there is a local Λ-morphism

$$R(\varphi)^* : R(A) \longrightarrow R(B)$$

such that

$$\begin{array}{ccc} t_{R(A)} & \longleftarrow & t_{R(B)} \\ \| & & \| \\ H^1(k,A,A) & \xleftarrow[{[\varphi^*]^{-1}\cdot\varphi_*}]{} & H^1(k,B,B) \end{array}$$

commutes.

Here, $t_{R(A)}$ is an abreviated notation for the tangent space $t_{h_{R(A)}}$.

Proof

By definition of $R(B)$, there is a lifting $\bar{B}$ of B to $R(B)$, called versal. By lemma 2.3 and by the definition of $R(A)$, there is a local Λ-homomorphism

$$R(A) \longrightarrow R(B) \ .$$

The commutative diagram for the tangent space follows from remark 2.

Q.E.D.

If we assume that B is a graded k-algebra and

$$\varphi : B \to A$$

a k-algebrahomomorphism (A not necessarily graded), the consider the diagram

$$(**)\qquad \begin{tikzcd} R \arrow[d] & & A' \arrow[d] \\ S \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\circ"] & B_1 \arrow[r, "\varphi_1"] \arrow[d] \arrow[r, phantom, shift right=4, "\circ"] & A_1 \arrow[d] \\ k \arrow[r] & B \arrow[r, "\varphi"'] & A \end{tikzcd}$$

where A_1 and A' are liftings and where B_1 is a graded lifting of B to S.

Look at the composed map

$$_0H^i(k,B,B) \longrightarrow H^i(k,B,B) \xrightarrow[\varphi_*]{} H^i(k,B,A)$$

and call it $\varphi_*/_0$.

LEMMA 2.5

If $\varphi_*/_0$ is bijective for $i = 1$ and injective for $i = 2$, then the given diagram (**) can be completed to

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\pi"'] & B' \arrow[r, "\varphi'"] \arrow[d] & A' \arrow[d] \\ S \arrow[r] & B_1 \arrow[r, "\varphi_1"'] & A_1 \end{tikzcd}$$

where B' is a graded lifting of B_1 to R. If $\varphi' : B' \to A'$ and $\varphi'' : B'' \to A'$ complete the diagram (**) in this way, then B' and B'' are equivalent graded liftings of B_1.

Proof

I claim that B_1 can be lifted to R.

For look at

$$\begin{array}{ccc} & & H^2(k,B,B) \otimes I \\ & & \downarrow \varphi_* \otimes id_I \\ H^2(k,A,A) \otimes I & \xrightarrow[\varphi^* \otimes id_I]{} & H^2(k,B,A) \otimes I \end{array}$$

where $I = \ker \pi$.

Since B_1 is graded, the obstruction for lifting is in

$$_oH^2(k,B,B) \otimes I$$

(chapter 1, theorem 1.5). Since the graded obstruction $o_o(B)$ in $_oH^2(k,B,B)\otimes I$ is mapped onto $o(B)$ in $H^2(k,B,B)\otimes I$, it is enough to prove that the composed map

$$_oH^2(k,B,B)\otimes_k I \to H^2(k,B,B)\otimes_k I \xrightarrow[\varphi_*\otimes id_I]{} H^2(k,B,A)\otimes_k I$$

is injective which is an assumption.

Let B'' be a graded lifting to R .

Since

$$_oH^1(k,B,B) \otimes I \longrightarrow H^1(k,B,A) \otimes I$$

is surjective, there is a $\lambda \in {}_oH^1(k,B,B)\otimes I$ such that

$$(\varphi_*\otimes id_I)(\lambda) = o(\varphi_1;B'',A') .$$

If we define B' by

$$\lambda = B''-B' \in {}_oH^1(k,B,B) \otimes I$$

then B' is a graded lifting of B_1 to R by theorem 1.5.

By remark 2

$$(\varphi_*\otimes id_I)(\lambda) = o(\varphi_1;B'',A') - o(\varphi_1;B',A') .$$

Hence $$o(\varphi_1;B',A') = 0 .$$

The same calculations will show uniqueness of B' . Q.E.D.

