On the spectrum of the analytic generator

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Abstract If $\{u_t\}$ is a strongly continuous one parameter group of unitaries on a Hilbert space $\mathcal{H}$, there is a unique selfadjoint operator h on $\mathcal{H}$, the infinitesimal generator, such that $u_t = \exp ith$ for all $t \in \mathbb{R}$. The operator $\Delta = \exp h$, defined by spectral theory, is positive self-adjoint. Roughly speaking, Δ is the analytic continuation of u_t to the point $z = -i$.

In the case of a strongly continuous one parameter group of isometries on a Banach space one still has an infinitesimal generator. Recently also the analogue of Δ has been defined and is called the analytic generator [2]. Implicitly this operator has played an important rôle in the Tomita-Takesaki theory for von Neumann algebras [12,14,15,16].

The spectrum of the infinitesimal generator is always real and one would expect, as in the case of unitaries, that the spectrum of the analytic generator would always be positive. After all, intuitively speaking, the analytic generator is the exponential of the infinitesimal generator. In this paper we give an example to show that even in fairly normal cases this is not true in general. In our example the whole complex plane is in the spectrum. We apply this result to obtain an example of two unbounded operators, both of which have positive spectrum, but such that the Banach space tensor product with respect to a certain cross-norm also has the whole complex plane in its spectrum.

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1. The analytic generator

Let $\mathcal{H}$ be a Hilbert space. By $\mathcal{B}(\mathcal{H})$ we will denote the set of all bounded linear operators in $\mathcal{H}$. It is well known that $\mathcal{B}(\mathcal{H})$ is the dual of a Banach space which is uniquely determined up to isometric isomorphism. It is called the predual of $\mathcal{B}(\mathcal{H})$ and it can be identified with the Banach space of all trace class operators in $\mathcal{H}$. The σ-weak topology on $\mathcal{B}(\mathcal{H})$ is the weak topology induced by the predual, i.e. the w^*-topology on $\mathcal{B}(\mathcal{H})$ as a dual space, [3,8].

Let $\{u_t, t \in \mathbb{R}\}$ be a strongly continuous one-parameter group of unitaries on $\mathcal{H}$. For any $t \in \mathbb{R}$ we define a linear operator α_t on $\mathcal{B}(\mathcal{H})$ by

$$\alpha_t(x) = u_t^* x u_t$$

Clearly every α_t is isometric and σ-weakly continuous. Moreover $\{\alpha_t, t \in \mathbb{R}\}$ is a one-parameter group of isometries and it is σ-weakly continuous in the sense that for all $x \in \mathcal{B}(\mathcal{H})$ the map

$$t \in \mathbb{R} \rightarrow \alpha_t(x)$$

is continuous with respect to the σ-weak topology on $\mathcal{B}(\mathcal{H})$.

For such groups one can define an "analytic generator" as follows [2].

<u>1.1 Definition</u> The analytic generator of the group $\{\alpha_t, t \in \mathbb{R}\}$ is an operator A on $\mathcal{B}(\mathcal{H})$ whose domain $\mathcal{D}(A)$ consist of all elements $x \in \mathcal{B}(\mathcal{H})$ for which there is an operator valued complex function F_x defined on the strip $-1 \leq \operatorname{Im} z \leq 0$, bounded and continuous on this strip, analytic in the interior,

and such that

$$F_x(t) = \alpha_t(x) \qquad \text{for all} \quad t \in \mathbb{R} .$$

Of course $\mathcal{B}(\mathcal{H})$ is considered here with its σ-weak topology.

Remark that such a function must be unique, because if there were two such functions, having the same value on $\mathbb{R}$, the difference would vanish on $\mathbb{R}$, one could use the reflection principle and obtain a function defined and analytic for $-1 < \operatorname{Im} z < 1$ and zero for $\operatorname{Im} z = 0$. Such a function must be zero.

Therefore given $x \in \mathcal{D}(A)$ one can define $A(x)$ by

$$A(x) = F_x(-i) .$$

It is immediately clear that $\mathcal{D}(A)$ is a linear space and that A is linear on $\mathcal{D}(A)$.

Roughly speaking the analytic generator A is the analytic extension of α_t to the point $z = -i$. This is easily seen in the following examples.

