Serre [5] proves the following duality theorem:

Let $X$ be a compact complex manifold, $\dim X = n$, and let $W$ be a holomorphic vector bundle on $X$, and $W^*$ the dual bundle of $W$. Then the vector spaces $H^{p,q}(X,W) = H^q(X,\mathcal{O}^p(W))$ and $H^{n-p,n-q}(X,W^*) = H^{n-q}(X,\mathcal{O}^{n-p}(W^*))$ are (canonically) dual to each other, in particular, they have the same (finite) dimension.

To prove this, he resolves the sheaf $\mathcal{O}^p(W)$ of germs of holomorphic $p$-forms with coefficients in a holomorphic vector bundle $W$ in two (fine) ways:

$$0 \to \mathcal{O}^p(W) \to \mathcal{L}^{(p,0)}(W) \xrightarrow{\delta} \mathcal{L}^{(p,1)}(W) \xrightarrow{\delta} \cdots \to 0$$

$$0 \to \mathcal{O}^p(W) \to \mathcal{D}^{(p,0)}(W) \xrightarrow{\delta} \mathcal{D}^{(p,1)}(W) \xrightarrow{\delta} \cdots \to 0,$$

where $\mathcal{A}^{(p,q)}(W)$ is the sheaf of germs of $\mathcal{C}^\infty$-forms of type $(p,q)$ with coefficients in $W$, and $\mathcal{D}^{(p,q)}(W)$ is the same kind of distributional forms.

Thus one can calculate $H^q(X,\mathcal{O}^p(W))$ from either sequence. Since $\mathcal{D}^!$ is dual to $\mathcal{A} = \mathcal{C}^\infty$, this is a natural procedure. The above result is a consequence of the well-known Grothendieck lemma, and of the fact that if $T \in \mathcal{D}^{(p,0)}(U)$, $U$ open in $\mathcal{O}^p$, satisfies $\partial T/\partial \bar{z}^j = 0$ for $j = 1, \ldots, n$, then $T$ is a holomorphic function.

Concerning the Grothendieck lemma for distributions, Serre refers to a paper of Dolbeault [1], in which this distributional Grothendieck lemma is stated. However in this paper Dolbeault gives no proof, but says that this is an unpublished result of Grothendieck.
We propose to give a proof of the distributional Grothendieck lemma below. The proof is modelled on the method in Narasimhan [4]. According to Narasimhan, this is the method of Grothendieck, as exposed by Serre. We also prove the statement about the distribution \( T \) above, and this proof is a generalization of the 1-dimensional proof in Gunning [2].

We first need the following lemma.

Let \( L', K, L \) be compact subsets of \( \mathbb{R}^s, \mathcal{C}, \mathcal{C}^n \) respectively. Denote by \( (t, z, w) \) points in \( \mathbb{R}^s \times \mathcal{C} \times \mathcal{C}^n \). Let \( g \in \mathcal{L}'(U) \), where \( U \) is an open subset of \( \mathbb{R}^s \times \mathcal{C} \times \mathcal{C}^n \) containing \( L' \times K \times L \), and suppose \( \partial g/\partial \overline{w}^k = 0 \) for \( 1 \leq k \leq n \), where \( w = (w^1, \ldots, w^n) \). Then there is a distribution \( f \in \mathcal{L}'(U') \), where \( U' \) is some open set contained in \( U \) and containing \( L' \times K \times L \), such that \( \partial f/\partial \overline{w}^k = 0 \) for \( 1 \leq k \leq n \) and \( \partial f/\partial \overline{z} = g \) in \( U' \).

