ISBN 82-553-0227-1
Mathematics
No 12 - June 11 1975

QUASI INVARIANT MEASURES,

SYMMETRIC DIFFUSION PROCESSES

by

Sergio Albeverio and Raphael Höegh-Krohn
Oslo Oslo

PREPRINT SERIES - Matematisk institutt, Universitetet i Oslo

QUASI INVARIANT MEASURES, SYMMETRIC DIFFUSION PROCESSES AND QUANTUM FIELDS*

by

Sergio Albeverio and Raphael Høegh-Krohn
Institute of Mathematics
University of Oslo
Blindern, Oslo (Norway)

ABSTRACT

We show that for a large class of quasi invariant probability measures μ on a separable Hilbert space with a nuclear rigging the Dirichlet form $\int \nabla \bar{f} \cdot \nabla g \, d\mu$ in $L_2(d\mu)$ is closable and its closure defines a positive self-adjoint operator H in $L_2(d\mu)$, with zero as an eigenvalue to the eigenfunction 1, which is simple if and only if μ is ergodic. The connection with the Hamiltonian formalism and canonical commutation relations is also studied.
We show moreover that, for a subclass of quasi invariant measures, H is the infinitesimal generator of a symmetric time homogeneous Markov process on the rigged Hilbert space, with invariant measure μ, and this process is ergodic if and only if μ is ergodic.
Moreover we study perturbations of H and μ as well as weak limits of quasi invariant measures μ_n and their associated Markov processes.
Finally we apply our result to quantum fields. In particular we show that for polynomial interactions in two space-time dimensions the physical vacuum restricted to the time zero fields is a measure μ in the above class of quasi invariant measures and the physical Hamiltonian coincides on a dense domain with the generator given by the Dirichlet form determined by μ.

June 1975

*Work supported by The Norwegian Research Council for Science and the Humanities.

1. Introduction

Within the general theory of Markov stochastic processes with continuous time parameter and finite dimensional state space the class of diffusion processes is of special importance due to its connection with second order partial differential equations. Since moreover every such Markov process is the solution of a stochastic differential equation, one has a beautiful interplay of the theory of partial differential equation, diffusion processes and stochastic differential equations. For this we refer to [1], [2], [3] and to [4] for potential theory.

In this paper we ε in a study of the extension of these subjects, and in particular of the theory of Markov diffusion processes, to the infinite dimensional case.

We first review shortly some previous work.

Early work which can be put in relation with this circle of problems was done, mainly by Friedrichs, Gelfand and Segal, in connection with the study of quantum fields and in particular of the representations of canonical commutation relations, see e.g. [5]. From another point of view Feynman's path integral formulation of quantum dynamics, has given much stimulus, see the references given in [6]. The work on quantum fields has been pursued vigorously in the last decade, within the framework of constructive quantum fiel theory, to which we shall come back later. Let us first however mention some other woek, which was originated primarily by other types of questions.

Daletskii has studied infinite dimensional elliptic operators of second order, parabolic equations and the corresponding stochastic equations on nuclear spaces, see [7], where also many references to related work by him and other investigators are to be found. The coefficients are assumed to be continuous with continuous uniformly bounded Fréchet derivatives and the Cauchy problem is shown to have a unique solution in the space of functions which are uniformly bounded and continuous together with their derivatives up to second order. The parametrix of a class of elliptic differential operators of higher order have been considered by Vishik on certain spaces of sequences [8].

Gross and Piech have studied potential theory on abstract Wiener space [9].

For Wiener processes on Banach manifolds see [10].
Kree has studied the extension of the theory of generalized functions in finitely many dimensions to the infinite dimensional case, with some applications to partial differential and variational equations [11].
Let us now shortly summarixe the content of our paper and indicate briefly the general methods used.
In section 2 we start by assembling some facts about Gelfand's representation of Weyl's canonical commutation relations by means of probability measures on N', quasi invariant with respect to translations by elements in N, where $N \subset K \subset N'$ is a real separable Hilbert space with a nuclear rigging. References to previous work on this representation are [5],2) and [12]. We then isolate a class of quasi invariant measures, which we call measures with first order regular derivatives and which in the finite dimensional case, correspond to the density function having L_2 derivatives. This class is suitable for the construction of the self adjoint positive operator H associated with the Dirichlet form $\int \nabla\bar{f}\cdot\nabla g \, d\mu$ and acting in the representation space $L_2(d\mu)$ for the canonical commutation relations.
The relations of Dirichlet forms with the canonical formalism has been discussed, modulo domain questions, by Araki, in his algebraic approach to the Hamiltonian formalism and canonical commutation relations [13]. Some of our results in this section can be looked upon as providing analytic versions of algebraic derivations of Araki, in particular by realizing the measures μ on a topological dual rather than an algebraic one.
We first define the Dirichlet form on finitely based functions which are continuously differentiable with bounded derivatives and on this domain it is closable. H is precisely the Friedrichs operator given by the closure of the Dirichlet form. We call H the diffusion operator generated by μ. H has the eigenvalue zero with the eigenfunction identically equal to 1 in $L_2(d\mu)$. Moreover the eigenspace to the eigenvalue zero consists precisely of all functions in $L_2(d\mu)$ which are invariant under translations by elements of N, hence in particular, zero is a simple eigenvalue of H iff μ is ergodic, which is equivalent to the representation of the canonical commutation relations given by μ being irreducible.

One has as decomposition of μ, $L_2(d\mu)$, H and the representation (U,V) of the canonical commutation relations into a direct sum of ergodic components.
The condition on μ to have regular first order derivatives is that the indinitesimal generator Px of the unitary group $V(tx)$ of translations in the x-direction, i.e. the canonical momentum operator, should contain the function 1 in its domain. If moreover the function obtained by applying Px to 1 has bounded components, then μ is by definition in the class of quasi invariant measures with bounded regular first order derivatives and we show that, in this case, e^{-tH} has positive kernel, hence it is a Markov contraction semigroup. So that in this case we have a time homogeneous Markov process on N' with invariant measure μ and infinitesimal generator H.
Finally we prove that this diffusion process is ergodic if and only if μ is ergodic.
In section 3 we study perturbations of quasi invariant measures μ with bounded regular first order derivatives and of the associated diffusion operator H.

We first show that if H is such a diffusion operator and if V is real measurable on N' and such that $H_1 = H + V$ is essentially self adjoint and lower bounded then e^{-tH_1} has non-negative kernel. The ergodic decomposition of $L_2(d\mu)$ and H carries over to V and H_1. If E_1 is the infinum of the spectrum of H_1 and is an eigenvalue of H_1, then the corresponding eigenfunction φ is strictly positive - almost everywhere, hence $d\mu_1 = \varphi^2 d\mu$ is quasi-invariant and a sufficient condition is given for $H_1 - E_1$ to be the infinitesimal generator for a unique diffusion process generated by μ_1.
Finally we find sufficient conditions for the stability under weak limits of the correspondence between quasi invariant measures with bounded regular first order derivatives and the associated diffusion process.
In section 4 we apply the general results of the proceeding section to the case of quantum fields. The stochastic approach to quantum fields is of course not a new one, but was initiated by work of Friedrichs, Gelfand and Segal, and more recently this approach has been emphasized by Symansik [14] and Nelson[15]. The latter also formulated an axiomatic framework in terms of generalized random fields with the more dimensional Markov property, corresponding to Levy's Markovian property of order 1 [16].

Gaussian generalized Markov random fields had been also considered by Wong [17] and Molchan [18]. Guerra adn Ruggiero [19], see also [20], pointed out the connection of free Euclidean Markov fields with the generalization of Nelson's stochastic mechanics to infinitely many degrees of freedom. Recently other connections between problems of quantum fields and the theory of generalized stochastic processes have been emphasized particularly by Klander [21], see also [22].

Concerning specific models studied intensively in constructive field theory we refer to [23]. For more recent work see [24]. Concerning the Markov property of the constructed generalized random fields see [25]. Many results with direct probabilistic implications are in [26].

Coming now to our present applications of the methods of sections 2 and 3 to the quantum fields, we first remark that the diffusion operator associated by the procedure of section 2 with the Dirichlet form given by the Gaussian measure μ_0 of the unit process on $S(R^d) \subset L_2(R^d) \subset S'(R^d)$ coincides with the infinitesimal generator of the Markov process of the free Markov time zero field.

Finally we consider the interacting case in two space-time dimensions, where the interaction is given by a polynomial of even degree with sufficiently small coefficients.

We first show that the measure μ, given by the physical vacuum, restricted to the time zero fields has regular first order derivatives hence belongs to the class of quasi invariant measures discussed in Section 2. By means of the perturbation theory given in Section 3 and direct estimates, we then show that the corresponding diffusion operator coincides on a dense domain with the physical Hamiltonian.

ACKNOWLEDGEMENTS

The first named author expresses his sincere thanks to the Institute of Mathematics, University of Oslo, for the standing hospitality and to the Norwegian Research Council for Science and the Humanities for financial support. Both authors have the pleasure to thank Mrs. Cordtsen and Mrs. Møller for their fine typing.

2. Symmetric diffusion processes

The Schrödinger equation in R^n is of the form

$$i \frac{\partial\psi}{\partial t} = -\Delta\psi + V\psi \tag{2.1}$$

where V is the operation of multiplication by the potential energy $V(x)$, $\psi(x,t)$ is a function on $R^n \times R$ and Δ is the Laplacian in R^n . In this section we shall see what happens when we replace R^n by a real separable Hilbert space $\mathbf{K}$. The method usually employed in the finite dimensional case $(K=R^n)$ is to consider

$$H = -\Delta + V \tag{2.2}$$

as a self adjoint operator on $L_2(R^n)$, which is possible under some mild regularity conditions on V (see for instance [ ]) in which case

$$\psi(x,t) = (e^{-itH}\varphi)(x) , \tag{2.3}$$

where $\varphi \in L_2(R^n)$ is the initial condition $\psi(x,0) = \varphi(x)$ and e^{-itH} is the unitary group generated by H . However, in the case h is infinite dimensional it is not possible to copy this procedure too closely because of the fact that there is no obvious candidate for the L_2 space.

Therefore let us now assume that H has at least one eigenfunction in $L_2(R^n)$, and we are still considering the case $K=R^n$, and that H is, as a self adjoint operator, bounded below. Then again under some quite general regularity conditions on V , the spectrum of H will end (to the left) in an eigenvalue E so that $H \geq E$ and the corresponding eigenfunction $\Omega(x)$ is positive almost everywhere i.e.

$$H\Omega = E\Omega \ , \tag{2.4}$$

$\Omega(x) \geq 0$ and $\Omega(x) = 0$ only on a set of measure zero. On the other hand if $\Omega(x)$ is an eigenfunction and $\Omega(x) \geq 0$ then $H \geq E$ and $\Omega(x) = 0$ only on a set of measure zero. These results are consequences of the ergodicity of the Markov semi-group generated by the Laplacian. For details concerning these results see ref.

Let us assume that Ω is normalized such that

$$(\Omega,\Omega) = \int_{R^n} |\Omega(x)|^2 \, dx = 1 \tag{2.5}$$

and set $\rho(x) = \Omega(x)^2$, then $d\mu(x) = \rho(x)dx$ is a probability measure on R^n . Since Ω is in the domain of H it must (again under slight regularity conditions on V) have locally integrable derivatives up to second order. So let $f(x)$ be a real smooth function of compact support, then

$$\begin{aligned}
\int (\nabla f)^2 d\mu &= (\nabla f\,\Omega, \nabla f\,\Omega) \\
&= (\nabla(f\Omega), \nabla(f\Omega)) - 2(\nabla f\cdot\Omega, f\cdot\nabla\Omega) - (f\nabla\Omega, f\nabla\Omega) \\
&= -(f\Omega, \Delta(f\Omega)) + 2(f, \nabla(\Omega f\nabla\Omega) - (f\nabla\Omega, f\nabla\Omega) \\
&= -(f\Omega, \Delta(f\Omega)) + (f\nabla\Omega, f\nabla\Omega) \\
&\qquad + 2(f\nabla f, \Omega\nabla\Omega) + 2(f, f\Omega\,\Delta\Omega) \\
&= -(f\Omega, \Delta(f\Omega)) + (f\nabla\Omega, f\nabla\Omega) \\
&\qquad + \tfrac{1}{2}(\nabla f^2, \nabla\Omega^2) + 2(f, f\Omega\,\Delta\Omega) \\
&= -(f\Omega, \Delta(f\Omega)) + (f\nabla\Omega, f\nabla\Omega) \\
&\qquad - \tfrac{1}{2}(f^2, \Delta\Omega^2) + 2(f, f\,\Omega\,\Delta\Omega) \ .
\end{aligned}$$

So that, since $\Delta\Omega = (V-E)\Omega$, we have

$$\int (\nabla f)^2 d\mu = (f\Omega, (-\Delta + V - E) f\Omega) \tag{2.6}$$

$$= (f\Omega, (H-E) f\Omega) \; .$$

Hence the correspondence $f \leftrightarrow f\Omega$ which is a unitary equivalence between $L_2(dx)$ and $L_2(d\mu)$ takes the form $(f,(H-E)f)$ into the form $(f\Omega,(H-E)f\Omega) = \int (\nabla f)^2 d\mu$. Hence we could define the operator $H-E$ as the self adjoint operator defined by the closable form

$$\int (\nabla f)^2 d\mu \tag{2.7}$$

in the Hilbert space $L_2(d\mu)$. The relation between the operator $H-E$ and the measure $d\mu = \rho dx$ is then given by

$$H-E = -\Delta + (V-E) \tag{2.8}$$

where

$$V-E = \frac{\Delta \rho^{\frac{1}{2}}}{\rho^{\frac{1}{2}}} \; . \tag{2.9}$$

Let now $d\mu(x)$ be an arbitrary probability measure on R^n which is quasi invariant with respect to translation. That is

$$d\mu(x+a) = \alpha(x,a) d\mu(x) \tag{2.10}$$

where $\alpha(x,a) \geq 0$, $\int \alpha(x,a) d\mu(x) = 1$ and

$$\alpha(x,a+b) = \alpha(x+a,b)\alpha(x,a) \; . \tag{2.11}$$

It is well known that in R^n any quasi invariant measure is equivalent to the Lebesgue measure, so that $d\mu(x) = \rho(x)dx$ and $\rho(x) \geq 0$ with $\rho(x) = 0$ only on a set of measure zero.