1.2 Examples. In the definition of A we have not used the explicit form of the α_t, We could as well have taken the unitary group $\{u_t\}$ on $\mathcal{H}$ itself, with the norm topology, to obtain an operator which we will denote by Δ. It is not hard to verify that in this case $\Delta = \exp h$ where h is the infinitesimal generator of $\{u_t\}$ in the sense that $u_t = \exp ith$ for all $t \in \mathbb{R}$. In particular Δ is a positive non-singular self-adjoint operator on $\mathcal{H}$ and $u_t = \Delta^{it}$ for all $t \in \mathbb{R}$.

In the case where the infinitesimal generator h of $\{u_t\}$ is bounded it is clear that u_t has an analytic extension to all of C, namely $\exp izh$. But then $\mathcal{D}(A) = \mathcal{B}(\mathcal{H})$ and for any $x \in \mathcal{B}(\mathcal{H})$ we will have

$$F_x(z) = e^{-izh} x e^{izh}$$

so that $A(x) = e^{-h} x e^{h} = \Delta^{-1} x \Delta$.

Cioranescu and Zsido [2] have proved that the operator A is σ-weakly closed and that its domain is σ-weakly dense (in fact in a much more general situation). In this particular case the σ-weak density of $\mathcal{D}(A)$ can easily be obtained from spectral theory (see also section 3). The closedness of A is more difficult to obtain, however the weaker condition that A is closed in the norm topology on $\mathcal{B}(\mathcal{H})$ follows in an easy way from the maximum modulus principle for the strip. The proof of this fact is the easy part in Cioranescu's and Zsido's proof of the σ-weakly closedness of A. We include it here for completeness.

<u>1.3 Proposition</u> The operator A is closed in the norm topology on $\mathcal{B}(\mathcal{H})$.

<u>Proof</u> Let $\{x_n\}$ be a sequence in $\mathcal{D}(A)$, and x and y operators in $\mathcal{B}(\mathcal{H})$ such that

i) $x_n \to x$

ii) $A(x_n) \to y$

both in the norm topology.

Then with the notations of 1.1 we have

$$\|F_{x_n}(t) - F_{x_m}(t)\| = \|\alpha_t(x_n) - \alpha_t(x_m)\| = \|x_n - x_m\| .$$

On the other hand, from the uniqueness of the analytic extension, it is easy to see that $a \in \mathcal{D}(A)$ implies $\alpha_t(a) \in \mathcal{D}(A)$ for all $t \in \mathbb{R}$ and

$$F_a(t-i) = A(\alpha_t(a)) = \alpha_t(A(a))$$

Then

$$\|F_{x_n}(t-i) - F_{x_m}(t-i)\| = \|\alpha_t(A(x_n)) - \alpha_t(A(x_m))\|$$
$$= \|A(x_n) - A(x_m)\| .$$

From the maximum modulus principle for the strip [7] it now follows that

$$\|F_{x_n}(z) - F_{x_m}(z)\| \leq \max\{\|x_n - x_m\| , \|A(x_n) - A(x_m)\|\} .$$

for all z in the strip $S = \{z \in \mathbb{C} \mid \operatorname{Im} z \in [-1,0]\}$.

So, from the assumptions on x_n, the functions F_{x_n} form a cauchy sequence uniformly on this strip S and therefore converge uniformly to a function F defined and bounded on S. Clearly F will also be continuous on and analytic inside the strip with respect to the σ-weak topology since F is the uniform limit in the norm topology of such functions and the norm topology is stronger than the σ-weak topology.

Now as $F(t) = \lim F_{x_n}(t) = \lim \alpha_t(x_n) = \alpha_t(x)$ we have that $x \in \mathcal{D}(A)$ and $F = F_x$. Finally $A(x) = F(-i) = \lim F_{x_n}(-i) = \lim A(x_n) = y$.

This proves that A is closed in the norm topology on $\mathcal{B}(\mathcal{H})$.

2. The spectrum of the analytic generator

For the examples 1.2 it is very easy to locate the spectrum of the analytic generator. In the case of the unitary group $\{u_t\}$ the analytic generator Δ is positive selfadjoint and therefore its spectrum $\sigma(\Delta)$ is contained in $\mathbb{R}^+$.