By now it is clear that

COROLLARY 2.6

With assumption from lemma 2.5, there is a local $\wedge$-homomorphism

$$R(\varphi)_* : R^o(B) \to R(A)$$

such that

$$\begin{array}{ccc} t_{R^o(B)} & \longleftarrow & t_{R(A)} \\ \| & & \| \\ {}_oH^1(k,B,B) & \xleftarrow[(\varphi_*/_o)^{-1}\cdot\varphi^*]{} & H^1(k,A,A) \end{array}$$

commutes.

We compose the morphism from 2.4

$$R(\varphi)^* : R(A) \to R(B)$$

with the canonical

$$R(B) \to R^o(B)$$

and call the composition $R(\varphi)^*$ too .

COROLLARY 2.7

Assume the conditions of lemma 2.3 and 2.5.

Then the local $\wedge$-morphisms

$$R(\varphi)^* : R(A) \to R^o(B)$$

$$R(\varphi)_* : R^o(B) \to R(A)$$

are isomorphisms.

<u>Proof</u>

To prove that

$$R(\varphi)_* : R^o(B) \to R(A)$$

is an isomorphism, it is enough to prove that

$$h_{R(A)}(-) \longrightarrow h_{R^o(B)}(-)$$

is an isomorphism on $\underline{C}$.

The corollaries say that we have isomorphisms for the tangent spaces.

I claim that we have isomorphisms on $\underline{C}_2$. For if $R \in$ ob $\underline{C}_2$ with maximal ideal m_R , then either

$$h_{R(A)}(R) \quad \text{and} \quad h_{R^o(B)}(R)$$

are empty, or we have commutative diagrams

$$\begin{array}{ccc} h_{R(A)}(R) & \longrightarrow & h_{R^o(B)}(R) \\ \wr\| & \circ & \wr\| \\ \mathrm{Der}^{\mathrm{cont}}_{\Lambda}(R(A),m_R) & \longrightarrow & \mathrm{Der}^{\mathrm{cont}}_{\Lambda}(R^o(B),m_R) \\ \| & \circ & \| \\ h_{R(A)}(k[m_R]) & \longrightarrow & h_{R^o(B)}(k[m_R]) \end{array}$$

when $k[m_R] = k \oplus m_R$ is the dual numbers.

We go on by induction.

Let $R \in$ ob $\underline{C}_n$ and look at the diagram

$$\begin{array}{ccc} h_{R(A)}(R) & \longrightarrow & h_{R^o(B)}(R) \\ \downarrow & \circ & \downarrow \\ h_{R(A)}(R/m_R^{n-1}) & \longrightarrow & h_{R^o(B)}(R/m_R^{n-1}) \end{array}$$

assuming that the lower horisontal map is an isomorphism.

But the fibers of the vertical maps are derivations. Since $(m_R^{n-1})^2 = 0$, we get

$$\begin{array}{ccc} \mathrm{Der}^{\mathrm{cont}}_{\Lambda}(R(A),m_R^{n-1}) & \longrightarrow & \mathrm{Der}^{\mathrm{cont}}_{\Lambda}(R^o(B),m_R^{n-1}) \\ \| & \circ & \| \\ \mathrm{Der}^{\mathrm{cont}}_{\Lambda}(R_2(A),m_R^{n-1}) & \longrightarrow & \mathrm{Der}^{\mathrm{cont}}_{\Lambda}(R_2^o(B),m_R^{n-1}) \end{array}$$

where

$$R_2(A) = R(A)/m^2_{R(A)} \qquad R^o_2(B) = R^o(B)/m^2_{R^o(B)}$$

are as usual.

Hence the fibers are isomorphic. This proves injectivity of

$$h_{R(A)}(R) \longrightarrow h_{R^o(B)}(R)$$

and surjectivity too if we use the existence of

$$R(\varphi)^* : R(A) \longrightarrow R^o(B)$$

(or simply, if we use lemma 2.3). Q.E.D.

This is the formal result we need. It should be remarked that corollary 2.7 becomes rather trivial when we use the theory developed in [L2], giving an explicit form for the hulls.