Proof: For simplicity, we assume that \( s = 0 \), as this does not affect the proof. We may also suppose that \( g \in \mathcal{C}'(\mathcal{C} \times \mathcal{C}^n) \), since we can multiply \( g \) by a \( \mathcal{C}_c^\infty \) function having support in \( U \) and being equal to 1 in a nbh. of \( K \times L \), and then consider \( g \) in this last nbh. Thus we assume \( g \) is a compactly supported distribution in \( \mathcal{C} \times \mathcal{C}^n \). The proof is somewhat technical, and we divide it into three parts:

I) A statement needed to define a distribution \( f \) in II).
II) Define \( f \) and prove that \( f \) is a distribution.
III) Show that \( f \) is the distribution we seek.
I) For \( \varphi \in C^\infty (\mathbb{C} \times \mathbb{C}^n) \), let \( \tau_{\alpha} \varphi \in C^\infty (\mathbb{C} \times \mathbb{C}^n) \) be the translation by \( \alpha \in \mathbb{C} \times \mathbb{C}^n \), thus \( (\tau_{\alpha} \varphi) (z,w) = \varphi((z,w)-\alpha) \).

If \( \varphi \in C^\infty_c (\mathbb{C} \times \mathbb{C}^n) \) and \( \xi \in \mathbb{C} \), put \( h(\xi) = \langle \xi , \tau_{(\xi,0)} \varphi \rangle \). Then \( h \in C^\infty_c (\mathbb{C}) \), and we have \( \frac{\partial h}{\partial \xi}(\xi) = -\langle \xi, \tau_{(\xi,0)} \frac{\partial \varphi}{\partial \xi} \rangle \).

We check this: If \( 0 \neq t \in \mathbb{R} \), then \( \frac{h(\xi+t)-h(\xi)}{t} \)

\[
\frac{\tau_{(\xi+t,0)} \varphi - \tau_{(\xi,0)} \varphi}{t} \rightarrow \frac{\tau_{(\xi+t,0)} \varphi(z,w) - (\tau_{(\xi,0)} \varphi)(z,w)}{t} \quad \text{as } t \to 0
\]

\[
\lim_{t \to 0} \frac{\varphi(z-\xi-t,w) - \varphi(z-\xi,w)}{t} = - \frac{\partial \varphi}{\partial x}(z-\xi,w) = - (\tau_{(\xi,0)} \frac{\partial \varphi}{\partial x})(z,w),
\]

where \( z = x+iy \), with \( x, y \in \mathbb{R} \). Similarly, if \( 0 \neq t \in \mathbb{R} \),

\[
\lim_{t \to 0} \frac{\tau_{(\xi+t,0)} \varphi(z,w) - (\tau_{(\xi,0)} \varphi)(z,w)}{t} \quad \text{as } t \to 0
\]

\[
= - (\tau_{(\xi,0)} \frac{\partial \varphi}{\partial y})(z,w).
\]

We now prove that \( \lim_{t \to 0} \frac{\tau_{(\xi+t,0)} \varphi - \tau_{(\xi,0)} \varphi}{t} = - \tau_{(\xi,0)} \frac{\partial \varphi}{\partial x} \) and

\[
\lim_{t \to 0} \frac{\tau_{(\xi+t,0)} \varphi - \tau_{(\xi,0)} \varphi}{t} = - \tau_{(\xi,0)} \frac{\partial \varphi}{\partial y}, \text{ for } t \in \mathbb{R}, \text{ that is that}
\]

these limits hold in the Fréchet space \( C^\infty_c (\mathbb{C} \times \mathbb{C}^n) \).

Consider the first limit. Let \( m \) be a non-negative integer and \( A \) a compact subset of \( \mathbb{C} \times \mathbb{C}^n \). Denote for the moment by \( D \)
a differentiation monomial in the real coordinates of \( C \times \mathbb{C}^n \),
and by \( \| \|_{A,m} \) the semi-norm in \( C^0(\mathbf{O} \times \mathbb{C}^n) \) given by \( A \) and \( m \), using monomials of order \( \leq m \). Then \( \| \frac{1}{t} (\tau(\xi + t,0) \varphi - \tau(\xi,0)\varphi) + \tau(\xi,0) \frac{\partial \varphi}{\partial x} \|_{A,m} \)

\[
= \sup_{\text{ord } D \leq m} \sup_{(z,w) \in A} |D(\frac{1}{t} (\tau(\xi + t,0) \varphi - \tau(\xi,0)\varphi) + \tau(\xi,0) \frac{\partial \varphi}{\partial x})(z,w)|.
\]