Hence in this case

$$\alpha(x,a) = \frac{\rho(x+a)}{\rho(x)} \; .$$

In $L_2(d\mu)$ we may now consider the form (2.7) and if it is closable we shall call H the self adjoint operator which is given by the corresponding closed form, so that with $(,)$ being the inner product in $L_2(d\mu)$,

$$(f,Hf) = \int \nabla f \cdot \nabla f \, d\mu \ , \tag{2.12}$$

and we shall say that

$$H = -\Delta + V \tag{2.13}$$

with

$$V = \sum_{i=1}^{n} \frac{\partial^2}{\partial a_i^{\,2}} \alpha^{\frac{1}{2}}(x,0) \ , \tag{2.14}$$

whether (2.14) defines a measurable function or not. In this way we obtain, from any quasi invariant probability measure $d\mu$ on R^n such that the form (2.12) is closable in $L_2(d\mu)$, a self adjoint operator $H \geq 0$ such that the constant function is an eigenfunction with eigenvalue 0 . Moreover if $d\mu(x) = \Omega(x)^2 dx$, where $\Omega(x)$ is the lowest eigenfunction for an operator of the form $-\Delta + V$, then $H = -\Delta + V$.

Example 1

Let $n = 3$ and take $d\mu$ to be the probability neasure in R^3 given by

$$d\mu(x) = \frac{m^3}{8\pi} \frac{e^{-2m|x|}}{|x|^2} \, dx \ .$$

We may verify that the form (2.7) is closable in $L_2(d\mu)$, so that H is well defined. In this case

$$\alpha^{\frac{1}{2}}(x,a) = \frac{|x|}{|x+a|} \, e^{-m|x+a|} \cdot e^{m|x|} \ .$$

For $x \neq 0$ we see that $\Delta_a \alpha^{\frac{1}{2}}(x,0) = m^2$. In fact we may easely prove that H is a self adjoint operator such that, when restricted to smooth functions which are zero at zero, then $H_m f = (-\Delta + m^2) f$

for $f(0) = 0$. However, $H_m - m^2$, when represented in $L_2(dx)$, has $\frac{e^{-m|x|}}{|x|}$ as an eigenfunction with eigenvalue $-m^2$, so $H_m - m^2 \neq -\Delta$. In fact $H_m - m^2$ form a one parametric family of self adjoint extensions of the restriction of $-\Delta$ to functions $f \in D(\Delta)$ such that $f(0) = 0$.

It is well known, at least in the case of sufficiently nice potential V , that H given by (2.13) is also the infinitesimal generator of a Markov semi group e^{-tH} which has $d\mu$ as an invariant measure. Moreover the stationary Markov process $\xi(t)$ in R^n given by the Markov semigroup e^{-tH} and its invariant measure $d\mu$ is the unique solution of the stochastic differential equation

$$d\xi(t) = \beta(\xi(t))dt + dw(t) \tag{2.15}$$

where $w(t)$ is the standard Wiener process in R^n , and $\beta(\xi)$ is the osmotic velocity

$$\beta(\xi) = \nabla \ln \rho(\xi) \ , \tag{2.16}$$

where $d\mu(x) = \rho(x)dx$. For more details on this we refer the reader to ref. [20] and the references contained there. It follows from the methods in ref. [20] that the stochastic process $\xi(t)$ is always a solution (2.15) although one can prove that this solution is unique only under regularity condition on the osmotic velocity $\beta(\xi)$, for instance is a Lipschitz condition good enough. For closer information on stochastic differential equations and their solutions see ref. [3].

We are now in the position to discuss the Schrödinger equation (2.1) on a separable real Hilbert space K. The setting which we shall use is given by a nuclear rigging

$$N \subset K \subset N' \qquad (2.17)$$

where N is a real nuclear space densely contained in K and N' is the dual of N. Moreover the inner product (x,y) in K when restricted to N coincides with the dualization between N and N'.

Let $d\mu(\xi)$ be a probability measure on N' which is quasi invariant under translations by elements in N.

We here recall Minlos theorem [29] that says that any continuous positive definite function φ on N such that $\varphi(0) = 1$ is given by a unique probability measure $d\mu$ on N' such that

$$\varphi(x) = \int_{N'} e^{i(x,\xi)}\, d\mu(\xi) \,, \qquad (2.18)$$

where the measure structure in N' is the one derived from its topology.

Since $d\mu$ is quasi invariant under translations with elements $x \in N$ we have that $d\mu(\xi)$ and $d\mu(\xi+x)$ are equivalent, hence

$$\alpha(\xi,x) = \frac{d\mu(\xi+x)}{d\mu(\xi)} \qquad (2.19)$$

is, for fixed x, positive μ- almost everywhere, and $\alpha(\xi,x) \in L_1(d\mu)$, in fact

$$\int \alpha(\xi,x)d\mu(\xi) = 1 \,. \qquad (2.20)$$

Furthermore for any x and y in N we have, for μ- almost all ξ, that

$$\alpha(\xi,x+y) = \alpha(\xi+x,y)\alpha(\xi,x) \,. \qquad (2.21)$$

Such a measure give rise to two unitary representations U and V of N on the Hilbert space $\mathcal{H} = L_2(d\mu)$ by

$$(U(x)f)(\xi) = e^{i(x,\xi)} f(\xi) \tag{2.22}$$

and

$$(V(x)f)(\xi) = \alpha^{\frac{1}{2}}(\xi,x)f(\xi+x) \ . \tag{2.23}$$

We have obviously that $U(x)$ and $V(x)$ are representations of N , namely

$$U(x)U(y) = U(x+y), \quad V(x)V(y) = V(x+y) \tag{2.24}$$

that satisfy the Weyl-commutation relations

$$V(x)U(y) = e^{\ i(x,y)}\ U(y)V(x) \tag{2.25}$$

for x and $y \in N$. Moreover we see that the mapping $x \to U(x)$ is strongly continuous from $N \to B(\mathcal{H})$ with the strong operator topology on $B(\mathcal{H})$, because for $y \in N$, $U(y)1$ is dense in $L_2(d\mu)$ and

$$\|(U(x)-1)U(y)1\|^2 = \|(U(x)-1)1\|^2$$
$$= 2\ \mathrm{Re}\ \varphi(x) - 2 \ ,$$

and by assumption $\varphi(x)$ is continuous on N and $\varphi(0) = 1$.

On the other hand we have by Minlos theorem that if $U(x), V(x)$ are unitary representations of the nuclear abelian group N such that $U(x)$ is weakly continuous in the topology of N (weakly referring to the weak operator topology) and with a cyclic element Ω and such that $U(x), V(x)$ satisfy the Weyl-commutation relations, then there is a measure $d\mu$ on N' which is quasi invariant under translations by elements in N . Moreover $U(x)$ and $V(x)$ are represented on $L_2(d\mu)$ by (2.22) and (2.23) respectively.

To see this, take

$$\varphi(x) = (\Omega, U(x)\Omega) \ , \tag{2.26}$$

then $\varphi(x)$ is a positive definite continuous function on N and by the Minlos theorem there is a probability measure $d\mu$ on N' such that (2.18) holds, and by the cyclicity of Ω under $U(x)$ we may take the representation space to be $L_2(d\mu)$ and $U(x)$ to be

$$(U(x)f)(\xi) = e^{i(x,\xi)} f(\xi) . \tag{2.27}$$

Consider now also the positive definite function

$$\varphi_y(x) = (V(y)\Omega, U(x)V(y)\Omega) \tag{2.28}$$

and the corresponding measure $d\mu_y(\xi)$. By the commutation relation (2.25) we see that

$$\varphi_y(x) = e^{-i(x,y)} \varphi(x) . \tag{2.29}$$

By the uniqueness of the Minlos representation (2.18) we therefore have that

$$d\mu_y(\xi) = d\mu(\xi+y) . \tag{2.30}$$

Now the subspace generated, for a fixed $y \in N$, by $U(x)V(y)\Omega$, as x runs through N , is a closed subspace which is equivalent to $L_2(d\mu_y)$, and the weakly closed subalgebra generated by $U(x)$ in this subspace is equivalent to $L_\infty(d\mu_y)$. However, the weakly closed algebra generated by $U(x)$ in the whole space is equivalent to $L_\infty(d\mu)$. This gives us then a continuous mapping from $L_\infty(d\mu)$ onto $L_\infty(d\mu_y)$ which implies that $d\mu_y$ is absolutely continuous with respect to $d\mu$. This proves the quasi invariance of $d\mu$ under translations by elements y in N .

Hence we have the following proposition, which is first proved in ref [5], 2).

Proposition 2.1

Let $N \subset K \subset N'$ be a nuclear rigging of K, and assume that we have two representations of N, $U(x)$ and $V(x)$ by unitary operators on a separable Hilbert space $\mathcal{H}$ such that $x \to U(x)$ is continuous from N into $B(\mathcal{H})$ with the weak operator topology and that there is a cyclic element Ω for the representation $U(x)$. If moreover U and V satisfy the Weyl commutation relation

$$V(x)U(y) = e^{i(x,y)} U(y)V(x) ,$$

then there is a probability measure $\mu \in \mathcal{M}(N')$ such that μ is quasi invariant with respect to translations by elements in N such that

$$(\Omega, U(x)\Omega) = \int_{N'} e^{i(x,\xi)} d\mu(\xi) ,$$

and $U(x)\Omega \leftrightarrow e^{i(x,\xi)}$ gives an identification of $\mathcal{H}$ with $L_2(d\mu)$ such that

$$(U(x)f)(\xi) = e^{i(x,\xi)} f(\xi) \quad \text{and} \quad (V(x)f)(\xi) = \alpha^{\frac{1}{2}}(\xi,x)f(\xi+x)$$

with

$$d\mu(\xi+x) = \alpha(\xi,x)d\mu(\xi) . \qquad \square$$

Consider now $\alpha(\xi,tx) = \frac{d\mu(\xi+tx)}{d\mu(\xi)}$. It is obviously simultaneously measurable in ξ and t, so that $\alpha^{\frac{1}{2}}(\xi,tx)$ is measurable as a function of $(\xi,t) \in N' \times R$ for fixed x. From this and (2.23) it follows that $V(tx)$ is weakly measurable in t and since $V(tx)$ is a unitary group as a function of t by (2.24) we have, by a standard theorem on unitary groups, that $V(tx)$ is strongly continuous. Let Px be its infinitesimal generator. Then Px is a self adjoint operator on $\mathcal{H} = L_2(d\mu)$, and by (2.24)

$$P(x+y) \supseteq Px + Py \tag{2.31}$$

and in fact it follows from what is known about the Weyl commutation relations on finite dimensional spaces that

$$P(x+y) = \overline{Px + Py} \; . \tag{2.32}$$

Let E be a orthogonal projection in K such that its range EK is a finite dimensional subspace contained in N, where $N \subset K \subset N'$ is the nuclear rigging of K. For $u \in K$, Eu is then of the form

$$Eu = \sum_{i=1}^{m} (e_i, u) \cdot e_i \; , \tag{2.33}$$

where $m = \dim EK$ and e_i, $i = 1, \ldots, m$ is an orthonormal base in EK. Since $e_i \in N$, we see from (2.33) that E extends by continuity to a projection $E : N' \to N$ given by $E\xi = \sum_{i=1}^{m} (e_i, \xi) e_i$. So we have proved that any orthogonal projection in K which has a finite dimensional range contained in N extends by continuity to a continuous projection from N' into N.

We shall now define some subspaces of $C(N')$, the Banach space of continuous bounded functions on N'. $f \in C(N')$ is said to be in F^n if there is some orthogonal projection E on K with a finite dimensional range $EK \subset N$ such that $f(\xi) = f(E\xi)$ and $f(x)$ for $x \in EK$ is in $C^n(EK)$, n is here $1, 2, \ldots$ or ∞. We also define $f \in F$ by requiring that $f(\xi) = f(E\xi)$ for some E of finite range in N and $f(x) \in C(EK)$ for $x \in EK$. Let $n < \infty$ and let $\| \; \|_n$ be the norm in C^n, then we define $\| \; \|_n$ in F^n by $\|f\|_n = \|f|_{EK}\|_n$ for some E such that $f(\xi) = f(E\xi)$ and EK of finite dimension in N. We see that $\|f\|_n$ does not depend on E so that $\|f\|_n$ is well defined and it organizes F^n to be a normed linear space. We shall refer to F as the space of continuous and finitely based functions, and F^n as the space of n-times differentiable and finitely based functions.

However, F^n is not complete in this norm and we shall denote by D^n the Banach space obtained by completion. We set $D^{\infty} = \bigcap_n D^n$ so that D^{∞} is a Fréchet space which contains F^{∞} as a dense subspace.

By $\mathcal{M}(N')$ we understand the Banach space of bounded complex measures on N', i.e. $\mathcal{M}(N') = C(N)^*$. It follows from the Minlos theorem that $\mathcal{M}(N')$ is closed under convolution of measures and since

$$\int_{N'} f\, d(\nu_1 * \nu_2) = \int_{N' \times N'} f(\xi_1+\xi_2)\, d\nu_1(\xi_1)\, d\nu_2(\xi_2) \qquad (2.34)$$

we get from the fact that $\mathcal{M}(N') = C(N)^*$ that $\|\nu_1 * \nu_2\| \leq \|\nu_1\| \cdot \|\nu_2\|$. So that $\mathcal{M}(N')$ is in fact a Banach algebra.

<u>Definition 2.1</u> We shall say that a probability measure $\mu \in \mathcal{M}(N')$ is quasi invariant if it is quasi invariant under translations by elements in N.

We have now the following proposition complementary to proposition 2.1.

<u>Proposition 2.2</u>

Let $N \subset K \subset N'$ be a nuclear rigging of K and $\mu \in \mathcal{M}(N')$ a quasi invariant probability measure, and consider in $\mathcal{H} = L_2(d\mu)$ the representation of the Weyl commutation relations over N given by

$$(U(x)f)(\xi) = e^{i(x,\xi)} f(\xi) \quad \text{and} \quad (V(x)f)(\xi) = \alpha^{\frac{1}{2}}(\xi,x) f(\xi+x)$$

where $\alpha^{\frac{1}{2}}(\xi,x)$ is the positive square root of $\alpha(\xi,x)$, with $d\mu(\xi+x) = \alpha(\xi,x)d\mu(\xi)$. Then the mappings $x \to U(x)$ and $x \to V(x)$ are strongly continuous unitary representations of N on $\mathcal{H}$. Moreover with $\Omega(x) = 1 \in L_2(d\mu)$, $U(x)$ is a cyclic representation with Ω as a cyclic vector.

Proof: For $f \in L_\infty(d\mu)$ we have that

$$\|(U(x)-1)f\Omega\|_2{}^2 \leq \|f\|_\infty^2 \|(U(x)-1)\Omega\|^2$$
$$= 2\|f\|_\infty^2(1-\mathrm{Re}(\Omega,u(x)\Omega))$$
$$= 2\|f\|_\infty^2(1-\mathrm{Re}\int_{N'} e^{i(x,\xi)}\, d\mu(\xi)) .$$

This proves that $x \to U(x)f\Omega$ is strongly continuous, and since $L_\infty(d\mu)$ is dense in $L_2(d\mu)$, we get that $U(x)$ is a strongly continuous representation of N with a cyclic element Ω. That $V(x)$ is also a strongly continuous representation follows from results proved by Hegerfeldt, see theorem 3.3 of ref. [12],3).