Let $B = A[T]$, where A is a given graded k-algebra. We consider the polynomial ring B in one variable as graded by claiming

$$\deg T = 1 \qquad (\text{respectively} \quad \deg T = -1)$$

Let

$$\varphi : B \to A$$

be the composed morphism

$$B = A[T] \longrightarrow A[T]/(T-1) \simeq A$$

$T-1$ is a regular element in B ; hence

$$H^i(B,A,A) = 0 \quad \text{for} \quad i \geq 2 .$$

By remark 1, the conditions of lemma 2.3 is verified for $\varphi : B \to A$.

The long exact sequence associated to

$$k \longrightarrow A \xrightarrow[j]{} B = A[T]$$

where j is the obvious morphism, proves that

$$H^i(k,B,M) \xrightarrow[j^*]{} H^i(k,A,M)$$

is an isomorphism for any B-module M and $i \geq 1$.

Look at the commutative diagram

$$\begin{tikzcd}
{}_oH^i(k,B,B) \arrow[r] \arrow[d, "j^*"] & H^i(k,B,B) \arrow[r, "\varphi_*"'] \arrow[d, "j^*"] & H^i(k,B,A) \arrow[d, "j^*"] \\
{}_oH^i(k,A,B) \arrow[r] & H^i(k,A,B) \arrow[r, "\varphi_*"'] \arrow[d, "\wr"'] & H^i(k,A,A) \arrow[d, equal] \\
& H^i(k,A,A)\otimes_k k[T] \arrow[r, "id\otimes_k\psi"'] & H^i(k,A,A)
\end{tikzcd}$$

where id is the identity on $H^1(k,A,A)$ and

$$\psi : k[T] \to k$$

is the composed map

$$k[T] \to k[T]/(T-1) \simeq k .$$

Since

$$ {}_oH^i(k,A,B) = (H^i(k,A,A)\otimes_k k[T])_o \simeq \coprod_{\nu=-\infty}^{o} {}_{\nu}H^i(k,A,A)T^{-\nu}$$

when $\deg T = 1$, it follows that $\varphi_*/_o$ is given by

$$\begin{tikzcd}
{}_oH^i(k,B,B) \arrow[rr, "\varphi_*/_o"] \arrow[d, equal] & & H^i(k,B,A) \arrow[d, "j^*"] \\
\coprod\limits_{\nu=-\infty}^{o} {}_{\nu}H^i(k,A,A)T^{-\nu} \arrow[r, "\simeq"'] & \coprod\limits_{\nu=-\infty}^{o} {}_{\nu}H^i(k,A,A) \arrow[r] & H^i(k,A,A)
\end{tikzcd}$$

where the lower horisontal isomorphism is induced by sending T to 1 and the morphism $\coprod_{\nu=-\infty}^{o} {}_{\nu}H^i(k,A,A) \to H^i(k,A,A)$ is given by theorem 1.9 from the first chapter.

When $\deg T = -1$, we get a diagram

$$\begin{tikzcd}
{}_0H^i(k,B,B) \arrow[r, "\varphi_*/_0"'] \arrow[d, equal] & H^i(k,B,A) \arrow[dr] & \\
\coprod\limits_{\nu=0}^{\infty} {}_\nu H^i(k,A,A)T^\nu \arrow[r, "\simeq"'] & \coprod\limits_{\nu=0}^{\infty} {}_\nu H^i(k,A,A) \arrow[r] & H^i(k,A,A) .
\end{tikzcd}$$

This proves

THEOREM 2.8

Let A be a graded k-algebra and let $B = A[T]$. If A has negative grading (respectively positive grading) and $\deg T = 1$ (respectively $\deg T = -1$), then there is a local Λ-isomorphism

$$R^o(B) \simeq R(A)$$

such that the diagram

$$\begin{tikzcd}
t_{R^o(B)} \arrow[r, dash, "\sim"] \arrow[d, equal] & t_{R(A)} \arrow[dr, equal] & \\
{}_0H^1(k,B,B) \arrow[r, "\sim", "\varphi_*/_0"'] & H^1(k,B,A) & H^1(k,A,A) \arrow[l, "\sim"', "\varphi^*"]
\end{tikzcd}$$