Consider for instance the case \( D = \partial / \partial u^k \), where \( w^k = u^k + iv^k \) with \( u^k, v^k \in \mathbb{R} \). Then \( \frac{\partial}{\partial u^k} (\frac{1}{t} (\tau(\xi + t,0) \varphi - \tau(\xi,0)\varphi) + \tau(\xi,0) \frac{\partial \varphi}{\partial x})(z,w) \)

\[
= (\frac{1}{t} (\tau(\xi + t,0) \varphi - \tau(\xi,0)\varphi) + \tau(\xi,0) \frac{\partial \varphi}{\partial x})(z,w) \frac{\partial \varphi}{\partial u^k}.
\]

Similar expressions hold for \( \partial / \partial v^k \), \( \partial / \partial x \), \( \partial / \partial y \), and other monomials \( D \). All these expressions tend uniformly to zero on the compact set \( A \) as \( t \) tends to zero. Hence the first limit expression in (2) above, and similarly also the second, is true. By the above expressions we therefore get, since \( g \) is continuous on \( C^0(\mathbf{O} \times \mathbb{C}^n) \), that \( \partial h / \partial \xi_1 \) and \( \partial h / \partial \xi_2 \) exist, where \( \xi = \xi_1 + i\xi_2 \) with \( \xi_1, \xi_2 \in \mathbb{R} \), and further

\[
\frac{\partial h}{\partial \xi_1} (\xi) = \langle g, \tau(\xi,0) \frac{\partial \varphi}{\partial x} \rangle \quad \text{and} \quad \frac{\partial h}{\partial \xi_2} (\xi) = \langle g, \tau(\xi,0) \frac{\partial \varphi}{\partial y} \rangle .
\]

This gives \( \frac{\partial h}{\partial \xi} (\xi) = \langle g, \tau(\xi,0) \frac{\partial \varphi}{\partial z} \rangle \), and also, when applied

several times,

\[
\frac{\partial^{\alpha + \beta} h}{\partial \xi_1^\alpha \partial \xi_2^\beta} (\xi) = (-1)^{\alpha + \beta} \langle g, \tau(\xi,0) \frac{\partial^{\alpha + \beta} \varphi}{\partial \xi_1^\alpha \partial \xi_2^\beta} \rangle .
\]

The last shows that \( h \) is \( C^\infty \). We must show that \( h \) has compact support. Choose \( R > 0 \) and a compact set \( B \) in \( \mathbb{C}^n \) such that \( \text{supp } g \cup \text{supp } \varphi \subset K_R \times B \), where \( K_R = \{ |z| \leq R \} \).

If \( |\xi| > 2R \) and \( (z,w) \in K_R \times B \), then \( |\xi - z| \geq |\xi| - |z| = |\xi| - |z| > R \), thus \( (z - \xi, w) \notin K_R \times B \). Hence \( \text{supp } g \cap \text{supp } \tau(\xi,0)\varphi = \emptyset \).
for $|\xi| > 2R$, and thus also $\langle \psi, \tau(\xi, 0) \psi \rangle = 0$ for $|\xi| > 2R$, which gives $\text{supp } h \circ K_{2R}$.

II) For $\psi \in C_0^\infty (C \times C^n)$, let $\langle f, \phi \rangle = -\frac{1}{4\pi} \int \frac{\langle e_{\tau(\xi, 0) \phi} \rangle}{\xi} \, d\mu(\xi)$, where $\mu$ is Lebesgue measure on $C$.