□

Definition 2.2

For any representation (U,V) of the Weyl commutation relations over N on a Hilbert space $\mathcal{H}$, we get a representation $(\tilde{U},\tilde{V})$ of the Weyl commutation relations over N on $\mathcal{H}$ by $\tilde{U}(x) = V(x)$ and $\tilde{V}(x) = U(-x)$. We shall call the representation given by $(\tilde{U},\tilde{V})$ the Fourier transform representation.

We remark that $\tilde{\tilde{U}}(x) = U(-x)$ and $\tilde{\tilde{V}} = V(-x)$ so that the mapping $(U,V) \to (\tilde{U},\tilde{V})$ is periodic with period 4.

Corollary to Proposition 2.2

Let μ be a quasi invariant probability measure on N' such that $d\mu(\xi+x) = \alpha(\xi,x)d\mu(\xi)$ for any $x \in N$, then $x \to \alpha^{\frac{1}{2}}(\xi,x)$

is continuous from N into $L_2(d\mu)$ and $x \to \alpha(\xi,x)$ is continuous from N into $L_1(d\mu)$.

Proof: By proposition 2.2, $x \to V(x)\Omega = \alpha^{\frac{1}{2}}(\xi,x)$ is strongly continuous. Moreover

$$\left[\int |\alpha(\xi,x)-\alpha(\xi,y)|\,d\mu(\xi)\right]^2 = \left[\int |(\alpha^{\frac{1}{2}}(\xi,x)+\alpha^{\frac{1}{2}}(\xi,y))(\alpha^{\frac{1}{2}}(\xi,x)-\alpha^{\frac{1}{2}}(\xi,y))|\,d\mu(\xi)\right]^2$$

$$\leq \int |\alpha^{\frac{1}{2}}(\xi,x)+\alpha^{\frac{1}{2}}(\xi,y)|^2\,d\mu \cdot \int |\alpha^{\frac{1}{2}}(\xi,x)-\alpha^{\frac{1}{2}}(\xi,y)|^2\,d\mu$$

$$\leq 4\int |\alpha^{\frac{1}{2}}(\xi,x)-\alpha^{\frac{1}{2}}(\xi,y)|^2\,d\mu(\xi) \ .$$

This proves the corollary.

<u>Definition 2.3</u>

We say that a quasi invariant probability measure $\mu \in \mathcal{M}(N')$ is ergodic iff the only functions $f \in L_\infty(d\mu)$ which are invariant with respect to translations by arbitrary elements $x \in N$ are the constant functions.

We remark that an obviously equivalent definition is that μ is ergodic iff all the N invariant measurable sets in N' have μ-measure zero or one.

We say that a representation (U,V) of the Weyl commutations relations is irreducible iff the only bounded operators that commute with all $U(x)$ and $V(x)$, $x \in N$, are the constants.

<u>Proposition 2.3</u>

A quasi invariant probability measure $\mu \in \mathcal{M}(N')$ is ergodic if and only if the representation of the Weyl commutation relations on $L_2(d\mu)$ given by proposition 2.1 is irreducible.

Proof: Let F be a bounded operator that commutes with $U(x)$, then F is given by the multiplication by a function $F(\xi) \in L_\infty(d\mu)$.

If F commutes with $V(x)$, then $F(\xi+x) = F(\xi)$ so that F is invariant under N. Hence if the representation is irreducible, the measure is ergodic and viceversa. □

<u>Proposition 2.4</u>

Let (U,V) be an irreducible representation of the Weyl commutation relation over N on a separable Hilbert space $\mathcal{H}$ such that $x \to U(x)$ is a strongly continuous representation of N. Then there is a quasi invariant probability measure $\mu \in \mathcal{M}(N')$ such that (U,V) is equivalent with the representation (U',V')

$$(U'(x)f)(\xi) = e^{i(x,\xi)} f(\xi) \quad \text{and} \quad V'(x)f(\xi) = z(\xi,x)\alpha^{\frac{1}{2}}(\xi,x)f(\xi+x)$$

on $L_2(d\mu)$, where $z(\xi,x)$ is a measurable function on N' such that, for almost every ξ, $|z(\xi,x)| = 1$ and

$$z(\xi,x+y) = z(\xi+x,y)z(\xi,x)$$

and $z(\xi,0) = 1$.

Proof: This proposition is an easy consequence of Theorem 6.2.2 and its corollary in ref. □

The following corollary is immediate:

<u>Corollary to Proposition 2.4</u>

Let (U,V) be an irreducible representation of the Weyl commutation relation over N on a separable Hilbert space $\mathcal{H}$, such that $x \to U(x)$ is strongly continuous, then there is a cyclic element Ω for the representation of N given by $U(x)$.

<u>Definition 2.4</u> We shall say that a probability measure $\mu \in \mathcal{M}(N')$ is L_p-differentiable iff it is quasi invariant and for the corresponding $\alpha(\xi,x) = d\mu(\xi+x)/d\mu(\xi)$ we have that $\frac{1}{t}(\alpha(\xi,tx) - 1)$

converges strongly in L_p as $t \to 0$. We shall say that it is weak L_p-differentiable if $\frac{1}{t}(\alpha(\xi,tx)-1)$ converges weakly in L_p as $t \to 0$.

Remark: In the finite dimensional case we have that a quasi invariant measure is of the form $\rho(x)dx$ for $\rho(x) \in L_1$. Then $\alpha(x,y) = \rho(x+y)/\rho(x)$ so we see that in the finite dimensional case the definition above amounts to the strong or weak L_1 differentiability of the function $\rho(x)$.

Now, if $\mu \in \mathcal{M}(N')$ is L_1-differentiable, then $\frac{1}{t}(\alpha(\xi,tx)-1) \to \beta(\xi)\cdot x$ in the L_1 norm and the derivative $\beta(\xi)\cdot x$ is then obviously a linear function from N into $L_1(d\mu)$, and we denote this linear function by $\beta(\xi)$. Since in the finite dimensional case it is actually given by (2.16), which was the osmotic velocity, we will also in the general case call it the osmotic velocity.

<u>Definition 2.5</u> We shall say that a probability measure $\mu \in \mathcal{M}(N')$ has regular first order derivatives iff it is quasi invariant and in the representation (U,V) of the Weyl commutation relations given by μ by proposition 2.1 we have that Ω is in the domain of Px for all $x \in N$, where Px is the infinitesimal generator of the unitary group $V(tx)$ in $L_2(d\mu)$.

<u>Proposition 2.5</u>

If $\mu \in \mathcal{M}(N')$ has regular first order derivatives, then μ is L_1-differentiable, and $2Px\Omega = \beta\cdot x$ where β is the osmotic velocity.

Proof: That μ has regular first order derivatives is obviously equivalent with the condition that $\frac{1}{t}(\alpha^{\frac{1}{2}}(\xi,tx)-1)$ converges in $L_2(d\mu)$ as $t \to 0$. Now we have that

$$\frac{1}{t}(\alpha(\xi,tx)-1) = \frac{1}{t}(\alpha^{\frac{1}{2}}(\xi,tx)-1)(\alpha^{\frac{1}{2}}(\xi,tx)+1) \tag{2.35}$$

and by proposition 2.2 we have that $\alpha^{\frac{1}{2}}(\xi,tx)$ converges to 1 in $L_2(d\mu)$. This then gives that the right hand side converges in L_1. We observe from (2.35) that $i(Px\,\Omega)(\xi) = \frac{1}{2}\beta(\xi)x$, and this proves the proposition.

Theorem 2.6

Let $N \subset K \subset N'$ be a nuclear rigging of K and let (U,V) be any representation of the Weyl commutation relations over N on a Hilbert space $\mathcal{H}$, such that $x \to V(x)$ is a strongly continuous representation of N. Let Px be the infinitesimal generator for the unitary group $V(tx)$, and $u \in \mathcal{H}$ in the domain of Px for each $x \in N$.

Then the mapping $x \to Px \cdot u$ is a continuous linear mapping from N into $\mathcal{H}$.

Proof: Set

$$\eta(x) = \|(V(x)-1)u\|$$

then

$$\eta(x+y) = \|(V(x+y)-1)u\| = \|V(x)(V(y)-1)u + (V(x)-1)u\|$$

$$\leq \|(V(y)-1)u\| + \|(V(x)-1)u\| = \eta(x) + \eta(y) .$$

So that $\eta(x)$ is a sublinear function on N, i.e. $\eta(x+y) \leq \eta(x) + \eta(y)$. Now, since $u \in D(Px)$ we have that

$$\lim_{t \searrow 0} \frac{1}{t}\,\eta(tx) = p(x) = \|Px \cdot u\| . \tag{2.36}$$

Since u is in the domain of Px for all $x \in N$, we have that $Px \cdot u$ is linear in x and (2.36) then gives that $p(x)$ is a semi norm on N i.e.

$$p(x+y) \leq p(x) + p(y) \quad \text{and} \quad p(\lambda x) = |\lambda|\,p(x) . \tag{2.37}$$

Since $\eta(x)$ is sublinear we get that $\eta(2x) \leq 2\eta(x)$ so that $2\eta(\frac{1}{2}x) \geq \eta(x)$ or

$$2^{n+1}\ \eta(2^{-n-1}x) \geq 2^{n}\eta(2^{-n}x)\ , \tag{2.38}$$

so that

$$p(x) = \sup_{n} 2^{n}\,\eta\,(2^{-n}x)\ . \tag{2.39}$$

Now, by assumption $\eta(x)$ is continuous on N so that $p(x)$ is lower semi continuous. But on any countable normed space a lower semi continuous semi norm is bounded in some neighborhood of zero. For this result see [5], 2) , Chapter I, section 1, theorem 1]. Since $\|Px{\cdot}u\|$ is bounded in some neighborhood of zero in N we have that $x \to Px{\cdot}u$ is a continuous mapping from $N \to \mathcal{H}$. This proves the theorem.

Remark. It follows from the proof that it holds for any strongly continuous representation of any countably normed space.

Corollary to theorem 2.6

Let $|x|_p$, $p = 1,2,\ldots$ be the countable set of norms that defines the topology on N . Under the assumption of theorem 2.6 there is a p such that $x \to Px{\cdot}u$ is continuous in the norm $|x|_p$, i.e. $\|Px{\cdot}u\| \leq C|x|_p$ where C depends only on p and u .

Proof: This follows from the theorem and theorem 5 of Ch I, section 3.5 of ref. [5], 2).

Proposition 2.7

Let $\mu \in \mathcal{M}(N')$ be a probability measure with regular first order derivatives. Then there are a p and a C such that, for Ω in the induced representation (U,V) of the Weyl commutation relations,

$$\|Px\,\Omega\| \leq C|x|_p$$

and the osmotic velocity $\beta(\xi)\cdot x$ is a continuous function from K_p into $L_2(d\mu)$, where K_p is the Hilbert space with norm $|\ |_p$ and $N = \bigcap_p K_p$.

Proof: This proposition is an immediate consequence of the theorem and its corollary above.

Proposition 2.8

Let $\mu \in \mathcal{M}(N')$ be a probability measure with regular/first order derivatives. Then there is a measurable function $\beta : N' \to N'$ such that the osmotic velocity $\beta(\xi)\cdot x$ is given by $\langle x,\beta(\xi)\rangle$.

Proof: This follows from proposition 2.7 and the Abstract Kernel Theorem (Ch I, section 3, theorem 3 of ref. [5], 2)). □

We shall now call the function $\beta : N' \to N'$ the osmotic velocity and $\beta(\xi)\cdot x = \langle x,\beta(\xi)\rangle$ the component in the x-direction of the osmotic velocity.

We want now to solve the stochastic differential equation

$$d\,\xi(t) = \beta(\xi(t))dt + dw(t) \ , \tag{2.40}$$

i.e. to find a stochastic process with values in N' which solves (2.40), where $w(t)$ is the standard Wiener process on N' given by the nuclear rigging $N \subset K \subset N'$. Of course we here must first introduce the standard Wiener process, but let us first recall some facts of Markov processes.

A homogeneous stochastic process on a measure space X may be described completely in terms of its transition probability $P_t(\xi,d\eta)$, where for any measurable set $A \subset X$ we have that

$$\Pr(\xi(0) = \xi \ \text{ and } \ \xi(t) \in A) = \int_A P_t(\xi,d\eta) \tag{2.41}$$

is the probability for the process $\xi(t)$ to start at the point ξ in X and be at the time t in the set $A \subset X$, so that, for fixed t, $P_t(\xi, d\eta)$ is a measurable function from X into the set of probability measures on X. Moreover one usually assumes $P_t(\xi, d\eta)$ to be measurable from $R \times X$ into $\mathcal{M}(X)$.

The condition for the corresponding process to be a Markov process is the Chapman-Kolmogoroff equation, namely that

$$P_{t+s}(\xi, d\eta) = \int P_t(\xi, d\zeta) P_s(\zeta, d\eta) \ . \tag{2.42}$$

The transition probability $P_t(\xi, d\eta)$ induces by (2.42) a semigroup on $C(X)$ and a dual semigroup on $\mathcal{M}(X)$ by, for $f \in C(X)$,

$$(P_t f)(\xi) = \int P_t(\xi, d\eta) f(\eta) \tag{2.43}$$

and, for $\mu \in \mathcal{M}(X)$.

$$(P_t \mu)(d\eta) = \int d\mu(\zeta) P_t(\zeta, d\eta) \ . \tag{2.44}$$

For a more detailed account on the theory of Markov processes see ref. [2], [30] (Ch. XIII).

We shall take $X = N'$ where N' is the dual of a nuclear space in a nuclear rigging $N \subset K \subset N'$ of a separable real Hilbert space K. Let us now define $P_t^W(0, d\eta)$ by the equation

$$e^{-\frac{t}{2}(x,x)} = \int_{N'} e^{i\langle x, \xi \rangle} P_t^W(0, d\eta) \ . \tag{2.45}$$

The existence of a unique $P_t^W(0, d\eta) \in \mathcal{M}(N')$ is secured by the Minlos theorem, since the left hand side is a positive definite continuous function on N. (x,x) is the inner product on K. $P_t^W(\xi, d\eta)$ is then defined as the translate by ξ of $P_t^W(0, d\eta)$, so that $P_t^W(\xi, d\eta) = P_t^W(0, d(\eta - \xi))$ and, by (2.45) the transition probability is translation invariant on N'. Since $e^{-\frac{t}{2}(x,x)}$ is a semigroup under multiplication we have that the corresponding measures $P_t^W(0,d\eta)$ form a semigroup under convolution, and due to the translation invariance, (2.42) only says that $P_{s+t}^W(0,d\eta)$ is the convolution of $P_t(0,d\eta)$ with $P_s(0,d\eta)$, which as already observed, is true. This proves that the corresponding process $w(t)$ is actually a Markov process which we call the standard Wiener process on N' given by the nuclear rigging $N \subset K \subset N'$.