commutes

This result implies the existence of a section of the canonical morphism

$$R(A) \to R^o(A) .$$

Indeed, if we let

$$\alpha : B \to A$$

be the morphism

$$B = A[T] \to A[T]/(T) \simeq A$$

then "a graded" lemma 2.3 guarantees a Λ-morphism

$$R^o(\alpha)^* : R^o(A) \to R^o(B) \; .$$

The composition

$$R^o(A) \xrightarrow{R^o(\alpha)^*} R^o(B) \xrightarrow{R(\varphi)_*} R(A) \xrightarrow[\text{canonical}]{} R^o(A)$$

is an isomorphism

since the corresponding

$$t_{R^o(A)} \longleftarrow t_{R^o(B)} \longleftarrow t_{R(A)} \longleftarrow t_{R^o(A)}$$

is an isomorphism

(the composition $t_{R^o(A)} \to t_{R(A)} \to t_{R^o(B)}$ maps $A' \in t_{R^o(A)}$ onto $A'[T] \in t_{R^o(B)}$) .

This proves

THEOREM 2.9

If A has negative or positive grading, then the canonical morphism

$$R(A) \longrightarrow R^o(A)$$

has a section which is a local Λ-homomorphism.

The converse is not true since there are k-algebras A satisfying $H^2(k,A,-) = 0$ which do not have negative nor positive grading.

REMARK 3

Theorem 2.8 can be generalized in the following way. Let A be a graded k-algebra and let

$$B = A[T_1,T_2]/(T_1T_2-1)$$

where $\deg T_1 = 1$ and $\deg T_2 = -1$. Then we have a Λ-isomorphism

$$R^o(B) \simeq R(A)$$

(without assuming anything about A). However, we do not get the nice application of theorem 2.9 in this case.

Chapter 3

APPLICATION TO HILBERT FUNCTORS AND LIFTING PROBLEMS OF PROJECTIVE GEOMETRY

We want to apply the results from chapter 2 to local Hilbert functors in order to generalize a result of Pinkham [P]. (our theorem 3.2). Again, our algebras are noetherian, but the hulls need not be.

Let

$$\psi : F \to A$$

be a graded k-algebrahomomorphism and assume F to be a free k-algebra. If $R \in \text{ob}\,\underline{C}$, we let F_R be the unique lifting of F to R. We define

$$\mathrm{Def}^o(\psi,R) = \left\{ \begin{array}{ccc} F_R & \xrightarrow{\psi'} & A' \\ \downarrow & o & \downarrow \\ F & \to & A \end{array} \;\middle|\; A' \text{ is a graded lifting; } \psi' \text{ graded} \right\} \Big/ \sim$$

where the equivalence relation is the usual one. Of course, $\mathrm{Def}^o(\psi,-)$ is a covariant functor on $\underline{C}$.

LEMMA 3.1

If $\psi : F \to A$ is surjective, then

$$\mathrm{Def}^o(\psi,-)$$

is prorepresentable.

Proof

This is easy since it is enough to prove that F-automorphisms of A can be lifted to F_R-automorphisms of A'. [S]

Q.E.D.

Let

$$B = A[T]$$

where $\deg T = 1$.

If F is a free k-algebra such that

$$\psi : F \to A$$

is surjective and graded, then let

$$\bar{\psi} = \psi \underset{k}{\otimes} \mathrm{id}_{k[T]} : \bar{F} = F[T] \to B = A[T]$$

where $\mathrm{id}_{k[T]}$ is the identity.

The canonical map

$$\mathrm{Def}^o(\bar{\psi},-) \to \mathrm{Def}^o(B/k,-)$$

gives a local Λ-morphism

$$R^o(\bar{\psi}) \leftarrow R^o(B)$$

The map

$$\mathrm{Def}^o(\bar{\psi},-) \to \mathrm{Def}^o(B/k,-)$$

is clearly smooth. Hence

$$h_{R^o(\bar{\psi})} \to h_{R^o(B)}$$

is smooth too. Indeed, if $R \overset{\pi}{\to} S$ is surjective such that

$$m_R \cdot \ker \pi = 0$$

where $m_R \subseteq R$ is its maximal ideal, then it is enough to prove that the morphism of the "fibers"

$$\mathrm{Der}_\Lambda^{\mathrm{cont}}(R^o(\bar{\psi}),I) \to \mathrm{Der}_\Lambda^{\mathrm{cont}}(R^o(B),I)$$

is surjective, where $I = \ker \pi$.