Then $\mathcal{L}(C \times C^n)$

We check this:

By I) $\langle f, \phi \rangle$ is well-defined. ($\xi^{-1}$ is integrable over $C$).

Clearly, $f$ is linear. To prove that it is continuous, it suffices to prove that it is continuous on $\partial_{K \times B}^\infty (C \times C^n)$, by properties of $L^p$ spaces [Trèves [6]], where $K_R = \{ |z| \leq R \}$, $B$ compact in $C^n$. We can also take $K_R \times B$ so big that $\text{supp } \psi \subset K_R \times B$, as in I) above. As in I) $\text{supp } \langle \psi, \tau(\xi, 0) \phi \rangle \subset K_{2R}$ for $\psi \in K_R \times B$. Introducing polar coordinates, $(r, \theta)$, on $C$, we get

$$\langle f, \phi \rangle = \frac{1}{4\pi} \int \frac{\langle e_{\tau(\xi, 0) \phi} \rangle}{\xi} \, d\mu(\xi) = \frac{1}{4\pi} \int_{K_{2R}} \frac{\langle e_{\tau(\xi, 0) \phi} \rangle}{\xi} \, d\mu(\xi)$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\langle e_{\tau(\xi, 0) \phi} \rangle}{\xi} \, d\mu_{e^{i\theta}}$$

$$\leq \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \langle e_{\tau(\xi, 0) \phi} \rangle \, d\xi \, d\theta \leq 4R^2 \langle e_{\tau(\xi, 0) \phi} \rangle$$

where $\langle e_{\tau(\xi, 0) \phi} \rangle$ attains its maximum on \{ $|\xi| \leq 2R$ \} $= K_{2R}$ at $\xi_0 \in K_{2R}$.

For $\xi \in K_{2R}$ we have $\text{supp } \tau(\xi, 0) \phi \subset K_3 \times B$, and the continuity of $e$ gives for some constant $C > 0$ that $|\langle e_{\tau(\xi_0, 0) \phi} \rangle | \leq C \sup_{D} \sup_{D} \sup_{D} \text{D}(\tau(\xi_0, 0) \phi)(z, w)$, where $D$ means
differentiation monomial in real coordinates of \( \emptyset \times \emptyset^n \), and the 
D-sup is taken over monomials of order less then some integer, we have 
\[ D(\tau(\xi_0,0)\varphi) = \tau(\xi_0,0) (D\varphi), \] 
and for \((z,w) \in \emptyset \times \emptyset^n\) we have 
\[ |\tau(\xi_0,0) (D \varphi) (z,w)| = |(D \varphi) (z - \xi_0,w)| \leq \sup_{(z,w) \in K_R \times B} |(D\varphi)(z,w)|, \]
since \(\text{supp } \varphi \subseteq K_R \times B\). Thus 
\[ |\langle \xi_0, \tau(\xi_0,0) \varphi \rangle| \leq \sup_{D} \sup_{(z,w) \in K_R \times B} |(D \varphi)(z,w)|. \]
By the above we then get 
\[ |\langle f, \varphi \rangle| \leq 4 R^2 C \sup_{D} \sup_{(z,w) \in K_R \times B} |(D \varphi)(z,w)|, \]
continuity, since \(\sup_{D} \sup_{(z,w) \in K_R \times B} |(D \varphi)(z,w)|\) is a semi-norm on 
\(C^{\infty}_{K_R \times B} (\emptyset \times \emptyset^n)\), which proves 
III) Remember now that in the beginning of the proof we multiplied 
the original \(g\), call it here \(\xi_0\), by a \(C^{\infty}_{\emptyset} (\emptyset \times \emptyset^n)\) - function 
with support in the given \(U\) and being equal to \(1\) in a nbh, \(U'\) of \(K \times L\). More accurately we do this as follows: Let \(U_1\) 
be open in \(\emptyset\) and \(U_2\) open in \(\emptyset^n\), such that \(K \times L \subseteq U_1 \times U_2 \subseteq U\), 
and let \(\theta_1 \in C^{\infty}_{\emptyset} (U_1)\) and \(\theta_2 \in C^{\infty}_{\emptyset} (U_2)\) be such \(\theta_1 = 1\) in a nbh, \(U_1'\) of \(K\) and \(\theta_2 = 1\) in a nbh, \(U_2'\) of \(L\). Then let 
\(\theta \in C^{\infty}_{\emptyset} (U)\) be \(\theta(z,w) : = \theta_1(z) \cdot \theta_2(w)\). We have then 
\(\theta = 1\) in \(U' := U_1' \times U_2'\).
We take our $g$ as $g_\epsilon = \epsilon g_0$. If $\varphi \in \mathcal{C}^\infty_c (\mathbb{C} \times \mathbb{C}^n)$ has support in $\mathbb{C} \times U'_2$, then we get $\langle \partial g / \partial w^k, \varphi \rangle = \langle \epsilon_1 \frac{\partial g_0}{\partial w^k}, \varphi \rangle +$