$w(t)$ is in fact the process studied by Gross, but for the fact that Gross prefers to study it relative for Banach rigging $B \subset K \subset B'$. This is possible since it easely follows that $w(t)$ actually takes values in a dual Banach space B' such that $K \subset B' \subset N'$. For the work of Gross see the references[9], 1), [31].

Having defined the standard Wiener process $w(t)$ on N' given by the nuclear rigging $N \subset K \subset N'$ we shall proceed to solve the stochastic differential equation (2.40).

We introduced earlier the space F^1 of functions f such that $f(\xi) = f(E\xi)$, where E was an orthogonal projection in K with finite dimensional range in N and $f(x)$ for $x \in EK$ was in $C^1(EK)$. For such functions we may consider the gradient $(\nabla f)(\xi)$, which is then a continuous mapping from N' into $(EK)^*$ and since EK is naturally self dual, we may consider ∇f as a mapping from N' into EK. For f and g in F^1 there is a common E of finite dimensional range in N so that $f(\xi) = f(E\xi)$ and $g(\xi) = g(E\xi)$, and we then denote by $(\nabla f \cdot \nabla g)(\xi)$ the inner product in EK of $\nabla f(\xi)$ with $\nabla g(\xi)$.

Definition 2.6

We shall say that μ has bounded regular first order derivatives if μ has regular first order derivatives and the components of β are bounded functions in N' i.e., for any $x \in N$, $\beta(\xi)x$ is a bounded measurable function.

We have now the following theorem

Theorem 2.9

Let $N \subset K \subset N'$ be a nuclear rigging of a real separable

Hilbert space K and let μ be a quasi invariant probability measure on N' with regular derivatives. Then the form $\int \nabla\bar{f}\cdot\nabla g\, d\mu$ defined for f and g in F^1 is closable in $L_2(d\mu)$. In fact if $(,)$ is the inner product in $L_2(d\mu)$ there is a symmetric operator H defined on F^2 such that for f and g in F^2

$$(f,Hg) = \int \nabla \bar{f} \cdot \nabla g \; d\mu$$

and for $f \in F^2$

$$(H f)(\xi) = -\Delta f(\xi) - \beta(\xi)\cdot\nabla f(\xi)$$

where $\beta(\xi)$ is the osmotic velocity.

Moreover the corresponding closed form gives us a self adjoint non negative operator H in $\mathcal{H} = L_2(d\mu)$ with $\Omega = 1$ as an eigenvector with eigenvalue zero. We call H the diffusion operator generated by μ . Furthermore, if μ has bounded derivatives, then the corresponding contraction semigroup e^{-tH} is a Markov semigroup, i.e. e^{-tH} has a positive kernel in $L_2(d\mu)$. This Markov semigroup then defines a stochastic Markov process in N' with invariant measure μ .

Proof: We know that, since μ has regular derivatives, $\beta(\xi)$ is a continuous function from N into $L_2(d\mu)$ and $f(\xi) = f(E\xi)$, where E is an orthogonal projection in H with finite dimensional range in N . We have seen already that such projections extend by continuity to projections from N' into N . Therefore $(\nabla f)(\xi)$ is a continuous bounded function from N' into $EK \subset N$ so that $\xi \to \beta(\xi)\nabla f(\xi)$ is in $L_2(d\mu)$. Moreover, since $f \in F^2$, we have that $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 \tilde{f}}{\partial x_i^2}$, where $x_i = (e_i,\xi)$ and $e_1,\dots,e_n$ is an orthonormal base in $EK \subset N$ and $f(\xi) = \tilde{f}(x_1,\dots,x_n)$, where $\tilde{f} \in C^2(R^n)$: Hence $\Delta f \in F \subset C(N')$, so that for f in F^2, which is dense in $L_2(d\mu)$, $Hf = -\Delta f - \beta(\xi)\cdot\nabla f(\xi)$ is in L_2 . Hence H is a densely defined symmetric operator. Let now f and g be in F^2 and let E be a projection with finite dimensional range in N so that we have both $f(\xi) = f(E\xi)$ and $g(\xi) = g(E\xi)$. Then

$$\int_{N'} \nabla\bar{f}\cdot\nabla g \, d\mu = \sum_{i=1}^{n} \int \lim_{t\to o} \overline{\frac{1}{t}(f(\xi+te_i) - f(\xi))} \cdot \frac{\partial g}{\partial x_i}(\xi) d\mu$$

which by dominated convergence is equal to

$$\sum_{i=1}^{n} \lim_{t\to o} \int \overline{\frac{1}{t}(f(\xi+te_i) - f(\xi))} \cdot \frac{\partial g}{\partial x_i}(\xi) d\mu$$

and by quasi invariance of μ this is equal to

$$\sum_{i=1}^{n} \lim_{t\to o} \int \bar{f}(\xi) \frac{1}{t}\left(\frac{\partial g}{\partial x_i}(\xi - te_i) - \frac{\partial g}{\partial x_i}(\xi)\right)\alpha(\xi, -te_i) d\mu$$

$$+ \sum_{i=1}^{n} \lim_{t\to o} \int \bar{f}(\xi) \frac{\partial g}{\partial x_i}(\xi) \frac{1}{t}(\alpha(\xi, -te_i) - 1) d\mu \; .$$

By the assumption that $g \in F^2$ the first term converges to $-\int \bar{f}\,\Delta g \, d\mu$ by dominated convergence, and by proposition 2.5 $\frac{1}{t}(\alpha(\xi, te_i) - 1)$ converges to $\beta(\xi)e_i$ strongly in L_1 , which implies that the second term converges to $-\int \bar{f}\beta\nabla g \, d\mu$. Hence we have proved that for f and g in F^2

$$(f, Hg) = \int \nabla\bar{f}\cdot\nabla g \, d\mu \; . \qquad (2.46)$$

From (2.46) we also get that H is symmetric and non negative on the domain F^2 . Hence the form is closable and its closure defines a self adjoint operator which we shall also denote by H , which then actually is the Friedrichs extension of $-\Delta - \beta\cdot\nabla$ on F^2 . So that $D(H^{\frac{1}{2}})$ is exactly the domain of the closed form.

We have obviously that $\Omega(\xi) \equiv 1$ is in F^2 so that $\Omega \in D(H)$ and $H\Omega = 0$. Hence the spectrum of H starts with an eigenvalue at zero, and e^{-tH} is a symmetric contraction semigroup which leaves Ω invariant. That, under the assumption of bounded regular first order derivatives for μ , e^{-tH} has a positive kernel, i.e. for any f and g non negative in $L_2(d\mu)$ we have that

$$(f\Omega,\, e^{-tH} g\Omega) \geq 0 \,, \tag{2.47}$$

is proven in the following way. Let $e_i \in N$ be an orthogonal base in K and let H_n be the Friedrichs extension of the form

$$(f,H_n g) = \sum_{i=1}^{n} \int \overline{e_i \cdot \nabla f}\; e_i \cdot \nabla g \; d\mu \,. \tag{2.48}$$

Then H_n is given on F^2 by

$$H_n f = -\Delta_n f - \beta \cdot \nabla_n f$$

where $\nabla_n f = P_n \nabla f$, P_n is the orthogonal projection onto the subspace of K generated by $\{e_1,\dots,e_n\}$ and $\Delta_n = \nabla_n \cdot \nabla_n$. So that (2.48) is actually closable and the Friedrichs extension exists. Moreover we have obviously that $H_n \leq H_m$ for $n \leq m$ and $H_n \leq H$ for all n . So that H_n forms a monotone sequence of self adjoint operators bounded above by H . Moreover for any f and g in $D(H^{\frac{1}{2}})$, i.e. such that f and $\frac{\partial f}{\partial x_i} \in L_2(d\mu)$ and

$$\int \overline{\nabla f} \cdot \nabla f \, d\mu = \sum_{i=1}^{\infty} \int \left| \frac{\partial f}{\partial x_i} \right|^2 d\mu < \infty$$

and the same for g , we have obviously that

$$(f,H_n g) = \sum_{i=1}^{n} \int \overline{\frac{\partial f}{\partial x_i}}\, \frac{\partial g}{\partial x_i}\, d\mu$$

converges to $(f,Hg) = \sum_{i=1}^{\infty} \int \overline{\frac{\partial f}{\partial x_i}}\, \frac{\partial g}{\partial x_i}\, d\mu(\xi)$.

This is actually a consequence of the Schwarz inequality on the space $L_2(d\mu;K)$. Using then the theorem on convergence from below of symmetric semibounded forms (Theorem 3.13, Ch. VIII Ref. [ ]) we get that $(z-H_n)^{-1}$ converges strongly to $(z-H)^{-1}$ for $z \notin [0,\infty]$. Hence we have resolvent convergence which by the semigroup convergence theorem (Theorem 2.16, Ch IX Ref. [ ]) implies that e^{-tH_n} converges strongly to e^{-tH} , hence that

$$(f,e^{-tH_n}g) \to (f,e^{-tH}g)$$

as $n \to \infty$. That $(f,e^{-tH_n}g)$ has a positive kernel follows immediately from the fact that $N' = P_nK \oplus N_1'$ where N_1' is the annihilator in N' of $P_nK \subset N$, so that $d\mu(\xi) = d\mu(x,\xi_1)$ on the product measure space $N' = P_nK \times N_1'$. Now we have by the quasi invariance of μ that

$$\begin{aligned} d\mu(x+y,\xi_1) &= \alpha(x+\xi_1,y)d\mu(x,\xi_1) \\ &= \frac{\alpha(\xi_1,x)}{\alpha(\xi_1,x+y)}\, d\mu(x,\xi_1) \ , \end{aligned} \tag{2.49}$$

so that $\alpha(\xi_1,x)d\mu(x,\xi_1)$ is translation invariant in x , and since P_nK is finite dimensional we have

$$\alpha(\xi_1,x)d\mu(x,\xi_1) = d\mu_1(\xi_1)dx \ , \tag{2.50}$$

where dx is the Lebesgue measure on P_nK . So that

$$d\mu(x,\xi_1) = \alpha(\xi_1,x)^{-1}\, d\mu_1(\xi_1)dx \ . \tag{2.51}$$

Consider now the correspondence

$$f(\xi) <-> \alpha^{-\frac{1}{2}}(\xi_1,x)f(x,\xi_1) \tag{2.51}$$

which gives a unitary correspondence between $L_2(d\mu)$ and

$L_2(d\mu_1 \times dx)$. Identifying P_nK with R^n we have that $L_2(d\mu_1 \times dx)$ is the direct integral $\mathcal{H} = \int \mathcal{H}_{\xi_1} \, d\mu_1(\xi_1)$ where $\mathcal{H}_{\xi_1} = L_2(R^n)$. Now the correspondence (2.52) takes the form (2.48) onto the direct integral of Sturm-Liouville forms

$$(f,H_n g) <\!\!-\!\!> \sum_{i=1}^{n} \int_{N_1'} [\int_{R^n} \frac{\partial}{\partial x_i}(\alpha^{-\frac{1}{2}}(\xi_1,x)\bar{f}(x,\xi_1)) \cdot \frac{\partial}{\partial x_i}(\alpha^{-\frac{1}{2}}(\xi,x)g(x,\xi_1))\alpha(\xi_1,x)dx]d\mu_1(\xi_1)$$

$$(f,g) <\!\!-\!\!> \int_{N_1'} [\int_{R^n} \bar{f}(x,\xi_1)g(x,\xi_1)dx]d\mu_1(\xi_1) \ . \tag{2.53}$$

Hence we have that $(f,H_n f)$ is the direct integral over N_1' of the Sturm-Liouville form

$$\sum_{i=1}^{n} \int_{R^n} \frac{\partial}{\partial x_i}(\alpha^{-\frac{1}{2}} \bar{f}) \frac{\partial}{\partial x_i}(\alpha^{-\frac{1}{2}} g)\alpha \, dx \tag{2.54}$$

in $L_2(R^n)$. The closability of this form for μ_1- almost all ξ_1 follows from the fact that the direct integral (2.53) is closable. In fact, since (2.53) is given by a self adjoint operator H_n we have that (2.54) is given by a self adjoint operator H_n^{ξ} for μ_1- almost all ξ_1 . The positivity of the kernel for $e^{-tH_n^{\xi}}$ in $L_2(R^n)$ follows from the assumption that the components of β are bounded by Stroock-Varadhan's work on diffusion processes in R^n , see Ref.[3],2). In fact from their work we get that H_n is essentially self-adjoint and generates a strong Markov process. This implies then that e^{-tH_n} has a positive kernel and therefore that e^{-tH} has a positive kernel. This Markov semigroup $P_t = e^{-tH}$ then defines a stochastic Markov process in N' with invariant measure μ which then proves the theorem. □

In what sense the Markov process $\xi(t)$ defined by theorem 2.9 actually solves the equation 2.40 will be discussed later.

Proposition 2.10

Let μ be a quasi invariant probability measure $\mu \in \mathcal{M}(N')$, then there is a standard Borel measure space (Z,dz) and a measurable mapping μ_z from Z into $\mathcal{M}(N')$ such that μ_z is quasi invariant and ergodic for almost all $z \in Z$ and

$$\mu = \int_Z \mu_z \, dz \, .$$

Proof: Let $L_\infty^{in}(d\mu)$ be the closed subspace of $L_\infty(d\mu)$ of functions f such that $f(\xi+x) = f(\xi)$ for all $x \in N$. $L_\infty^{in}(d\mu)$ is obviously closed under multiplication so that it is a commutative C^*-algebra. Hence by the Gelfand representation theorem $L_\infty^{in}(d\mu) \cong C(Z)$, moreover $d\mu$ restricted to $L_\infty^{in}(d\mu)$ defines a positive continuous linear functional on $C(Z)$ which again defines a measure dz on Z. Since $L_\infty^{in}(d\mu)$ is weakly closed in $L_2(d\mu)$ we have that $C(Z)$ is weakly closed in $L_2(dz)$. Hence $C(Z) = L_\infty(dz)$, so that in particular all measurable sets in Z are open. The representation of μ by $\int \mu_z \, dz$ follows from this. □

This proposition together with proposition 2.3 gives us a decomposition of the corresponding representation of the Weyl commutation relation (U,V) as a direct integral over irreducible representations. Namely for $\mathcal{H} = L_2(d\mu)$ and $\mathcal{H}_z = L_2(d\mu_z)$

$$\mathcal{H} = \int_Z \mathcal{H}_z \, dz \tag{2.55}$$

and

$$(U,V) = \int_Z (U_z,V_z) dz \, . \tag{2.56}$$

Theorem 2.11

Let μ be a quasi invariant probability measure with regular first order derivatives on N' relative to a nuclear rigging $N \subset K \subset N'$ and let H be the self adjoint operator on $\mathcal{H} = L_2(d\mu)$ of theorem 2.9. Let $\mathcal{H}_0$ be the eigensubspace of $\mathcal{H}$ corresponding to the eigenvalue zero of H. Then $f \in \mathcal{H}_0$ if and only if $f \in L_2(d\mu)$ and $f(\xi+x) = f(\xi)$ for all $x \in N$.