This is true since

$$\begin{array}{ccc}
\mathrm{Der}_{\Lambda}^{\mathrm{cont}}(R^o(\bar{\psi}),I) & \to & \mathrm{Der}_{\Lambda}^{\mathrm{cont}}(R^o(B),I) \\
\| & & \| \\
\mathrm{Der}_{\Lambda}^{\mathrm{cont}}(R_2^o(\bar{\psi}),I) & \to & \mathrm{Der}_{\Lambda}^{\mathrm{cont}}(R_2^o(B),I) \\
\| & & \| \\
h_{R^o(\bar{\psi})}(k[I]) & \to & h_{R^o(B)}(k[I])
\end{array}$$

commutes.

If A has negative grading, then theorem 2.8 gives a diagram

$$\begin{array}{ccccc}
h_{R^o(\bar{\psi})} & \to & h_{R^o(B)} & \tilde{\to} & h_{R(A)} \\
\wr\downarrow & & \downarrow & & \downarrow \\
\mathrm{Def}^o(\bar{\psi},-) & \to & \mathrm{Def}^o(B/k,-) & & \mathrm{Def}(A/k,-)
\end{array}$$

which proves that there is a smooth morphism of functors

$$\mathrm{Def}^o(\bar{\psi},-) \to \mathrm{Def}(A/k,-)$$

We shall enter into projective geometry. Assume therefore that F and A are positively graded, that

$$F_o = A_o = k$$

and that the elements of degree one generate the algebras.

We denote by

$$X = \mathrm{Proj}(A)$$

and $$Y = \mathrm{Proj}(B) = \mathrm{Proj}(A[T])$$

its projective cone.

In a moment we shall prove that

$$\mathrm{Def}^o(\bar{\psi},-) \simeq \mathrm{Hilb}_Y(\mathrm{Proj}(\bar{\psi}),-)$$

when X is normally projective. Hence

THEOREM 3.2

Let $X = \mathrm{Proj}(A)$ be a normally projective scheme in $\mathbb{P}^n_k$ and let $Y = \mathrm{Proj}(A[T])$ be its projective cone in $\mathbb{P}^{n+1}_k$. Let

$$g : Y \to \mathbb{P}^{n+1}_k$$

be the induced embedding.

If A has negative grading, then there is a smooth morphism of functors

$$\mathrm{Hilb}_Y(g,-) \to \mathrm{Def}(A/k,-)$$

on $\underline{C}$.

Loosely speaking, the morphism

$$\mathrm{Hilb}_Y(g,-) \to \mathrm{Def}(A/k,-)$$

is induced by sending T to 1.

It remains to establish the isomorphism

$$\mathrm{Def}^o(\mathbb{V},-) \to \mathrm{Hilb}_Y(g,-)$$

Recall that if

$$X' \to \mathrm{Spec}(R)$$

$R \in \mathrm{ob}\,\underline{C}$, is proper and flat and if

$$H^o(X',O_{X'}(\nu)) \underset{R}{\otimes} k \to H^o(X' \underset{R}{\otimes} k, O_{X'}(\nu) \underset{R}{\otimes} k)$$

is surjective where ν is an integer, then it is an isomorphism and

$$H^o(X',O_{X'}(\nu)) \quad \text{is} \quad R\text{-flat}$$

This can be used to prove

PROPOSITION 3.3

Let R be a local ring with residue field k. Let

$$X = \mathrm{Proj}(A)$$

be a projective k-scheme such that

$$\mathrm{depth}_I A \geq 2$$

where I is the irrelevant maximal ideal. If $X' = \mathrm{Proj}(A')$ is a deformation of X to R, then A' is a graded lifting of A to R given by

$$A' \simeq \coprod_{\nu=0}^{\infty} H^0(X', O_{X'}(\nu))$$

Proof

We follow the proof given by Ellingsrud in [E].