$\langle \epsilon_1 \partial_2 \frac{\partial g_0}{\partial w^k}, \varphi \rangle = 0$, since $\partial g_0 / \partial w^k = 0$ and since

$\partial \partial_2 / \partial w^k = 0$ in $U'_2$.

With $g$ constructed in this way, $f$ is the distribution we seek. (4)

Check: Let $\varphi \in \mathcal{C}^\infty_c (U')$. Then for any $\xi \in \mathcal{C}$ we have supp $\tau (\xi, 0) \subset \mathbb{C} \times U'_2$, and the above gives: $\langle \partial f / \partial w^k, \varphi \rangle =$

$\langle f, \partial \varphi / \partial w^k \rangle$

$= \int_0^1 \frac{\langle \xi, \tau (\xi, 0) \varphi \rangle}{\xi} \, \text{dm} (\xi) = \int_0^1 \frac{\langle \xi, \tau (\xi, 0) \varphi \rangle}{\xi} \, \text{dm} (\xi)$

$= - \int_0^1 \frac{\langle \partial g / \partial w^k, \tau (\xi, 0) \varphi \rangle}{\xi} \, \text{dm} (\xi) = - \int_0^1 \frac{0}{\xi} \, \text{dm} (\xi) = 0.$

Thus $\frac{\partial f}{\partial w^k} = 0$ in $U'$, which is part of what we need. Further, for $\varphi \in \mathcal{C}^\infty_c (U')$, we have $\langle \partial f / \partial \bar{z}, \varphi \rangle = - \langle f, \partial \varphi / \partial \bar{z} \rangle$

$= \int_0^1 \frac{\langle \xi, \tau (\xi, 0) \partial \varphi / \partial \bar{z} \rangle}{\xi} \, \text{dm} (\xi) = \int_0^1 \frac{\langle \xi, \tau (\xi, 0) \partial \varphi / \partial \bar{z} \rangle}{\xi} \, \text{dm} (\xi)$

$= - \int_0^1 \frac{1}{\xi} \int_{\mathbb{C} \setminus \{0\}} \frac{1}{\xi} \langle \xi, \tau (\xi, 0) \varphi \rangle \, \text{dm} (\xi), \text{ by I}. \text{ Since } d \xi \wedge d \xi = 2 \, \text{idm}(\xi),$

we have $\langle \partial f / \partial \bar{z}, \varphi \rangle = - \frac{1}{2 \pi i} \int_{\mathbb{C} \setminus \{0\}} \frac{\partial}{\partial \bar{\xi}} \langle \xi, \tau (\xi, 0) \varphi \rangle d \xi \wedge d \xi$
\[ \frac{1}{2\pi i} \int_{\mathcal{C} - \{0\}} \frac{1}{\xi} \left( \langle \xi, \tau(\xi,0) \phi \rangle \right) \wedge d\xi. \]