In the second example one can show that $\sigma(A) = \sigma(\Delta^{-1})\sigma(\Delta)$; from spectral theory it is not difficult to obtain that at least $\sigma(A) \supseteq \sigma(\Delta^{-1})\sigma(\Delta)$. To obtain the other inclusion one can argue as follows. We define two operators A_1 and A_2 on $\mathcal{B}(\mathcal{H})$ by

$$A_1(x) = \Delta^{-1}x \quad , \quad A_2(x) = x\Delta \quad ,$$

then $A = A_1 \cdot A_2$ while A_1 and A_2 commute. Then we know that $\sigma(A) \subseteq \sigma(A_1) \cdot \sigma(A_2)$. Now one can easily verify that $\sigma(A_1) = \sigma(\Delta^{-1})$ and $\sigma(A_2) = \sigma(A)$.

Thus one might expect that $\sigma(A) \subseteq \mathbb{R}^+$ in general, after all roughly speaking A is the exponential of the infinitesimal generator which always has real spectrum as we are working with isometries.

We will now proceed to show that in general one will not have that $\sigma(A) \subseteq \mathbb{R}^+$. In fact in our example $\sigma(A) = \mathbb{C}$. All this is closely related to a counterexample in [16] where the question of the spectrum came up.

We will deal with a very specific example.

2.1 Notation Let $\mathcal{H}$ be the Hilbert space $\mathcal{L}_2(\mathbb{R})$ of (equivalence classes of) square integrable functions on $\mathbb{R}$ with respect to Lebesque measure. For the group of unitaries we will take the translations. So

$$(u_t \xi)(s) = \xi(s-t)$$

where $\xi \in \mathcal{L}_2(\mathbb{R})$ and $s,t \in \mathbb{R}$.

It is well known that the $\{u_t\}$ defined in this way form a strongly continuous one-parameter group of unitaries on $\mathcal{H}$.

Now we are going to determine certain operators in the domain $\mathcal{D}(A)$ of the analytic generator of the group $\{\alpha_t\}$ associated to the group $\{u_t\}$ as in section 1.

For any $g \in \mathcal{L}_\infty(\mathbb{R})$, the bounded measurable functions on $\mathbb{R}$, there is a bounded operator M_g in $\mathcal{B}(\mathcal{H})$ such that

$$(M_g \xi)(s) = g(s)\xi(s)$$

where $\xi \in \mathcal{L}_2(\mathbb{R})$ and $s \in \mathbb{R}$. Moreover $\|M_g\| = \|g\|_\infty$ where $\|g\|_\infty$ is the essential supremum of g .

We will now give a sufficient condition for an operator of the form M_g with $g \in \mathcal{L}_\infty(\mathbb{R})$ to be in the domain $\mathcal{D}(A)$ of A .

2.2 Lemma Let $g : \mathbb{R} \to C$ be bounded and continuous, and suppose that g has a bounded and continuous extension $\tilde{g}$ to the strip $S = \{z \in \mathbb{C} \mid \operatorname{Im} z \in [-1,0]\}$ which is analytic inside S , then $M_g \in \mathcal{D}(A)$ and

$$A(M_g) = M_{g^{-i}}$$

where $g^{-i}(t) = \tilde{g}(t-i)$ for $t \in \mathbb{R}$.

Proof For each $z \in S$ let g^z denote the function defined by

$$g^z(t) = \tilde{g}(t+z) \qquad \text{for all } t \in \mathbb{R} .$$

Then define $F(z) = M_{g^z}$. This is possible because every g^z is bounded since $\tilde{g}$ is bounded on S.

We claim that $F(t) = \alpha_t(M_g)$. Indeed

$$\begin{aligned}(\alpha_t(M_g)\xi)(s) &= (u_t^* M_g u_t \xi)(s) = (M_g u_t \xi)(s+t) \\ &= g(s+t)(u_t\xi)(s+t) \\ &= g(s+t)\xi(s) \\ &= g^t(s)\xi(s) \\ &= (M_{g^t}\xi)(s)\end{aligned}$$

where $\xi \in \mathcal{L}_2(\mathbb{R})$ and $t,s \in \mathbb{R}$.