The morphism

$$A' \to \coprod_{\nu} H^0(X', O_{X'}(\nu))$$

give a commutative diagram

$$\begin{array}{ccc} A' \underset{R}{\otimes} k & \to & \coprod_{\nu} [H^0(X', O_{X'}(\nu)) \underset{R}{\otimes} k] \\ \wr\downarrow & & \downarrow \\ A & \to & \coprod_{\nu} H^0(X, O_X(\nu)) \end{array}$$

where

$A' \underset{R}{\otimes} k \simeq A \simeq \coprod_{\nu} H^0(X, O_X(\nu))$ are isomorphisms by the depth condition.

Hence the vertical map on the right is surjective. By base change theorem, it is an isomorphism and

$$H^0(X', O_{X'}(\nu)) \quad \text{is} \quad R\text{-flat}$$

for every ν.

The flatness of $H^0(X', O_{X'}(\nu))$ and Nakayana's lemma imply that the morphism

$$A' \to \coprod_{\nu} H^0(X', O_{X'}(\nu))$$

is an isomorphism. Hence A' is R-flat.

Q.E.D.

COROLLARY 3.4

Let

$$\psi : F \to A$$

be a graded surjection of k-algebras and

$$f = \mathrm{Proj}(\psi) : X \to \mathbb{P}^n_k$$

the corresponding embedding.

If

$$\mathrm{depth}_I A \geq 2$$

where I is the irrelevant maximal ideal, then the canonical morphism

$$\mathrm{Def}^0(\psi,-) \to \mathrm{Hilb}_X(f,\cdot)$$

is an isomorphism.

Proof

Let $R \in \mathrm{ob}\,\underline{C}$ and let

$$f' : X' \to \mathbb{P}^n_R$$

be a deformation of $f : X \to \mathbb{P}^n_k$. Then if F_R is the unique lifting of F to R, we have a surjection

$$\psi' : F_R \to A'$$

corresponding to the embedding above and such that

$$\psi' \otimes_R k \simeq \psi$$

Moreover, by proposition 3.3, A' is a graded lifting given by

$$A' = \coprod_{\nu} H^0(X', O_{X'}(\nu))$$

By the commutative diagram

$$\begin{array}{ccc} \coprod_{\nu} H^{o}(\mathbb{P}^n_R, O_{\mathbb{P}^n_R}(\nu)) & \rightarrow & \coprod_{\nu} H^{o}(X', O_{X'}(\nu)) \\ \| & & \| \\ F_R & \xrightarrow{\psi'} & A' \end{array}$$

we conclude that the map

$$\mathrm{Def}^{o}(\psi, R) \rightarrow \mathrm{Hilb}_X(f, R)$$

is bijective.

Q.E.D.

With this corollary, we are through with theorem 3.2.

We will ask. What kind of relationship do we have between the cohomology groups

$${}_oH^i(F, A, A)$$

and

$$A^i(k, f, O_X)$$

where $\psi : F \rightarrow A$ is surjective and

$f : X \rightarrow \mathbb{P}^n_k$

is the corresponding embedding. And we can ask the same question for the cohomology groups

$${}_oH^i(k, A, A)$$

and

$$A^i(k, X, O_X)$$

For the definition of the groups $A^i(k, f, O_X)$ and $A^i(k, X, O_X)$, called the global cohomology groups of algebras, see [L2].
Recall however, that if X is k-smooth, then

$$A^i(k, X, O_X) = H^i(X, \theta_X)$$

where θ_X is the sheaf of derivations on X.

By [L2] we note that the groups

$$A^2(k,X,O_X) \underset{k}{\otimes} (-)$$

contain the obstructions for deforming X as a scheme and $A^1(k,X,O_X) \underset{k}{\otimes} (-)$ measures how many deformations we have. If we want to deform the embedding $f : X \to \mathbb{P}^n_k$, a similar result is true if we use the groups $A^i(k,f,O_X) \underset{k}{\otimes} (-)$ for $i = 1,2$.