In particular 0 is a simple eigenvalue with eigenvector $\Omega(\xi) = 1$ if and only if μ is ergodic, which by proposition 2.3 is the case if and only if the representation of the Weyl commutation relation (U,V) is irreducible.

In fact there is a natural isomorphism of $\mathcal{H}_0$ with $L_2(Z)$ where Z is given in proposition 2.10. The direct integral representation (2.55) gives a direct integral representation also of H, in fact

$$H = \int_Z H_z \, dz ,$$

where each H_z has the unique lowest eigenvector $\Omega_z(\xi) = 1$ in $\mathcal{H}_z = L_2(d\mu_z)$ and

$$(f, H_z g)_z = \int \nabla \bar{f} \cdot \nabla g \, d\mu_z$$

for all f and g in $D(H_z) \subset \mathcal{H}_z$. Moreover the decomposition of the measure $\mu = \int_Z \mu_z dz$ is the N-ergodic decomposition of μ. In particular $\mu_{z_1} \perp \mu_{z_2}$ for $z_1 \neq z_2$.

Proof: Let $x \in N$ and f and g in F^2. H^x is the self adjoint operator given by the closable form

$$(f, H^x f) = \int (x \cdot \nabla \bar{f})(x \cdot \nabla f) \, d\mu , \tag{2.57}$$

where $x \cdot \nabla f$ is the derivative in the direction x of f. We have obviously for $f \in F^2$ that

$$(f, H^x f) \leq (f, Hf) = \|H^{\frac{1}{2}} f\|^2 . \tag{2.58}$$

Hence the same inequality must hold for the closure of these forms. Now, if $f \in \mathcal{H}_0$, then $f \in D(H) \subset D(H^{\frac{1}{2}})$ and $H^{\frac{1}{2}} f = 0$.

Hence f is in the domain of the closed form (2.57) and

$$\int (x\cdot\nabla\bar{f})(x\cdot\nabla f)\,d\mu = 0 \qquad (2.59)$$

for arbitrary $x \in N$. Hence $(x\cdot\nabla f)(\xi) = 0$ for all ξ , which gives that $f(\xi) = f(\xi+tx)$ for all t . Hence $f(\xi) = f(\xi+x)$ for all $x \in N$. On the other hand let χ be the characteristic function for an invariant set A of measure different from zero and one. Then by proposition 2.11 χ_1 and $\chi_2 = 1-\chi_1$ are projections onto two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ which are actually given by

$$\mathcal{H}_i = \int_Z \tilde{\chi}_i(z)\,\mathcal{H}_z\,dz ,$$

where $\tilde{\chi}_i$ is the image of χ_i by the mapping $L_\infty^{in}(d\mu) \leftrightarrow L_\infty(dz)$ utilized in the proof of proposition 2.11. Hence we have the non trivial decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ which immediately gives rise to the non trivial decompositions $H = H_1 \oplus H_2$, and $\Omega = \Omega_1 \oplus \Omega_2$, where in fact $\Omega_i(\xi) = \chi_i(\xi)$, and Ω_1 as well as Ω_2 are , by the decomposition $H = H_1 \oplus H_2$, in $\mathcal{H}_0$. Hence we see that there is a natural one to one correspondence between characteristic functions in $L_\infty^{in}(d\mu)$ and elements in $\mathcal{H}_0$. This immediately extends to a one to one correspondence between $L_2(dz)$ and $\mathcal{H}_0$ by the identification used in proposition 2.11 of $L_2(dz)$ with the closure of $L_\infty^{in}(d\mu)$ in $L_2(d\mu)$. The rest of the theorem follows immediately from earlier results. □

<u>Theorem 2.12</u>

Let μ be a quasi invariant probability measure on N' . If μ has weak L_2-derivatives, i.e. if $\frac{1}{t}(\alpha(\xi,tx)-1)$ converges weakly in $L_2(d\mu)$ for each $x \in N$, then μ has regular first order derivatives.

Proof: Since $\frac{1}{t}(\alpha(\xi,tx)-1)$ is weakly L_2-convergent, it is uniformly bounded in L_2, and since $\alpha^{\frac{1}{2}}(\xi,tx)+1 \geq 1$ we have that

$$\frac{1}{t}(\alpha^{\frac{1}{2}}(\xi,tx)-1) = \frac{1}{t}(\alpha(\xi,tx)-1)(\alpha^{\frac{1}{2}}(\xi,tx)+1)^{-1} \qquad (2.60)$$

is uniformly in L_2. Now by the corollary to proposition 2.2 we have that $\alpha^{\frac{1}{2}}(\xi,tx)$ congerves strongly to 1 in $L_2(d\mu)$, so that

$$(\alpha^{\frac{1}{2}}(\xi,tx)+1)^{-1} - \frac{1}{2} = -\frac{1}{2}(\alpha^{\frac{1}{2}}(\xi,tx)-1)(\alpha^{\frac{1}{2}}(\xi,tx)+1)^{-1} \quad (2.61)$$

and since $(\alpha^{\frac{1}{2}}(\xi,tx)+1)^{-1} \leq 1$ we get that $(\alpha^{\frac{1}{2}}(\xi,tx)+1)^{-1}$ converges strongly to $\frac{1}{2}$ in $L_2(d\mu)$. Hence (2.60) also converges weakly in $L_2(d\mu)$, that is $\frac{1}{t}(V(tx)-1)\Omega$ converges weakly in $\mathcal{H} = L_2(d\mu)$ to some limit $u \in \mathcal{H}$. Then Ω must be in the domain of the adjoint $(Px)^*$ of the infinitesimal operator Px. This is so because for $v \in D(Px)$ we have that

$$(v,u) = \lim_{t\to o} \frac{1}{t}(v,(V(tx)-1)\Omega)$$

$$= \lim_{t\to o} \frac{1}{t}(V(-tx)-1)v,\Omega)$$

so that

$$(v,u) = i(Px\,v, \Omega) \qquad (2.62)$$

which says that $(Px\,v,\Omega)$ is a continuous function of v, i.e. $\Omega \in D(Px)^*)$. But since $V(tx)$ is a strongly continuous unitary group, Px is self adjoint so that $\Omega \in D(Px)$, i.e. μ has regular derivatives. This proves the theorem. □

Remark. By this theorem we then have that the conclusions of proposition 2.8, theorem 2.9 and theorem 2.11 hold under the condition of weak L_2-derivatives instead of regular derivatives. In fact, we have the implications

weak L_2-derivatives => regular derivatives => strong L_1-derivatives.

Definition 2.7

A homogeneous process $\eta(t)$ on some measure space $(X,d\nu)$ is called ergodic if for any measurable sets A and B both with positive measure, $\nu(A) > 0$ and $\nu(B) > 0$, there is some $t \geq 0$ such that

$$\Pr\{\eta(0) \in A \ \& \ \eta(t) \in B\} > 0 .$$

Lemma 2.13

Let $\xi(t)$ be a homogeneous self adjoint Markov process on a measure space $(X,d\mu)$ such that μ is an invariant measure. Then $\xi(t)$ is ergodic if and only if $\Omega(\xi) \equiv 1$ is the only eigenfunction corresponding to the eigenvalue zero of the infinitesimal generator.

Remark: If $\xi(t)$ is a process so is $\xi(-t)$ and we call $\xi(-t)$ the adjoint process, and we say that $\xi(t)$ is self adjoint if $\xi(t)$ and $\xi(-t)$ are equivalent processes.

Proof of lemma 2.13 Let $P_t(\xi,\eta)$ be the transition probabilities of $\xi(t)$, i.e. $P_t(\xi,\eta)$ is the kernel of e^{-tH}, the semigroup generated by the process. Assume that $\xi(t)$ is not ergodic. Then there exist two measurable sets A and B with positive μ-measure such that

$$(\chi_A, e^{-tH}\chi_B) = 0 \tag{2.63}$$

for all $t \geq 0$, where χ_A and χ_B are the characteristic functions for A and B. Let now A_1 be the union of the supports of $e^{-tH}\chi_A$ for all $t \geq 0$ i.e.

$$A_1 = \bigcup_{t \geq 0} \operatorname{supp} e^{-tH}\chi_A . \tag{2.64}$$

Let A_2 the complement of A_1, we then have that

$$(\chi_{A_2}, e^{-tH}\chi_A) = 0 \tag{2.65}$$

for all t. Let now A_t be the support of $e^{-tH}\chi_A$.
By the positivity of e^{-sH} we have that

$$\text{supp } e^{-sH}\chi_{A_t} \subseteq \text{supp } e^{-(s+t)H}\chi_A \tag{2.66}$$

so that

$$(\chi_{A_2}, e^{-sH}\chi_{A_t}) = 0 \tag{2.67}$$

for all s and t. Hence by taking supremum over $t \geq 0$ we then get

$$(\chi_{A_2}, e^{-sH}\chi_{A_1}) = 0 \tag{2.68}$$

for all $s \geq 0$,
from which it follows that

$$e^{-sH}\chi_{A_i} = \chi_{A_i} \, . \tag{2.69}$$

To see this we use the fact that

$$e^{-sH}(\chi_{A_1}+\chi_{A_2}) = \chi_{A_1}+\chi_{A_2} \, , \tag{2.70}$$

since $\chi_{A_1}+\chi_{A_2} = 1$. (2.70) together with (2.68) give

$$(\chi_{A_1}, e^{-sH}\chi_{A_2}) = (\chi_{A_1}, \chi_A \;) \tag{2.71}$$

and since $H \geq 0$ we get from the spectral resolution theorem that χ_{A_1} is an eigenvector with eigenvalue zero of H.

Let us now assume that Ω_1 and Ω_2 both are eigenvectors of eigenvalue zero of H. Let us assume $(\Omega_1, \Omega_2) = 0$. Then

$$(\Omega_i, e^{-sH}\Omega_i) = (\Omega_i, \Omega_i) \tag{2.72}$$

and by the positivity of e^{-sH}

$$(|\Omega_i|, e^{-sH}|\Omega_i|) \geq (\Omega_i, e^{-sH}\Omega_i) \, ,$$

but since $H \geq 0$ we then have

$$(|\Omega_i|, e^{-sH}|\Omega_i|) = (|\Omega_i|, |\Omega_i|) \tag{2.73}$$

and by spectral resolution we get that $|\Omega_i|$ are eigenvectors of eigenvalue zero.

Now in case we do not have already that both Ω_1 and Ω_2 are proportional to $|\Omega_1|$ and $|\Omega_2|$, with positive proportionality factors, then for $i = 1$ or 2 we still have that $|\Omega_i| \pm \Omega_i$ are two positive eigenfunctions which are orthogonal. If both Ω_1 and Ω_2 are proportional to $|\Omega_1|$ and $|\Omega_2|$, with positive proportionality factors, then $|\Omega_1|$ and $|\Omega_2|$ are two orthogonal positive eigenfunctions. In any case we see that if there are two eigenfunctions with eigenvalue zero, we may find two positive orthogonal eigenfunctions.

So let now Ω_1 and Ω_2 be two positive orthogonal eigenfunctions with eigenvalue zero. Then obviously for all $t \geq 0$

$$(\Omega_1, e^{-tH}\Omega_2) = 0 \ .$$

Let A_i be the support of Ω_i, then by positivity we also have

$$(\chi_{A_1}, e^{-tH}\chi_{A_2}) = 0$$

for all $t \geq 0$. Hence $\xi(t)$ is non ergodic.
This proves the lemma.

Theorem 2.14

Let $\mu \in \mathcal{M}(N')$ be a quasi invariant probability measure on N', such that μ has regular bounded derivatives. Let $\xi(t)$ be the Markov diffusion process that is generated by μ, relative to the nuclear rigging $N \subset K \subset N'$. Then $\xi(t)$ is ergodic if and only if μ is ergodic in the sense of definition 2.3.

Proof: This theorem is an immediate consequence of lemma 2.13 and theorem 2.11.

3. Perturbations of symmetric diffusion processes

Let now $N \subset K \subset N'$ be a real nuclear rigging of the real separable Hilbert space K and μ a quasi invariant probability
bounded first order
measure on N' with/regular/derivatives and let H be the infinitesimal generator in $L_2(d\mu)$ for the corresponding diffusion process. H is the diffusion operator given by μ. Let $V(\xi)$ be a real measurable function on N' such that

$$H_1 = H + V \tag{3.1}$$

is essentially self adjoint and bounded below. Consider now for $k < l$

$$H_1^{k,l} = H + V^{k,l} \tag{3.2}$$

where

$$V^{k,l}(\xi) = \begin{cases} k & \text{if } V(\xi) < k \\ V(\xi) & \text{if } k \leq V(\xi) \leq l \\ l & \text{if } V(\xi) > l \,. \end{cases} \tag{3.3}$$

Let $V^k(\xi) = \lim\limits_{l \to \infty} V^{k,l}(\xi)$ and

$$H_1^k = H + V^k \,. \tag{3.4}$$

We have obviously that $H_1^{k,l}$ converges monotoneously to H_1^k as $l \to \infty$. Hence by the theorem on convergence from below of symmetric semibounded forms (Theorem 2.13, Ch VIII, Ref. [27],1) we get that

$$\operatorname{strong} \lim_{l \to \infty} e^{-tH_1^{k,l}} = e^{-tH_1^k} \,. \tag{3.5}$$

On the other hand $H_1^k \geq H_1$ and H_1^k converges monotoneously to H_1 as $k \to -\infty$. From which it follows that the corresponding resolvents converge monotoneously, so that

$$\operatorname{strong} \lim_{l \to -\infty} e^{-tH_1^k} = e^{-tH_1} \,. \tag{3.6}$$

Now e^{-tH} has a positive kernel and by Trotter's product formula

$$e^{-tH_1^{k,l}} = \operatorname{st}\lim_{n\to\infty} \left[e^{-\frac{t}{n}H} e^{-\frac{t}{n}V^{k,l}} \right]^n \tag{3.7}$$

we get that $e^{-tH_1^{k,l}}$ has a positive kernel. Hence, by (3.5) and (3.6), e^{-tH_1} has a positive kernel. We have thus the following theorem.

<u>Theorem 3.1</u>

Let H be the diffusion operator given by μ and let V be a measurable real function on N', where $N \subset K \subset N'$ is the muclear rigging. If

$$H_1 = H + V$$

is essentially self adjoint and bounded below, then e^{-tH_1} has a non negative kernel. □

H_1 is assumed to be bounded below, but contrary to H it need not have any eigenvectors. If however, its spectrum ends in an eigenvalue, this eigenvalue must, under weak regularity conditions, have at most the same umotiplicity as the corresponding eigenvalue zero of H which in the precise meaning of theorem 2.11 was the same as the number of irreducible components in the representation of the commutation relations induced in $L_2(d\mu)$. We have in fact the following theorem.