So $\alpha_t(M_g) = M_{g^t} = F(t)$.

Next $\|F(z)\| = \|M_{g^z}\| = \|g^z\|_\infty$ which is uniformly bounded in $z \in S$ as $\tilde{g}$ was assumed to be bounded on S.

Clearly $F(-i) = M_{g^{-i}}$. So it suffices to proof that F is continuous on S and analytic inside S.

Since the σ-weak and weak topology coincide on bounded sets [3,8] it is sufficient to show that for any pair $\xi_1, \xi_2 \in \mathcal{L}_2(\mathbb{R})$ the function

$$z \in S \to \langle F(z)\xi_1, \xi_2\rangle = \int_{-\infty}^{+\infty} \tilde{g}(t+z)\xi_1(t)\overline{\xi_2(t)}dt$$

is continuous and that it is analytic inside S.
The continuity follows from the continuity of $\tilde{g}$, the boundedness of $\tilde{g}$ and the dominated convergence theorem. The analyticity can be proved from Fubini's theorem and Morera's theorem.

As in this case $\sigma(\Delta) = \mathbb{R}^+$ it is not so hard to show that $\sigma(A) \supseteq \mathbb{R}^+$. In fact using the previous lemma one can easily see that $\mathbb{R}^+ \subseteq P\sigma(A)$ where $P\sigma(A)$ is the pointspectrum of A. However we prove more:

2.3 Lemma $P\sigma(A) = \mathbb{R}^+$.

Proof. Define a function $g(t) = \exp(i\lambda t)$ for λ fixed in $\mathbb{R}$ and $t \in \mathbb{R}$. Then clearly g satisfies the conditions of the previous lemma and $\tilde{g}(z) = \exp i\lambda z$. Then

$$g^{-i}(t) = \tilde{g}(t-i) = e^{\lambda} g(t)$$

and therefore

$$A(M_g) = M_{g^{-i}} = e^{\lambda} M_g$$

showing that $\mathbb{R}^+ \subseteq P\sigma(A)$.

To prove the converse inclusion take $\lambda \in \mathbb{C}$ and assume the existence of $x \in \mathcal{D}(A)$ such that $x \neq 0$ and $A(x) = \lambda x$.

Then $F_x(t) = \alpha_t(x)$ and $F_x(t-i) = \alpha_t(A(x)) = \lambda\, \alpha_t(x) = \lambda\, F_x(t)$.

If $\lambda = 0$ this would imply $F_x(t-i) = 0$ for all $t \in \mathbb{R}$ which again by Schwartz reflection principle would yield $F_x = 0$ and $x = 0$.

So $\lambda \neq 0$ and we may put $\lambda = \exp(a+bi)$ with $a,b \in \mathbb{R}$.

Define $G(z) = \exp(-iaz) \cdot F_x(z)$ for z in the strip S.

Then $G(t-i) = \exp(-iat) \exp(-a) \exp(a+bi) F_x(t)$

$$= \exp(bi)\, G(t).$$

Clearly $\|G(z)\| \leq \max \{1, \exp(-a \operatorname{Im} z)\} \|F(z)\|$ so that also G

is uniformly bounded on the strip S .

Then using the relation $G(t-i) = \exp(bi)\,G(t)$ the function G can be extended periodically, "twisted" with the factor $\exp bi$, to a function analytic over $\mathbb{C}$ and still bounded as $|\exp bi| = 1$. Then it has bo be constant so that either G is identically zero, or $\exp bi = 1$ and $\lambda = \exp a \in \mathbb{R}^+$. However $G = 0$ would imply $F = 0$ and $x = 0$ which is a contradiction. This proves the lemma.

2.4 Remarks From the proof of the lemma we see that if $A(x) = (\exp a)x$ with $x \neq 0$, then $x = F_x(0) = \exp(-iat)F_x(t) = \exp(-iat)\alpha_t(x)$ so that

$$\alpha_t(x) = \exp(iat)x$$

which was to be expected.

The fact that $P\sigma(A) \subseteq \mathbb{R}^+$ was implicitly used in [14,15]

The more difficult part however is to show that also $\mathbb{C} \setminus (0,\infty) \subseteq \sigma(A)$. To do this we first define an analytic branch of the log function.