REMARK 1

Let $R \overset{\pi}{\to} S$ satisfy $m_R \cdot \ker \pi = 0$ where $m_R \subseteq R$ is the maximal ideal. Let $X' = \mathrm{Proj}(A')$ be a deformation of X to S. By 3.3 and 3.4 we conclude that if

$$\mathrm{depth}_I A \geq 2$$

then the obstruction

$$\sigma_o(A) \in {}_oH^2(k,A,A) \underset{k}{\otimes} \ker \pi$$

is zero if and only if X' can be lifted to R as a projective scheme. And moreover, if $\sigma_o(A) = 0$, then the set of non-equivalent projective R-schemes which lift X' is a principal homogeneous space over ${}_oH^1(k,A,A) \underset{k}{\otimes} \ker \pi$.

We shall see that 3.4 and this remark has much to do with our question when $i = 1$ or $i = 2$.

Let us first prove a general theorem about the relationship. As usual we let $A = \coprod_{\nu=0}^{\infty} A_\nu$ be a graded k-algebra such that $A_o = k$ and such that A is generated by A_1. Then $I = \coprod_{\nu=1}^{\infty} A_\nu$ is its irrelevant maximal ideal. If $X = \mathrm{Proj}(A)$ and M is any finitely generated and graded A-module, then we can prove

THEOREM 3.5

There is a canonical morphism

$$_{\nu}H^{i}(k,A,M) \rightarrow A^{i}(k,X,\widetilde{M}(\nu))$$

for every $i \geq 0$ and every ν.

If

$$\operatorname{depth}_I M \geq n+2$$

then the morphism above is injective for $i = n$ and bijective for $1 \leq i < n$

Proof

We let

$$Y = \operatorname{Spec}(A)$$

$$Z = \operatorname{Spec}(A/I)$$

and $$U = Y - Z$$

Let

$$e : U \rightarrow X$$

be the canonical morphism. e is both smooth and affine.

In [L2] we find two long exact sequences

(1) $$\rightarrow A_Z^{i}(k,Y,\widetilde{M}) \rightarrow A^{i}(k,Y,\widetilde{M}) \rightarrow A^{i}(k,U,\widetilde{M}) \rightarrow A_Z^{i+1}(k,Y,\widetilde{M})$$

and

(2) $$\rightarrow A^{i}(k,e,\widetilde{M}) \rightarrow A^{i}(k,U,\widetilde{M}) \rightarrow A^{i}(k,X,e_*\widetilde{M}) \rightarrow A^{i+1}(k,e,\widetilde{M}) \rightarrow$$

Since

$$H^{i}(k,A,M) = A^{i}(k,Y,\widetilde{M})$$

it is trivial that there are canonical morphisms

$$H^{i}(k,A,M) \rightarrow A^{i}(k,X,e_*\widetilde{M})$$

defined by the composition

$$\begin{array}{ccc} H^i(k,A,M) = A^i(k,Y,\widetilde{M}) & & \\ \downarrow & & \\ A^i(k,U,\widetilde{M}) & \to & A^i(k,X,e_*\widetilde{M}) \end{array}$$

Since

$$e_*\widetilde{M} = \coprod_{\nu=-\infty}^{\infty} \widetilde{M}(\nu)$$

we have canonical morphisms

$$\coprod_{\nu=-\infty}^{\infty} [{}_\nu H^i(k,A,M)] \to \coprod_{\nu=-\infty}^{\infty} A^i(k,X,\widetilde{M}(\nu))$$

which clearly factors through

$${}_\nu H^i(k,A,M) \to A^i(k,X,\widetilde{M}(\nu))$$

for every $i \geq 0$ and every ν.

We are through if we can prove that

$$A^i_Z(k,Y,\widetilde{M}) = 0 \quad \text{for} \quad i \leq n+1$$

and that

$$A^i(k,e,\widetilde{M}) = 0 \quad \text{for} \quad 1 \leq i \leq n$$

This will be a consequence of the depth condition. To see this, we use two spectral sequences which we find in [L2].

First there is a spectral sequence

$$E_2^{p,q} = A^p(k,Y,H^q_Z(\widetilde{M}))$$

converging to

$$A^{\bullet}_Z(k,Y,\widetilde{M})$$

where $H^q_Z(\widetilde{M})$ are local cohomology groups with support in Z.