<u>Theorem 3.2</u>

If $H_1 = H + V$ is essentially self adjoint, then the decomposition of theorem 2.11

$$L_2(d\mu) = \int_Z L_2(d\mu_z)dz ,$$

where $\mu = \int_Z \mu_z \, dz$ is the N-ergodic decomposition of μ, is a direct decomposition also of V

$$V = \int V_z \, dz$$

as well as of

$$H_1 = \int H_1^z \, dz \ .$$

Moreover

$$H_1^z = H_z + V_z$$

is essentially self adjoint for almost all z.

Proof: From theorem (2.11) we have that $H = \int H_z \, dz$, where H_z is the diffusion operator generated by the measure μ_z. Since $\mu = \int \mu_z \, dz$ is the N-ergodic decomposition of μ, we have that $\mu_{z_1} \perp \mu_{z_2}$ for $z_1 \neq z_2$, so that the direct decomposition also reduces V, because V is the multiplication by a measurable function $V(\xi)$. From this it follows that H_1 is reduced, and that each component H_1^z is self adjoint and also equal to the closure of $H_z + V_z$ for almost all z. This proves the theorem. $\square$

In view of theorem 3.2 we may restrict our considerations to the case where μ is N-ergodic or, equivalently, to the case where zero is a simple eigenvalue of H, and in this case we have the following theorem:

Theorem 3.3

Let $H_1 = H + V$ be essentially self adjoint and zero a simple eigenvalue of H. If there is an eigenvalue E_1 of H_1 such that $H_1 \geq E_1$ and $H_1 - V$ is essentially self adjoint, then E_1 is a simple eigenvalue of H_1. Moreover we may take the corresponding eigenfunction to be positive almost everywhere.

Proof: Let us assume that there are two eigenfunctions f and g of H_1 with eigenvalue E_1. Using now that e^{-tH_1} has a non negative kernel we may use the technique in the proof of lemma 2.13 to construct two non negative and orthogonal eigenfunctions and then further on to prove that the process generated by $H_1 - E_1$ is non ergodic. Hence there are disjoint measurable sets A and B of μ measure different from zero and one, such that χ_A and χ_B give projections that reduce H_1. χ_A is the characteristic function of A. In fact if f and g are two positive orthogonal eigenfunctions, we may take A and B as the support of f and g respectively.
Hence $\chi_A\, e^{itH_1} = e^{itH_1}\chi_A$, so by Trotter's product formula $\chi_A\, e^{itH} = e^{itH}\chi_A$, since obviously $\chi_A\, e^{itV} = e^{itV}\chi_A$, because χ_A and e^{itV} both are multiplication operators. But then, since 1 is an eigenfunction of eigenvalue zero of H, we get that so is χ_A. But $\mu(A)$ being different from zero and one, this is contrary to the assumptions. Hence there is only one eigenfunction.
Let now A be the support of this eigenfunction. It follows then exactly as above that χ_A is an eigenfunction of H with eigenvalue zero. By assumption we must then have $\chi_A = 1$. This proves the theorem. □

Let us still assume that μ is N-ergodic i.e. that zero is a simple eigenvalue of H. Hf, for $f \in F^2$, is given by

$$Hf = -\Delta f - \beta(\xi)\cdot\nabla f(\xi) \,. \qquad (3.8)$$

Assume that $H_1 = H + V$ has an eigenvalue E_1 such that $H_1 \geq E_1$, then the corresponding eigenfunction φ of H_1 must satisfy the equation, where β is the osmotic velocity for H,

$$(V - E_1)\varphi = \Delta\varphi + \beta(\xi)\nabla\varphi(\xi) , \qquad (3.9)$$

in a weak sense if $V \in L_2(d\mu)$. The weak sense in which it is satisfied is of course that

$$((V-E_1)f,\varphi) = (\Delta f - \beta\cdot\nabla f,\varphi) \qquad (3.10)$$

for all $f \in F^2$. Let us normalize φ such that $\varphi > 0$ and $\int\varphi^2 d\mu = 1$. Since φ is positive almost everywhere we also have that

$$V-E_1 = \frac{\Delta\varphi}{\varphi} + \beta(\xi)\cdot\frac{\nabla\varphi}{\varphi} , \qquad (3.11)$$

which gives the relation in the weak sense between the function $V-E_1$ and the eigenfunction φ . Since V is a multiplication with a measureable function we have that, for any $f \in F^2$,

$$[H_1,f] = [H,f] = -2\nabla f\cdot\nabla - \Delta f \qquad (3.12)$$

on the domain $F^2\Omega$, if $V \in L_2(d\mu)$, because if $V \in L_2(d\mu)$ then $F^2\Omega \subset D(H\ ) \cap D(V) \subset D(H_1)$.

Let us now assume that $H_1 = H+V$ is essentially self adjoint and that $\Omega_1 = \varphi\Omega$ is in $D(H)$ as well as in $D(V)$, and that the measure $d\mu_1 = \varphi^2 d\mu$ has regular first order derivatives with corresponding osmotic velocity β_1 . Let $f \in F^2$ then $f\Omega_1$ is in $D(V)$ since $f\Omega_1 \in D(V)$ is equivalent with $Vf\varphi \in L_2(d\mu)$. Now $\|f\|_\infty < \infty$ and by assumption $V\varphi \in L_2(d\mu)$ so that $f\Omega_1 \in D(V)$. Moreover by (3.12) we have

$$[H,f]\Omega_1 = -\Delta f\Omega_1 - 2\nabla f\cdot\nabla\Omega_1 . \qquad (3.13)$$

That is

$$[H,f]\Omega_1 = -\Delta f\Omega_1 - \beta_1\cdot\nabla f\Omega_1 \qquad (3.14)$$

and $-\Delta f - \beta_1\cdot\nabla f \in L_2(d\mu_1)$ since the components of β_1 are in $L_2(d\mu_1)$ by assumption. Hence, since $\Omega_1 \in D(H)$ so that if $H\Omega_1$

is well defined, we have that $f\Omega_1 \in D(H)$.

Since now $D(H_1) \supset D(H) \cap D(V)$ we have therefore that $f\Omega_1 \in D(H_1)$. But then again by (3.12) and (3.14) we have

$$H_1 f\Omega_1 = [H_1, f]\Omega_1 + E_1 f\Omega_1 \tag{3.15}$$

i.e.

$$H_1 f\Omega_1 = (-\Delta f + E_1 f)\Omega_1 - \beta_1 \cdot \nabla f\Omega_1 \tag{3.16}$$

Hence $H_1 - E_1$ coincides on $F^2\Omega_1$ with the unique diffusion operator given by μ_1 . We have therefore the following theorem.

Theorem 3.4

Let μ be a quasi invariant, N-ergodic probability measure on N' with bounded regular first order derivatives. Let V be measurable and in $L_2(d\mu)$ such that $H_1 = H + V$ is essentially self adjoint with eigenvalue E_1 such that $H_1 \geq E_1$. Then the corresponding eigenfunction φ is positive μ - almost everywhere, such that $d\mu_1 = \varphi^2 d\mu$ is quasi invariant. If moreover μ_1 has regular first order derivatives and $\Omega_1 = \varphi\Omega$ are in $D(H) \cap D(V)$, then $F^2\Omega_1 \subset D(H) \cap D(V)$ and on $F^2\Omega_1$ we have that $H_1 - E_1$ coincides with the infinitesimal generator for the unique diffusion process generated by μ_1 .

We have also the following theorem

Theorem 3.5

Let the assumptions be as in the previous theorem. If in addition $H_1 = H + V$ is self adjoint i.e. $D(H_1) = D(H) \cap D(V)$ then $H_1 - E_1$ is the infinitesimal generator for the unique diffusion process generated by μ_1 .

Proof: By the previous theorem we have that if H' is the diffusion operator generated by μ_1 , then H' coincides with $H_1 - E_1$

on $F^2\Omega_1$. Hence $H' = H + V - E_1$ on $F^2\Omega_1$, so by definition H' is the Friedrichs extension of $H + V - E_1$ on $F^2\Omega_1$. Hence the domain of $H'^{\frac{1}{2}}$ is exactly the elements for which the form $(f\Omega_1, (H+V-E_1)f\Omega_1)$ makes sense as continued from $F^2\Omega_1$. From this it follows that

$$D(H'^{\frac{1}{2}}) \supseteq D(H) \cap D(V) . \qquad 3.17)$$

Now if $H_1 = H + V$ is self adjoint we have that

$$D(H_1) = D(H) \cap D(V) . \qquad 3.18)$$

Therefore

$$D(H_1) \subseteq D(H'^{\frac{1}{2}}) . \qquad 3.19)$$

Now by a well known theorem ([27], 1),Ch.VI,Th.2.11. we have that of all lower bounded self adjoint extensions of the operator $H_1 - E_1$ restricted to $F^2\Omega_1$, only the Friedrichs extension which has domain contained in the domain of the form i.e. in $D(H'^{\frac{1}{2}})$. Hence by (3.19) H_1 is the Friedrichs extension. This proves the theorem. □

These theorems then lead us to another type of perturbation of symmetric diffusion processes which we shall now consider. Let μ be a quasi invariant measure on N' with regular first order derivatives and let $\rho(\xi) > 0$ be a measurable function that is positive μ - almost everywhere such that $d\mu' = \rho d\mu$ is a probability measure. Then μ' is obviously quasi invariant, and let us now further assume that μ' has regular first order derivatives. We get then that the osmotic velocity β' for μ' is given in terms of the osmotic velocity β of μ by

$$\beta'(\xi)x = x\cdot\nabla \ln \rho + \beta(\xi)\cdot x \ , \qquad (3.20)$$

and the assumption is then that $\beta'(\xi)x \in L_2(d\mu')$. We see this is the case if for instance $\nabla\rho^{\frac{1}{2}}$ as well as $\rho^{\frac{1}{2}}\beta(\xi)\cdot x$ are both in $L_2(d\mu)$. For such perturbations we have the following theorem.

Theorem 3.6

Let μ and μ' be two equivalent quasi invariant measures on N' both of which have regular first order derivatives. Let H and H' be the corresponding diffusion operators, then zero is an eigenvector of the same multiplicity for both operators. In fact there is a natural one-to-one isomorphism of the respective eigenspaces corresponding to the eigenvalue zero.

Proof: By theorem 2.11 we have that the eigenspace for the eigenvalue zero is in a one-to-one correspondence with the set of functions in $L_2(d\mu)$ which are invariant under translations by elements in N . Since by assumption μ and μ' are equivalent, there is a natural one-to-one isomorphism between $L_2(d\mu)$ and $L_2(d\mu')$, which takes N-invariant functions of $L_2(d\mu)$ into N-invariant functions of $L_2(d\mu')$. This isomorphism then induces a one-to-one isomorphism of the eigenspaces of H and H' , to the eigenvalue zero. This proves the theorem. □

Let μ still be a quasi invariant measure on N' with regular bounded first order derivatives and let now H_μ be the corresponding diffusion operator, and $\xi_\mu(t)$ the corresponding diffusion process. If μ_n converge weakly to some measure μ, then $H_{\mu_n} \to H_\mu$ in the sense that for any f and $g \in F^2$ we have that $(f\Omega_n, H_{\mu_n} g\Omega_n) \to (f\Omega, H_\mu g\Omega)$. We do not know however whether $\xi_{\mu_n}(t)$ converge weakly to $\xi_\mu(t)$, but we shall see that if $\xi_{\mu_n}(t)$ converge weakly to some Markov process then under slight regularity conditions this Markov process is the process ξ_μ generated by H_μ. In fact we have the following theorem.

Theorem 3.7

Let μ_n be a sequence of quasi invariant probability measures on N', where $N \subset K \subset N'$ is a nuclear rigging of a real separable Hilbert space H, such that μ_n has regular bounded first order derivatives and converges weakly to a measure μ, where μ is also quasi invariant and with regular first order derivatives.

Then for all f and g in F^2 we have that

$$(f\Omega_{\mu_n}, H_{\mu_n} g\Omega_{\mu_n}) \to (f\Omega_\mu, H_\mu g\Omega_\mu) .$$

If moreover the osmotic velocities $\beta_n(\xi)$ of μ_n have components uniformly bounded in L_2, i.e. for any $x \in N$ there is a $c_x > 0$ independent of n such that

$$\int |\beta_n(\xi)\cdot x|^2 d\mu_n(\xi) \le c_x^2$$

then, for any f and g in F^2, $E(f(\xi_{\mu_n}(0))g(\xi_{\mu_n}(t)))$ has a uniform bounded second derivative with respect to t. If moreover the process $\xi_{\mu_n}(t)$ converges weakly to some process $\eta(t)$ in the sense that the joint distribution measure of $\{\xi_{\mu_n}(t_1),\ldots,\xi_{\mu_n}(t_k)\}$ converges weakly to $\{\eta(t_1),\ldots,\eta(t_k)\}$ for any k and any $t_1 \leq \ldots \leq t_k$ then, $E[(f(\eta(0))g(\eta(t))]$ is a twice differentiable function of t and

$$\frac{d}{dt} E\,[f(\eta(0))g(\eta(t))/_{t=0} = (f\Omega_\mu, H_\mu g\Omega_\mu)$$

for any f and g in $F^2(N')$. In particular if $\eta(t)$ is a Markov process, then $H_\eta = H_\mu$ on F^2, where H_η is the infinitesimal generator of η.

Proof. Since

$$(f\Omega_{\mu_n}, H_{\mu_n} g\Omega_{\mu_n}) = \int_{N'} \nabla\bar{f}\cdot\nabla g\, d_{\mu_n} \tag{3.21}$$

the first convergence is obvious. Since for $f \in F^2$ we have that $f\Omega_n$ is in $D(H_{\mu_n})$ and

$$H_{\mu_n} f\Omega_{\mu_n} = (-\Delta f - \beta_n(\xi)\nabla f)\Omega_{\mu_n}. \tag{3.22}$$

By the assumption on f there is a orthogonal projection P_E of finite dimensional range $E \subset N$ such that $f(\xi) = f(P_E\xi)$. We then have

$$\|H_{\mu_n} f\Omega_{\mu_n}\| \leq \|f\|_2 + \sum_i C_i \|f\|_1 \tag{3.23}$$

where $C_i = C_{e_i}$ and $e_1,\ldots,e_k$ is an orthonormal base of E in K and $\|f\|_2$ and $\|f\|_1$ are the D^2 and D^1 norms of f respectively, where D^2 and D^1 are the Banach spaces defined in section 2 below formula (2.33). We see that the estimate (3.30) is independent of n, so that

$$(f\Omega_{\mu_n}, e^{-tH_{\mu_n}} g\,\Omega_{\mu_n}) = E[f(\xi_{\mu_n}(0))g(\xi_{\mu_n}(t))] \tag{3.24}$$

is continuously twice differentiable with a uniformly bounded second derivative

$$(H_{\mu_n} f\Omega_{\mu_n}, e^{-tH_{\mu_n}} H_{\mu_n} g\Omega_{\mu_n}) . \tag{3.25}$$

If $\xi_{\mu_n}(t)$ converge weakly we have in particular that (3.24) converge and the limit is $E[f(\eta(0))g(\eta(t))]$. Since the second derivative is uniformly bounded the first derivatives

$$-(H_{\mu_n} f\Omega_{\mu_n}, e^{-tH_{\mu_n}} g\Omega_{\mu_n}) \tag{3.26}$$

converge uniformly to the first derivative of the limit. This gives us then that

$$\frac{d}{dt} E[f(\eta(0))g(\eta(t))]/_{t=0} = (f\Omega_\mu, H_\mu g\Omega_\mu) . \tag{3.27}$$

Now assume that $\eta(t)$ is a Markov process. Then by the convergence of the processes $\xi_{\mu_n} \to \eta$ and their invariant measures $\mu_n \to \mu$ we see that $\eta(t)$ is homogeneous with invariant measure μ , and since the ξ_{μ_n} are symmetric under time reflection so is η . Hence the infinitesimal generator H_η of η is a positive self adjoint operator in $L_2(d\mu)$ with $\Omega(x) \equiv 1$ as an eigenfunction of eigenvalue zero of H_η . Hence

$$E(f(\eta(0))g(\eta(t))) = (f\Omega, e^{-tH_\eta} g\Omega) . \tag{3.28}$$

From (3.27) we then get that $H_\eta = H_\mu$ on F^2 . This proves the theorem.