Now if \( \alpha \in \mathcal{O}^\infty(\mathcal{C}) \), then in \( \mathcal{C} - \{0\} \) we have \( d(\alpha \xi^{-1} d\xi) = d(\alpha \xi^{-1}) \wedge d\xi = \xi^{-1} \alpha d\xi - \alpha \xi^{-2} d\xi \wedge d\xi = \xi^{-1} \alpha d\xi \wedge d\xi \). In the above calculation this gives

\[ \langle \alpha f / \partial \overline{\omega}, \phi \rangle = - \frac{1}{2\pi i} \int_{\mathcal{C} - \{0\}} d(\langle \xi, \tau(\xi,0) \phi \rangle \frac{1}{\xi} d\xi) \]

\[ = \lim_{\epsilon \to 0} - \frac{1}{2\pi i} \int_{|\xi| \geq \epsilon} d(\langle \xi, \tau(\xi,0) \phi \rangle \frac{1}{\xi} d\xi) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int \langle \xi, \tau(\xi,0) \phi \rangle \frac{1}{\xi} d\xi \]

\[ = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\mathcal{C} - \{0\}} \langle \xi, \tau(\epsilon e^{i0},0) \phi \rangle d\epsilon = \langle \xi, \tau(0,0) \phi \rangle = \langle \xi, \phi \rangle \text{, since } \xi \tau(\epsilon,0) \phi \text{ is continuous.} \]

We see now that \( \partial f / \partial \overline{\omega} = \gamma \) in \( U' \). Since this \( \gamma \) equals the original \( \gamma \), called \( \gamma_0 \) in part III), in \( U' \), the lemma is proved.

We now need some notation:

For \( U \) open in \( \mathbb{C}^n \), let \( D'(p,q)(U) \) be the forms of type \( (p,q) \) with distributional coefficients ("currents"). Thus \( \omega \in D'(p,q)(U) \) can be written \( \omega = \sum a_{IJ} dz^I \wedge d\overline{z}^J \), with \( a_{IJ} \in D'(U) \), \( I \) and \( J \) multi-indices of length \( p \) and \( q \) respectively, and \( dz^I = dz^{i_1} \wedge \ldots \wedge dz^{i_p} \) if \( I = (i_1, \ldots, i_p) \) etc.

The operator \( \delta \) acts as usual: \( \delta \omega = \sum_{IJ} \frac{\partial a_{IJ}}{\partial \overline{z}^j} d\overline{z}^j \wedge dz^I \wedge d\overline{z}^J \).

We now have Grothendieck's lemma, the proof of which is found in Narasimhan [4]. (In the \( \mathcal{O}^\infty \)-case) Since it is short, we give it for completeness.

\textbf{Grothendieck's lemma}

Let \( K_1, \ldots, K_n \) be compact sets in \( \mathcal{C} \) and \( S = K_1 \times \ldots \times K_n \subset \mathbb{C}^n \).

Let \( \omega \in D'(p,q)(U) \), where \( q \geq 1 \) and \( U \) is a nbh. of \( S \). If \( \delta \omega = 0 \), then there exists \( \omega' \in D'(p,q-1)(U) \) such that \( \delta \omega' = \omega \) in a nbh. of \( S \).
Proof: If \( 1 \leq v \in \mathbb{Z} \), let \( A^p,q_v = A^p,q_v(S) \) be the space of elements \( w \in \mathcal{D}^p(p,q)(U) \) defined in a nbh. \( U = U(w) \) of \( S \) and such that \( w \) does not involve \( dz^v, \ldots, dz^n \). Thus \( w = \sum_{IJ} a_{IJ} dz^I \wedge dz^J \) where \( J = (j_1, \ldots, j_q) \) with \( 1 \leq j_1 < \ldots < j_q \leq v-1 \).