Define $\log z = \ln|z| + i \arg z$ where $|\arg z| < \pi$. So this function is defined and analytic except for z negative real. Fix $\alpha \in \mathbb{C}$ and take $0 < \epsilon < 1$. Then define functions $\tilde{g}_\epsilon$ by

$$\tilde{g}_\epsilon(z) = -\log(z-i) + \log(z-i\epsilon) - \alpha \log(z+i+i\epsilon) + \alpha \log(z+2i)$$

These functions are defined and analytic for $-\epsilon - 1 < \operatorname{Im} z < \epsilon$. Denote the restriction of $\tilde{g}_\epsilon$ to $\mathbb{R}$ by g_ϵ .

2.5 Lemma The functions g_ϵ are bounded on $\mathbb{R}$ and satisfy the conditions of lemma 2.2, so that $M_{g_\epsilon} \in \mathcal{D}(A)$.

Moreover $\|A(M_{g_\epsilon}) + \alpha M_{g_\epsilon}\|$ remains bounded as $\epsilon \to 0$ while

$$\|M_{g_\epsilon}\| \to \infty .$$

In particular $-\alpha \in \sigma(A)$.

Proof. To show that g_ϵ satisfies the conditions of lemma 2.2 we have to verify that $\tilde{g}_\epsilon$ is uniformly bounded on the strip $S = \{z \in \mathbb{C} \mid \operatorname{Im} z \in [-1,0]\}$.

Now $|\tilde{g}_\epsilon(z)| \leq \left|\ln\left|\frac{z-i\epsilon}{z-i}\right|\right| + |\alpha| \left|\ln\left|\frac{z+2i}{z+i+i\epsilon}\right|\right| + 2\pi(1+|\alpha|)$

and if now $\operatorname{Im} z \in [-1,0]$ this function remains bounded at infinity. (This is the reason why we had to add the additional terms without the ϵ).

$$\begin{aligned} \text{Then} \quad g_\epsilon(t-i) + \alpha g_\epsilon(t) = &-\log(t-2i) + \log(t-i-i\epsilon) \\ &- \alpha \log(t-i) + \alpha \log(t-i\epsilon) \\ &- \alpha \log(t+i\epsilon) + \alpha \log(t+i) \\ &- \alpha^2 \log(t+i+i\epsilon) + \alpha^2 \log(t+2i) \end{aligned}$$

$$\begin{aligned} \text{So} \quad |g_\epsilon(t-i) + \alpha\, g_\epsilon(t)| &\leq \left(\frac{1}{2} + \frac{|\alpha|^2}{2}\right) \ln \frac{t^2+4}{t^2+(1+\epsilon)^2} + 2\pi(1+|\alpha|)^2 \\ &\leq (1+|\alpha|^2) \ln 2 + 2\pi(1+|\alpha|)^2 \end{aligned}$$

If we call this last number N then we have that

$$\|A(M_{g_\epsilon}) + \alpha\, M_{g_\epsilon}\| \leq N \quad \text{for all} \quad 0 < \epsilon < 1$$

while on the other hand

$$|g_\epsilon(t)| \geq \frac{1}{2} \ln \frac{t^2+1}{t^2+\epsilon^2} - |\alpha| \ln \left(\frac{t^2+4}{t^2+(1+\epsilon)^2}\right) - (1+|\alpha|)2\pi .$$

and $\|g_\epsilon\|_\infty \geq |\ln \epsilon| - |\alpha| \ln 2 - 2\pi(1+|\alpha|)$

so that $\|M_{g_\epsilon}\| = \|g_\epsilon\|_\infty \to \infty$ when $\epsilon \to 0$.

This completes the proof.

So we have obtained the following result.

2.6 Theorem With the notations 2.1 and definitions 1.1 we have $\sigma(A) = \mathbb{C}$ while $P\sigma(A) = \mathbb{R}^+$.

Remark In a forthcoming paper it will be shown by Zsido that in general either $\sigma(A) = \mathbb{C}$ or $\sigma(A) \subseteq \mathbb{R}^+$, he will also give other examples for which $\sigma(A) = \mathbb{C}$ [17]. We would like to thank Prof. Zsido for discussions concerning this material.