4. The Euclidean Markov fields as diffusion processes

The free Euclidean Markov field in $d+1$ dimensions is the generalized random field $\xi(x)$ on R^{d+1} such that

$$E\left[e^{i\int \xi(x)\psi(x)dx}\right] = e^{-\frac{1}{2}(\psi,\psi)_{-1}} \tag{4.1}$$

where

$$(\psi,\psi)_{-1} = \int_{R^{d+1}} (p^2+m^2)^{-1} |\hat{\psi}(p)|^2 dp \tag{4.2}$$

and

$$\hat{\psi}(p) = (2\pi)^{-\frac{d+1}{2}} \int e^{-ipx} \psi(x)dx \ , \tag{4.3}$$

and $m \geq 0$ is a constant called the mass of the free Euclidean Markov field. If $d = 0$ or 1 we have to take $m > 0$ in order for (4.2) to be well defined. The right hand side of (4.1) is obviously a continuous positive definite function on the real nuclear Schwartz space $S(R^{d+1})$ so that (4.1) defines a measure on its dual $S'(R^{d+1})$, i.e. the space of tempered distributions on R^{d+1}. Hence the generalized random field $\xi(x)$ is a random field of tempered distributions.

It is well known that $\xi(x)$ is a Markov field, but we shall not need that here.

However let $\varphi \in S(R^d)$, then $(\varphi \otimes \delta_\tau)(\vec{x},t) = \varphi(\vec{x})\cdot\delta(t-\tau)$ is in the Sobolev space $\mathcal{H}_{-1}$, in fact

$$(\varphi\otimes\delta_\tau, \varphi\otimes\delta_\tau)_{-1} = \tfrac{1}{2}(\varphi,\varphi)_{-\frac{1}{2}} \ , \tag{4.4}$$

where

$$(\varphi,\varphi)_{-\frac{1}{2}} = \int_{R^d} (\vec{p}^2+m^2)^{-\frac{1}{2}} |\hat{\varphi}(\vec{p})|^2 d\vec{p} \tag{4.5}$$

with

$$\hat{\varphi}(\vec{p}) = (2\pi)^{-\frac{d}{2}} \int e^{-i\vec{p}\vec{x}} \varphi(\vec{x})d\vec{x} \ .$$

From (4.1) we get that

$$E\left[e^{i\int \xi(\vec{x},t)\varphi(\vec{x})d\vec{x}}\right] = e^{-\frac{1}{4}(\varphi,\varphi)_{-\frac{1}{2}}} . \tag{4.6}$$

Hence since the right hand side of (4.6) is a positive definite continuous function on the real nuclear space $S(R^d)$ we have that the conditional expectation of the measure with respect to the σ-algebra generated by functions of the form $\langle \xi,\varphi\otimes\delta_\tau\rangle$ exists and defines a measure on $S'(R^d)$. The corresponding random variable with values in $S'(R^d)$ we have already denoted by $\xi(\vec{x},t)$. Hence $t \to \xi(\vec{x},t)$ is a stochastic process with values in $S'(R^d)$.
Let now $\mu_o \in \mathcal{M}(S'(R^d))$ be the probability measure whose Fourier transform is given by (4.6) , i.e. ,

$$\int e^{i\langle \xi,\varphi\rangle} d\mu_o(\xi) = e^{-\frac{1}{4}(\varphi,\varphi)_{-\frac{1}{2}}} \tag{4.7}$$

where $\langle \xi,\varphi\rangle$ is the dualization between $S'(R^d)$ and $S(R^d)$. μ_o is then a Gaussian measure on $S'(R^d)$ and we see easely that it is quasi invariant with respect to translations from $S(R^d)$, in fact if

$$\alpha(\xi,\varphi) = \frac{d\mu_o(\xi+\varphi)}{d\mu_o(\xi)} \tag{4.8}$$

then

$$\alpha(\xi,\varphi) = e^{-(\varphi,\varphi)_{\frac{1}{2}}} e^{2\langle \omega\varphi,\xi\rangle} \tag{4.9}$$

where

$$\widehat{\omega\varphi}(\vec{p}) = (\vec{p}^2+m^2)^{\frac{1}{2}} \hat{\varphi}(\vec{p}) \tag{4.10}$$

and

$$(\varphi,\varphi)_{\frac{1}{2}} = \langle \varphi,\omega\varphi\rangle .$$

From (4.9) it easely follows that μ_o has regular first order derivatives and that the osmotic velocity $\beta(\xi)$ is given by

$$\beta(\xi)\cdot\varphi = -2\langle\omega\,\varphi,\xi\rangle \ , \qquad (4.11)$$

which is obviously in $L_2(d\mu)$.

It is well known that $t \to \xi(x,t)$ is a Markov process in $S'(R^d)$: We shall see now that this process is the diffusion process given by the nuclear rigging

$$S(R^d) \subset L_2(R^d) \subset S'(R^d) \qquad (4.12)$$

and the quasi invariant measure μ_o with regular first order derivatives in the sense of theorem 2.9. We formulate this in the following theorem

Theorem 4.1

Consider the nuclear rigging

$$S(R^d) \subset L_2(R^d) \subset S'(R^d)$$

and the measure $\mu_o \in \mathcal{M}(S'(R^d))$ given by

$$\int e^{i\langle\xi,\varphi\rangle} d\mu_o(\xi) = e^{-\frac{1}{4}(\varphi,\varphi)_{-\frac{1}{2}}} .$$

Then μ_o is quasi invariant with regular first order derivatives and the diffusion process given by μ_o and the nuclear rigging by theorem 2.9 is the free Euclidean Markov field in $d+1$ dimensions.

Proof:

Since the free Euclidean Markov field induces a Markov process $t \to \xi(\vec{x},t)$ on $S'(R^d)$, we have only to show that this process has the same infinitesimal generator as the diffusion process given by theorem 2.9.

By (4.1) we have that

$$E\left[e^{-i\int\xi(\vec{x},0)\varphi_1(\vec{x})d\vec{x}+i\int\xi(\vec{x},t)\varphi_2(\vec{x})d\vec{x}}\right] = e^{-\frac{1}{4}[(\varphi_1,\varphi_1)_{-\frac{1}{2}}+(\varphi_2,\varphi_2)_{-\frac{1}{2}}]}\cdot e^{\frac{1}{2}(\varphi_1,e^{-t\omega}\varphi_2)_{-\frac{1}{2}}} \tag{4.13}$$

where

$$(\varphi_1, e^{-t\omega}\varphi_2)_{-\frac{1}{2}} = \int\frac{e^{-t\omega}}{\omega}\bar{\hat{\varphi}}_1(\vec{p})\hat{\varphi}_2(\vec{p})d\vec{p} \tag{4.14}$$

where $\omega(\vec{p}) = \sqrt{\vec{p}^2+m^2}$. Taking the derivative of (4.13) with respect to t at $t = 0$ we get

$$-\frac{d}{dt}\int e^{-i\langle\xi(0),\varphi_1\rangle}e^{i\langle\xi(t),\varphi_2\rangle}d\mu(\xi)\ \Big|\ t=0 = \langle\varphi_1,\varphi_2\rangle\int e^{-i\langle\xi(0),\varphi_1\rangle}e^{i\langle\xi(0),\varphi_2\rangle}d\mu(\xi)\ . \tag{4.15}$$

From this it follows that for f and g in $F^1(S'(R^d))$ we have that

$$-\frac{d}{dt}\int\bar{f}(\xi(0))g(\xi(t))d\mu(\xi) = \int\overline{\nabla f(\xi(0))}\cdot\nabla g(\xi(0))d\mu(\xi), \tag{4.16}$$

which proves that the infinitesimal generators coincide on F^2 . Moreover let e^{-tH_o} be the semigroup generated by the free Euclidean Markov field, then we have the following well known formula

$$e^{-tH_o}:e^{i\langle\xi,\varphi\rangle}: = :e^{i\langle\xi,e^{-t\omega}\varphi\rangle}: \tag{4.17}$$

where

$$:e^{i\langle\xi,\varphi\rangle}: = e^{\frac{1}{2}(\varphi,\varphi)_{-\frac{1}{2}}}e^{i\langle\xi,\varphi\rangle} \tag{4.18}$$

Hence the linear span of $e^{i\langle\xi,\varphi\rangle}$ for $\varphi\in S(R^d)$ is invariant under the semigroup e^{-tH_o} and it is obviously dense in $L_2(d\mu)$, therefore it is a core for the infinitesimal generator H_o. Now this core is obviously contained in F^2 which proves the theorem. □

Now in one space dimension i.e. $d = 1$ the perturbations of the free Euclidean Markov field by local interactions of different types have been intensively studied. For simplicity we shall here restrict our attention to the polynomial interactions. So let $p(s)$ be a real polynomial of one real variable s such that $p(s)$ is bounded below. Moreover let $P^{(\leq n)}$ be the closed subspace of $L_2(d\mu_o)$ generated by polynomials of degree at most n on S', i.e. by functions of the type $\langle\varphi_1,\xi\rangle\ldots\langle\varphi_k,\xi\rangle$ where $\varphi_1,\ldots,\varphi_k$ are in $S(R)$ and $\xi \in S'(R)$ and $k \leq n$. Let $P^{(n)}$ be the orthogonal complement of $P^{(\leq n-1)}$ in $P^{(\leq n)}$.

Let us now define for any $h \in L_2(R)$

$$:\xi^n:(h) \equiv \int_R :\xi(x)^n: h(x)dx \tag{4.19}$$

as the unique element in $P^{(n)}$ such that

$$(:\xi^n:(h), \langle\varphi_1,\xi\rangle\ldots\langle\varphi_n,\xi\rangle) = n!\int\ldots\int \prod_{i=1}^{n} G(y_i-x)\varphi_i(y_i)h(x)dy_i dx, \tag{4.20}$$

where

$$G(x) = \frac{1}{2}\int_R (p^2+m^2)^{-\frac{1}{2}} e^{ipx}\, dp\ . \tag{4.21}$$

Since the projections of $\langle\varphi_1,\xi\rangle\ldots\langle\varphi_n,\xi\rangle$ on $P^{(n)}$ obviously span a dense subset of $P^{(n)}$, (4.19) gives us a densely defined linear functional on $P^{(n)}$. In fact this linear functional is bounded in as much as its square norm may be computed to be

$$(:\xi^n:(h)\ ,:\xi^n:(h)\ ) = n!\iint G(x-y)^n h(x)h(y)dxdy\ , \tag{4.22}$$

which is finite since $G(x)$ has only a logarithmic singularity. In fact since $G(x)^n$ is in $L_1(R)$ (4.21) is finite for any $h \in L_2(R)$.

Now if

$$p(s) = \sum_{k=o}^{2n} a_k s^k \tag{4.23}$$

we define

$$:p:(h) = \sum_{k=o}^{2n} a_k :\xi^k:(h) \tag{4.24}$$

for any $h \in L_2(R)$, and since $P^{(i)}$ are orthogonal for different i we have that (4.23) is an orthogonal sum in $L_2(d\mu_o)$. In particular we have that

$$\| :p:(h) \|_2^2 = \sum_{k=o}^{2n} a_k^2 \, k! \iint G(x-y)^k h(x)h(y)dxdy . \tag{4.25}$$

Let now H_o be the diffusion operator generated by μ_o and the real rigging $S(R) \subset L_2(R) \subset S'(R)$. We have seen that H_o is the infinitesimal generator of the Markov process given by the free Euclidean Markov field.

Let $V_k(\xi)$ be the real function in $L_2(d\mu_o)$ given by

$$V_1 = :p:(\chi_1) \tag{4.26}$$

where χ_1 is the characteristic function for $[-1,1]$ and $p(s)$ is a real polynomial which is bounded below. It is well known, see for instance ref. [23] 3) that

$$H_1 = H_o + V_1 \tag{4.27}$$

is essentially self adjoint and bounded below and has an isolated simple eigenvalue E_1 such that $H_1 \geq E_1$. The corresponding eigenfunction $g_1(\xi)$ may be chosen positive μ_o - almost everywhere. The measure

$$d\mu_1 = g_1^2 \, d\mu_o \tag{4.28}$$

is therefore equivalent with μ_o , hence quasi invariant with respect to translation in S . Now let $\Omega_1 = g_1 \Omega_o$ where $\Omega_o(x) \equiv 1$ in $L_2(d\mu_o)$.

Lemma 4.1

Let $\varphi \in S(R)$ and $P\varphi$ the intinitesimal generator for the one parameter unitary group of translations by $t\varphi$ in $L_2(d\mu_o)$. Then $i[P\varphi, H_l]$ is a densely defined operator whose closure is given by

$$\overline{i[P\varphi, H_l]} = :p':(\chi_l \cdot \varphi) + \langle \xi, (-\Delta + m^2)\varphi \rangle$$

where p' is the derivative of p and χ_l the characteristic function for $[-l, l]$.

Proof: The proof follows immediately from the fact that $i[P\varphi, H_l]$ is the derivative at $t = 0$ of $e^{itP\varphi} H_l e^{itP\varphi} = H_l^{t\varphi}$, where

$$H_l^{\varphi} = H_o + \langle \xi, (-\Delta+m^2)\varphi \rangle + \tfrac{1}{2}\langle \varphi, (-\Delta+m^2)\varphi \rangle + :p_{\varphi}:(\chi_l) \tag{4.29}$$

and

$$:p_{\varphi}:(\chi_l) = \sum_{k=1}^{2n} a_k \int_{-l}^{l} :(\xi+\varphi)^k(x): dx \;, \tag{4.30}$$

with

$$\int_{-l}^{l} :(\xi+\varphi)^n(x): \, dx = \sum_{j=1}^{n} \binom{n}{j} :\xi^j:(\varphi^{n-j}\chi_l) \;. \quad \Box \tag{4.31}$$

Theorem 4.1

μ_l is a quasi invariant measure on $S'(R)$ which has regular first order derivatives. Moreover the components of the corresponding osmotic velocity β_l have $L_2(d\mu_l)$ norms which are bounded uniformly in l if the coefficients of l are small enough.