If \( v > n \), then \( A^p,q_v \) is the space of elements \( w \in \mathcal{D}^p(p,q)(U) \) defined in a nbh. \( U = U(w) \) of \( S \). Also if \( w \in A^p,q_1 \) and \( q \geq 1 \), then clearly \( w = 0 \), and thus the lemma is trivial if \( w \in A^p,q_1 \).

Suppose the lemma is true for all \( w \in A^p,q_v \), and let \( w \in A^p,q_{v+1} \). We can write \( w = dz^v \wedge w_1 + w_2 \), with \( w_1 \in A^p,q^{-1} \) and \( w_2 \in A^p,q \).

If \( \delta w = 0 \), then \( -dz^v \wedge \delta w_1 + \delta w_2 = 0 \). Since \( w_1 \) and \( w_2 \) do not involve \( dz^v, \ldots, dz^n \), we have \( \delta w_1 /\delta z^j = 0 \) and \( \delta w_2 /\delta z^j = 0 \)
(componentwise differentiation), for \( j = v+1, \ldots, n \). By our first lemma there exists \( \chi' \in \mathcal{D}^p(p,q^{-1})(U) \) in a nbh. \( U' \) of \( S \), with \( \delta \chi' /\delta z^j = 0 \) for \( j = v+1, \ldots, n \) and \( \delta \chi' /\delta z^v = w_1 \). Multiplying \( \chi' \) by a \( C^\infty(U') \)-function which is equal to 1 in a nbh. of \( S \), we see that there exists \( \chi \in \mathcal{D}^p(p,q^{-1})(\mathbb{C}^n) \) with \( \delta \chi /\delta z^j = 0 \) for \( j = v+1, \ldots, n \) and \( \delta \chi /\delta z^v = w_1 \) in a nbh. of \( S \). This implies that \( w - \delta \chi \in A^p,q_v \). Since \( \delta(w - \delta \chi) = \delta w = 0 \), there is, by the induction hypothesis, an element \( \psi \in \mathcal{D}^p(p,q^{-1})(\mathbb{C}^n) \) with \( w - \delta \chi = \delta \psi \) in a nbh. of \( S \).

We further need the following theorem, a proof of which in the case of a Riemann surface can be found in Gunning [2]. To generalize that proof to the case of arbitrary dimension, we need a Cauchy formula in several variables. Since we will use differential forms, Stoke's theorem etc., it is convenient to use the Cauchy-Martinelli formula. This reads as follows: If \( f \) is a holomorphic function in a nbh. of \( \xi + K_R \subset \mathbb{C}^n \), where \( K_R = \{ z \in \mathbb{C}^n | |z| \leq R \} \), then
\[ f(\xi) = (-1)^{\frac{n(n+1)}{2}} \times \frac{(n-1)!}{(2\pi i)^n} \int_{\xi+S_R} \frac{f(z)}{|z-\xi|^{2n}} w(z-\xi), \text{ where } S_R = \{|z| = R\} \]

and \[ w(z) = \sum_{k=1}^{n} (-1)^k z^k dz_1 \wedge \cdots \wedge dz^n \wedge \overline{dz}_1 \wedge \cdots \wedge \overline{dz}_n. \] \(^*(\wedge\text{ means "omission", as usual})*\)

There is a simple proof in Lang [3].

The generalization of the theorem in Gunning is:

Let \( T \in \mathcal{D}'(U) \), where \( U \) is open in \( \mathbb{C}^n \), and assume that \( \partial T/\partial z^j = 0 \) for \( j = 1, \ldots, n \). Then \( T \) is a holomorphic function in \( U \).