3. The tensor product case

The previous example can be used to show that even under nice situations the tensor product of two operators with positive spectrum can have the whole complex plane in its spectrum.

Let X and Y be two Banach spaces, and let A and B be two bounded linear operators on X and Y respectively. If $X \otimes Y$ is the completion of the algebraic tensor product of X and Y with respect to some uniform crossnorm, there is a unique bounded linear operator $A \otimes B$ on $X \otimes Y$ such that $(A \otimes B)(x \otimes y) = Ax \otimes By$ for all $x \in X$ and $y \in Y$. Moreover $\|A \otimes B\| = \|A\| \, \|B\|$. [9]

To find the spectrum $A \otimes B$ one considers $A \otimes B = (A \otimes 1)(1 \otimes B)$ where 1 denotes the identity operator both on X and Y, and since one can show that $\sigma(A \otimes 1) = \sigma(A)$ and $\sigma(1 \otimes B) = \sigma(B)$, it follows from the fact that $A \otimes 1$ and $1 \otimes B$ commute that $\sigma(A \otimes B) \subseteq \sigma(A)\,\sigma(B)$.

In fact Brown & Pearcy [1] have proved equality when X and Y are Hilbert spaces, and $X \otimes Y$ the Hilbert space tensor product, while Schecter [11] has extended this result to Banach spaces and any uniform reasonable cross norm. See also [5].

In the unbounded case the situation is quite different. It has been considered by Ichinose [5] and by Simon and Reed [6]. As was remarked by Ichinose, in general it is not to be expected that $\sigma(A \otimes B) = \sigma(A)\,\sigma(B)$ because the product of two closed sets need not be closed. Take his example with $\sigma(A) = \mathbb{N}$ and $\sigma(B) = \{n^{-1}, n \in \mathbb{N}\}^{-}$ so that $\sigma(A)\,\sigma(B) = \mathbb{Q}^{+}$ which is not closed. Then one might expect that $\sigma(A \otimes B) = \overline{\sigma(A)\,\sigma(B)}$. In the same paper however an example is given of two operators A and B such that $\sigma(A) = \{0\}$ and $\sigma(B) = \mathbb{N}$ while $\sigma(A \otimes B) = \mathbb{C}$.

In Ichinoses example A and B are operators on Hilbert spaces, B selfadjoint but A nilpotent. To prove that $\sigma(A \otimes B) = \mathbb{C}$ he uses a result due to Taylor on operators with non-empty resolvent set, [13]. We will now give an other example based on our previous results.

Therefore let $\mathcal{H}$ be a Hilbert space, and $\{u_t\}$ a strongly continuous one parameter group of unitaries. Let Δ be the unique non-singular positive self-adjoint operator on $\mathcal{H}$ such that $\Delta^{it} = u_t$ for all $t \in \mathbb{R}$.

Denote by $\mathcal{H}'$ the conjugate Hilbert space of $\mathcal{H}$, i.e. the set with addition as in $\mathcal{H}$ but new scalar multiplication and scalar product defined as

$$(\lambda,\xi) \in \mathbb{C} \times \mathcal{H} \quad \to \quad \bar{\lambda}\,\xi$$

$$(\xi,\eta) \in \mathcal{H} \times \mathcal{H} \quad \to \langle \overline{\xi,\eta} \rangle$$

Then $\{u_t\}$ as well as Δ may be considered as operators on $\mathcal{H}'$ because $\mathcal{H}$ and $\mathcal{H}'$ coincide as sets. Clearly also on $\mathcal{H}'$ we will have that $\{u_t\}$ is a strongly continuous one parameter group of unitaries, and that Δ is a positive non-singular self-adjoint operator. However on $\mathcal{H}'$ we will have $u_t = \Delta^{-it}$ instead!

Now let $\mathcal{H} \otimes \mathcal{H}'$ denote the algebraic tensor product of $\mathcal{H}$ and $\mathcal{H}'$. If $x \in \mathcal{B}(\mathcal{H})$ and $\psi = \sum_{i=1}^{n} \xi_i \otimes \eta_i \in \mathcal{H} \otimes \mathcal{H}'$ we denote $\langle \psi, x \rangle = \sum_{i=1}^{n} \langle x\,\xi_i, \eta_i \rangle$.