The depth condition imply that

$$H^q_Z(\widetilde{M}) = 0 \qquad \text{for} \quad q \leq n+1$$

Hence the spectral sequence proves that

$$A^i_Z(k,Y,\widetilde{M}) = 0 \qquad \text{for} \quad i \leq n+1$$

Moreover, there is a spectral sequence

$$E_2^{p,q} = H^p(X,\underline{A}^q(e,\tilde{M}))$$

converging to

$$A^{\bullet}(k,e,\tilde{M})$$

The sheaf

$$\underline{A}^q(e,\tilde{M})$$

is a O_X-Module defined by

$$\underline{A}^q(e,\tilde{M})(V) = A^q(B,e^{-1}(V),\tilde{M}(e^{-1}(V)))$$

where

$$V = \mathrm{Spec}(B) \subseteq X$$

is any open affine set in X.

Since e is smooth, we conclude that

$$A^i(k,e,\tilde{M}) = H^i(X,\underline{A}^o(e,\tilde{M}))$$

If $a \in A_\nu$ and

$$V = \mathrm{Spec}(A_{(a)}) \subseteq X = \mathrm{Proj}(A)$$

where $A_{(a)}$ is the elements of degree zero in A_a, then

$$e^{-1}(V) = \mathrm{Spec}(A_a)$$

Hence

$$\underline{A}^o(e,\tilde{M})(V) = A^o(A_{(a)},A_a,M_a) = \mathrm{Der}_{A_{(a)}}(A_a,M_a) \simeq M_a$$

In fact

$$\underline{A}^o(e,\tilde{M}) = e_*\tilde{M}$$

Since $\mathrm{depth}_I M \geq n+2$, then

$$H^i(X,\underline{A}^o(e,\tilde{M})) = 0 \quad \text{for} \quad 1 \leq i \leq n$$

and we are done.

Q.E.D.

For deformation problems it can be useful to see that

COROLLARY 3.6

If

$$\operatorname{depth}_I A \geq 4$$

then the canonical morphism

$${}_oH^2(k,A,A) \to A^2(k,X,O_X)$$

is injective and

$${}_oH^1(k,A,A) \to A^1(k,X,O_X)$$

is bijective.

FINAL COMMENT

Let M be a finitely generated and graded A-module. It is not difficult to see that we have canonical morphisms

$${}_\nu H^i(F,A,M) \to A^i(k,f,\tilde{M}(\nu))$$

for every $i \geq 0$ and every ν. If we assume that

$$\operatorname{depth}_I A \geq 2$$

then by 3.4 and the commutative diagram

$$\begin{array}{ccc} \operatorname{Def}^o(\psi,k[\epsilon]) & \simeq & \operatorname{Hilb}_X(f,k[\epsilon]) \\ \| & & \| \\ {}_oH^1(F,A,A) & \to & A^1(k,f,O_X) \end{array}$$

where $k[\epsilon]$ is the dual numbers, we conclude that

$${}_oH^1(F,A,A) \to A^1(k,f,O_X)$$

is an isomorphism.

Moreover, by remark 1 the morphism

$${}_oH^1(k,A,A) \to A^1(k,X,O_X)$$

is injective.

REMARK 2

If $R \overset{\pi}{\to} S$ satisfy $m_R \cdot \ker\pi = 0$ and if $X' \to \mathbb{P}^n_S$ is a deformation of f to S, then the morphisms

$$_oH^2(k,A,A) \underset{k}{\otimes} \ker\pi \to A^2(k,f,O_X) \underset{k}{\otimes} \ker\pi$$

and

$$A^2(k,f,O_X) \underset{k}{\otimes} \ker\pi \to A^2(k,X,O_X) \underset{k}{\otimes} \ker\pi$$

take obstructions to obstructions.

We have morphisms

$$_oH^2(k,A,A) \simeq {}_oH^2(F,A,A) \to A^2(k,f,O_X)$$

By remark 1 and 2 we can assume "for all obstruction questions" that the morphism above is injective.

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