Proof: Let $\varphi \in S(R)$, then $\beta_l \cdot \varphi$ is equal to twice the derivative of $e^{itP\varphi}\Omega_l$ at $t = 0$, if it exists, so that $\beta_l \cdot \varphi$ is in $L_2(d\mu_l)$ iff $\Omega_l \in D(P\varphi)$ and

$$\beta_l \cdot \varphi = 2P\varphi\Omega_l \;. \tag{4.32}$$

Now

$$P\varphi\Omega_l = -(H_l - E_l)^{-1}[P\varphi, H_l - E_l]\Omega_l \tag{4.33}$$

so that

$$P\varphi\Omega_l = -\frac{H_l - E_l + C}{H_l - E_l}(H_l - E_l + C)^{-1}[P\varphi, H_l]\Omega_l \ . \tag{4.34}$$

But $(H_l-E_l+C)^{-1}[P\varphi, H_l-E_l]\Omega_l$ is in the range of H_l-E_l, hence orthogonal to Ω_l. Now, for fixed $C > 0$, $(H_l-E_l+C)(H_l-E_l)^{-1}$ is bounded in norm on the complement of Ω_l by a constant that depends only on the distance m_l from E_l to the rest of the spectrum of H_l. This distance m_l is called the mass gap for H_l and it is well known (see [36] that if all the coefficients of p are small enough this distance is bounded from below by a positive constant. Hence in that case $(H_l-E_l+C)(H_l-E_l)^{-1}$ is bounded in norm uniformly in l. Therefore

$$\|P\varphi\Omega_l\| \leq C_1\|(H_l-E_l+C)^{-1}[P\varphi, H_l](H_l-E_l+C)^{-1}\Omega_l\| \tag{4.35}$$

where C_1 is a constant that depends only on p and C. By lemma 4.1 it is therefore enough to prove that, if $:p_1:(h) = :p':(h) + \langle \xi, (-\Delta+m^2)\varphi\rangle$, then

$$(H_l-E_l+C)^{-1} \ :p_1:(\chi_l\varphi)(H_l-E_l+C)^{-1} \tag{4.36}$$

is norm bounded uniformly in l. But this follows from

$$\pm \ :p_1:(\chi_l\varphi) \leq C_2(H_l-E_l+C) \tag{4.37}$$

where C_2 is independent of l. This is proved by resolution of the identity from ref. [34]. We also remark that recently Glimm and Jaffe have proved similar inequalities for the polynomially interacting fields with Dirichlet boundary conditions [35].

From (4.36) we have that

$$(H_l-E_l+C)^{-\frac{1}{2}} :p_1:(\chi_l\varphi)(H_l-E_l+C)^{-\frac{1}{2}} \tag{4.38}$$

is a bounded operator with norm independent of l . Hence (4.35) is bounded with norm independent of l . This proves the theorem.

□

Now it follows from ref. [36] that if the coefficients of p are small enough then the process $\xi_l(t)$ converges weakly to a process $\xi(t)$, however it is not known whether $\xi(t)$ is a Markov process.

Consider now for f and g in $F^2(S')$

$$(f\Omega_l, e^{-t(H_l-E_l)} g\Omega_l) = E[\bar{f}(\xi_l(0))g(\xi_l(t))] \tag{4.39}$$

which by the results of ref. [36] converge to

$$(f\Omega, e^{-tH} g\Omega) = E[\bar{f}(\xi(0))g(\xi(t))] , \tag{4.40}$$

where H is the physical Hamiltonian.

By theorem 4.1 and theorem 3.7 we have that (4.39) is twice differentiable with respect to t and the first derivative converges uniformly to the first derivative of (4.40). Hence we have in particular that μ_l converges weakly to a measure μ which is actually the physical vaccuum Ω restricted to the time zero fields i.e.

$$\int e^{i\langle \xi,\varphi\rangle} \, d\mu(\xi) = (\Omega, e^{i\langle\varphi,\xi(0)\rangle}\Omega) . \tag{4.41}$$

Now from (4.36) it follows by standard method [37, [23], 3)

$$\pm :p_1:(\varphi) \le C_2(H+C) \tag{4.42}$$

and from lemma 4.1 that

$$i\overline{[P\varphi,H]} = :p_1:(\varphi) . \tag{4.43}$$

Hence in the same way as for μ_l we get that μ has regular/first order derivatives in particular that μ is quasi invariant. Therefore we have the following theorem.

Theorem 4.2

Let μ be the physical vacuum restricted to the time zero fields as defined by (4.40). Then μ is a quasi invariant measure with regular first order derivatives. Moreover the physical Hamiltonian H restricted to $F^2\Omega$ coincides with the diffusion operator generated by μ, by theorem 2.9.

Proof: This follows by what is said above and theorem 3.6.

Remark: Bounds of the form (4.37) and (4.41) have been resently proved also for the Dirichlet boundary conditions on the fields by Glimm and Jaffe [35]. Hence theorem 4.1 and theorem 4.2 will also hold for the Dirichlet boundary conditions and their infinite volume limits, which also exist, by the method of Nelson [38], for arbitrary even polynomial p. In this case there is no smallness condition on the coeffisients of p.

References

[1] 1) A.N. Kolmogorov, Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. 104 (1931), 415-458.

2) W. Feller, Some new connections between probability and classical analysis, Proc. Intern. Congr. Math. 1958, pp 69-86.

[2] 1) E.B. Dynkin, Die Grundlagen der Theorie der Markoffschen Prozesse, Springer, Berlin (1961).

2) E.B. Dynkin, Markov processes, Vol. I and II, Springer, Berlin (1965).

3) K. Ito - H.P. McKean, Jr., Diffusion processes and their sample paths, Springer, Berlin (1965).

[3] 1) I.I. Gihman - A.V. Skorohod, Stochastic differential equations, Springer, Berlin (1972) (transl.)

2) D.W. Stroock - S.R.S. Varadhan, Diffusion processes with continuous coefficients, I, II, Commun. Pure Appl. Math. 22, 345-400 and 479-530 (1969).

3) P. Priouret, Processus de diffusion et équations différentielles stochastiques, In P.A. Meyer, P. Priouret, F. Spitzer, Ecole d'Eté de Probabilités de Saint Flour III, 1973, Springer (1974).

[4] R.M. Blumenthal, R.K. Getoor, Markov processes and potential theory, Academic Press (1969)

Z. Cieselski, Lectures on Brownian motion, heat conduction and potential theory, Aarhus University Lectures Notes, No 3 (1965).

P. A. Meyer, Probability and potentials, Blaisdell, 1966.

[5] 1) K.O. Friedrichs - H.N. Shapiro, Proc. Nat. Ac. of Sciences, 43, No 4, 336-338 (1957).

2) I.M. Gelfand - N. Ya. Vilenkin, Generalized functions, Vol. 4, Academic Press (1964) (transl.)

3) I. Segal, Mathematical problems of relativistic physics, Am. Math. Soc., Providence (1963).

[6] 1) S. Albeverio - R. Høegh-Krohn, Mathematical theory of Feynman path integrals, Oslo Univ. Preprint, October 1974 (to be publiched).

2) S. Albeverio - R. Høegh-Krohn, Oscillatory integrals in infinitely many dimensions and applications to the study of the classical limit of quantum mechanics I., Oslo Univ. Preprint, May 1975 (to be published).

[7] Yu.L. Daletskii, Infinite - dimensional elliptic operators and parabolic equations connected with them, Rull. Math. Surv. 22, No 4, 1-53 (1967) (transl.).

Yu. L. Daletskii, Ya.I. Shnaiderman, Diffusion and quasi-invariant measures on infinite-dimensional Lie groups, Funct. Anal. and its Appl. 3, No 2, 88-90 (1969).

[8] M.I. Vishik, The parametrix of elliptic operators with infinitely many independent variables, Rull. Math. Surv. 26, No 2, 91-112 (1971) (transl.)

M.I. Vishik, les opératours différentiels et pseudo différentiels a une infinité de variables des problèmes elliptiques et paraboliques, ét les équations aux dérivées partielles, CNRS (1973), 342-362.

[9] 1) L. Gross, Potential theory on Hilbert space, J. Funct. Anal. 1, 123-181 (1967).

2) A. Piech, Some regularity properties of diffusion processes on abstract Wiener space, J. Funct. Anal. 8, 153-172 (1971).

[10] J. Eells, K.D. Elworthy, Wiener integration on certain manifolds, in Problems in nonlinear analysis, C.I.M.E. IV, 67-94 (1970).

[11] P. Kree, Solutions faibles d'équations aux dérivées fonctionnelles II, Séminaire P. Lelong, (Analyse) 14^{e} anuée, 1973/74.

[12] 1) Y. Umemura, Measures on infinite dimensional vector spaces, Publ. Res. Inst. Kyoto Univ. A.1, 1-54 (1965).

2) Xia Dao-Xing, Measure and integration theory on intinite-dimensional spaces, Academic Press, New York (1972).

3) G.C. Hegerfeldt, On canonical commutation relations and infinite-dimensional measures, J. Math. Phys. 13, 45-50 (1972).

[13] H. Araki, Hamiltonian formalism and the canonical commutation relations in quantum field theory, J. Math. Phys. 1, 492-504 (1960).

[14] K. Symanzik, Euclidean quantum fild theory, in local quantum theory, R. Jost Ed., Academic Press, New York (1969).

[15] 1) E. Nelson, Construction of quantum fields from Markoff fields, J. Funct. Anal. 12, 97-112 (1973).

2) E. Nelson, The free Markoff field, J. Funct. Anal. 12, 211-227 (1973).

[16] P. Levy, Processus stochastiques et mouvement brownien, Gauthier - Villars, Paris, (1965).

[17] E. Wong, Homogeneous Gauss-Markov random fields, Annals of Math. Stat. 40, 1625-1634 (1969).

[18] G.M. Molčan, Characterization of Gaussian fields with the Markov property, Dokl. Ak. Nauk 197, No 4, 784-787 (1971).

[19] 1) F. Guerra, P. Ruggiero, New interpretation of the Euclidean-Markov field in the framework of physical Minkowski space-time, Phys. Rev. Letts. 31, 1022-1025 (1973).

2) F. Guerra, On stochastic field theory, J. de Phys. Suppl. T. 34, Face. 11-12, Collogne, C - 1 - 95 - 98.

[20] S. Albeverio, R. Høegh-Krohn, A remark on the connection between stochastic mechanics and the heat equation, J. Math. Phys. 15, 1745-1747 (1974).

[21] 1) J.R. Klander, Stochastic processes and quantum theory, lecture given at XIV International Universitätwochen für Kernphysik, Schladming, Austria, (1975).

2) J.R. Klander, Solvable models and the meaning of nonrenormalizability, Intern. Sympos. Math. Probl. Theor. Phys., Kyoto (1975).

3) J.R. Klander, Field theory as noice theory, Phys. letts.

[22] G.C. Hegerfeldt, Probability measures on distribution spaces and quantum field theoretical models, Göttingen Preprint.

[23] 1) J. Glimm, A. Jaffe, Boson quantum field models, in Mathematics of contemporary physics, R.F. Streater Ed., Academic Press (1972), pp. 77-143.

2) G. Velo, A. Wightman, Edts, Constructive quantum field theory, Springer (1973)

3) B. Simon, The $P(\varphi)_2$ Euclidean (quantum) field theory, Princeton Univ. Press, Princeton (1974).

[24] R. Høegh-Krohn, Relativistic quantum statistical mechanics in two-dimensional space time, Commun. math. Phys. 38, 195-224 (1974).

A. Jaffe, Status of constructive field theory, Int. Conference High Energy Physics, London, 1974).

J. Glimm, A. Jaffe, Particles and bound states and progress toward unitarity and scaling, Int. Symp. Mathem. Probl. Theor. Physics, Kyoto, 1975).

B. Simon, Bose quantum field theory as an Ising ferromagnet: recent developments, Preprint.

[25] C. Newman, The construction of stationary two-dimensional Markoff fields with an application to quantum field theory, J. Funct. Anal. 14, 44-61 (1973).

J. Fröhlich, Schwinger functions and their generating functionals, to appear in Adv. Math.

B. Simon, Ref. [23], 3).

[26] R.L. Dobrushin, R. Minlos, Construction of a one-dimensional quantum field by means of a continuous Markov field, Funct. Anal. and its Appl. 7, No 4, 81-82 (1973) (transl.).

R.L. Dobrushin, R. Minlos, Markov fields theory and application to quantum boson fields, to appear in Proc. Karpacz School "Probabilistic and Functional Methods in Quantum Field Theory".

[27] 1) T. Kato, Perturbation theory of linear operators, Springer, Berlin ( ).

2) W.G. Faris, Self-adjoint operators, Springer, Heidelberg (1975).

[28] 1) B. Simon, R. Høegh-Krohn, Hypercontractive semi-groups and two-dimensional self-coupled Bose fields, J. Funct. Anal. 9, 121-180 (1972).

2) W. Faris, B. Simon, Degenerate and non-degenerate ground states for Schrödinger operators, Univ. Arizona and Princeton Preprint.

[29] R.A. Minlos, Generalized random processes and their extension in measure, in "Selected translations in mathematical statistics and probability", No 3, Am. Math. Soc., Providence, (1963).

[30] K. Yosida, Functional analysis, Springer, Berlin (1968).

[31] L. Gross, Abstract Wiener space, in Proc. Fifth Berkeley Symp. Mathem. Stat. and Prob., 1965-1966, Univ. Calif. Press, Berkeldy.

[32] K.R. Parthasaraty, Probability measures on metric spaces, Academic Press, New York (1967).

[33] S. Albeverio, R. Høegh-Krohn, The Wightman axioms and the mass gap for strong interactions of exponential type in two-dimensional space-time, J. Funct. Anal. 16, 39-82,(1974).

[34] T. Spencer, Perturbation of the $P(\varphi)_2$ quantum field Hamiltonian, J. Math. Phys., Vol. 14, No 7 (1973).

[35] J. Glimm, A. Jaffe, φ^j bounds in $P(\varphi)_2$ quantum field models, Preptint.

[36] J. Glimm, A. Jaffe, T. Spencer, The wightman axioms and particle structure in the $P(\varphi)_2$ quantum field model, Ann. Math.

[37] J. Glimm, A. Jaffe, On the approach to the critical point, Ann. Inst. H. Poincaré 22.

[38] E. Nelson, Probability theory and Euclidean field theory, in Ref. [23], 2).