Clearly $\langle \psi, x \rangle$ is bilinear and one can show that

$$\|\psi\| = \sup_{\|x\| \leq 1} |\langle \psi, x \rangle|$$

defines a uniform cross norm on $\mathcal{H} \otimes \mathcal{H}'$ [9] and we will let $\mathcal{H} \hat{\otimes} \mathcal{H}'$ denote the completion of $\mathcal{H} \otimes \mathcal{H}'$ with respect to this norm. Then we have identified $\mathcal{B}(\mathcal{H})$ as the dual spase of $\mathcal{H} \hat{\otimes} \mathcal{H}'$, see also [10].

Now we define α_t on $\mathcal{B}(\mathcal{H})$ as before, and an operator, denoted by $\Delta \otimes \Delta^{-1}$ on $\mathcal{H} \hat{\otimes} \mathcal{H}'$ by

$$\mathcal{D}(\Delta \otimes \Delta^{-1}) = \{\psi = \sum_{i=1}^{n} \xi_i \otimes \eta_i \mid \xi_i \in \mathcal{D}(\Delta), \eta_i \in \mathcal{D}(\Delta^{-1})\}$$

and $(\Delta \otimes \Delta^{-1})\psi = \sum_{i=1}^{n} \Delta\,\xi_i \otimes \Delta^{-1}\eta_i$

We then have the following lemma

<u>3.1 Lemma</u> The analytic generator A of $\{\alpha_t\}$ is contained in the adjoint of $\Delta \otimes \Delta^{-1}$.

Proof. Let $x \in \mathcal{D}(A)$, and $\xi \in \mathcal{D}(\Delta)$, $\eta \in \mathcal{D}(\Delta^{-1})$.
Then by the definition of A there is a complex function f defined, bounded and continuous on the strip $S = \{z \in C,\ \mathrm{Im}\, z \in [-1,0]\}$, analytic inside and such that

$$f(t) = \langle \alpha_t(x)\xi,\eta\rangle, \quad f(-i) = \langle A(x)\xi,\eta\rangle$$

Now $\langle \alpha_t(x)\xi,\eta\rangle = \langle u_t^* x u_t\ \xi,\eta\rangle = \langle x u_t\ \xi, u_t\ \eta\rangle$. So $\langle \alpha_t(x),\xi\otimes\eta\rangle = \langle x, u_t\,\xi\otimes u_t\,\eta\rangle$.

Because $\xi \in \mathcal{D}(\Delta)$ and $\eta \in \mathcal{D}(\Delta^{-1})$ and because Δ is the analytic generator for u_t on $\mathcal{H}$ and Δ^{-1} is the analytic generator for u_t on $\mathcal{H}'$ we have that there are vector valued functions p and q on the strip S, bounded and continuous on S, analytic inside S and such that

$$p(t) = u_t\,\xi \qquad q(t) = u_t\,\eta$$
$$p(-i) = \Delta\,\xi \qquad q(-i) = \Delta^{-1}\eta$$

By uniqueness of analytic extensions we must have that

$$f(z) = \langle x, p(z)\otimes q(z)\rangle$$

so that in particular

$$\langle A(x)\xi,\eta\rangle = f(-i) = \langle x,p(-i)\otimes q(-i)\rangle = \langle x,\Delta\,\xi\otimes\Delta^{-1}\eta\rangle$$
$$= \langle x,(\Delta\otimes\Delta^{-1})(\xi\otimes\eta)\rangle .$$

Then the following is an easy application.

3.2 Theorem With $\mathcal{H}$ and u_t as in section 2 we have $\sigma(\Delta\otimes\Delta^{-1}) = \mathbb{C}$.

Proof. Denote A_1 the adjoint of $\Delta\otimes\Delta^{-1}$ as an operator on $\mathcal{B}(\mathcal{H})$. Then $\sigma(\Delta\otimes\Delta^{-1}) = \sigma(A_1)$ [4]. By the previous lemma

$A \subseteq A_1$.

But as A_1 is an extension of A lemma 2.5 is still valid for A_1 so that $\sigma(A_1) = C$.

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