Proof: Rewriting the Cauchy-Martineti formula, interchanging \( z \) and \( \xi \), and putting \( A = (-1)^{\frac{n(n+1)}{2}} \times \frac{(n-1)!}{(2\pi i)^n} \), we have \( f(z) = A \int_{S_R} f(z+\xi) w(\xi) |\xi|^{2n} \) for \( f \) holomorphic near \( z + K_R \). In particular, for \( f = 1 \) we get \( 1 = A \int_{S_R} \frac{w(\xi)}{|\xi|^{2n}} \). If \( f \) is only \( C^\infty \) in a nbh. of \( z \), we get then:

\[ f(z) - A \int_{S_R} f(z+\xi) w(\xi) |\xi|^{2n} = A \int_{S_R} f(z) w(\xi) |\xi|^{2n} - A \int_{S_R} f(z+\xi) w(\xi) |\xi|^{2n} \]

\[ = A \int_{S_R} (f(z) - f(z+\xi)) w(\xi) |\xi|^{2n} \]

This quantity can be made arbitrarily small by taking \( R \) small enough, since \( f \) is continuous.

Thus we see

\[ f(z) = \lim_{R \to 0} A \int_{S_R} f(z+\xi) w(\xi) |\xi|^{2n} \text{ for } f C^\infty \text{ near } z. \] (5)

To prove that \( T \) is holomorphic it is sufficient to prove that it is \( C^\infty \), and to show that, we will write any \( \varphi \in C^\infty_c(U) \) in a special form so that we can use the conditions given on \( T \). Let then, for \( \varepsilon > 0 \), \( U_\varepsilon = \{z \in \mathbb{C}^n | \text{dist}(z, \mathbb{C}^n - U) > \varepsilon\} \), and let \( \varphi \in C^\infty_c(U_\varepsilon) \subset C^\infty_c(U) \). Further let \( \rho \in C^\infty_c(\mathbb{C}^n) \) be such that \( \rho(\xi) = 1 \) for
\[|\xi| < \epsilon/2, \text{ and } \text{supp } \rho \subset \{|\xi| < \epsilon\}. \] By (5) we get, for \( z \in U_\epsilon \),
\[ \varphi(z) = \lim_{R \to 0} \int_{S_R} \frac{\varphi(z + \xi)}{|\xi|^{2n}} w(\xi), \]
since \( \rho = 1 \) on \( S_R \) for \( R < \epsilon/2 \).

By Stokes we get, since the orientation of \( S_R \) is outward, observing that \( \xi \mapsto f(z + \xi)\rho(\xi) \) is defined for all \( \xi \in \mathbb{C} \) since \( z \in U_\epsilon \):
\[ \varphi(z) = -\lim_{R \to 0} \int_{|\xi| \geq R} d[\varphi(z + \xi) - \frac{\rho(\xi)}{|\xi|^{2n}} w(\xi)] \tag{6} \]

Here the \( d \) is w.r.t. \( \xi \). We have
\[ d[\varphi(z + \xi) - \frac{\rho(\xi)}{|\xi|^{2n}} w(\xi)] =
\[ d\varphi(z + \xi) \wedge \frac{\rho(\xi)}{|\xi|^{2n}} w(\xi) + \varphi(z + \xi) d[\frac{\rho(\xi)}{|\xi|^{2n}} w(\xi)]. \]
Since each term of \( w \)
contains \( d\xi^1 \wedge \cdots \wedge d\xi^n \), we see
\[ d\varphi(z + \xi) \wedge \frac{\rho(\xi)}{|\xi|^{2n}} w(\xi) = \delta \varphi(z + \xi) \]
\[ \wedge \frac{\rho(\xi)}{|\xi|^{2n}} w(\xi) = \sum_{j=1}^n \delta \varphi(z + \xi) \frac{\partial \rho(\xi)}{\partial \xi^j} \] \[d\xi^j \wedge \frac{\rho(\xi)}{|\xi|^{2n}} w(\xi). \]

Further,
\[ d[\frac{\rho(\xi)}{|\xi|^{2n}} w(\xi)] = d[\frac{\rho(\xi)}{|\xi|^{2n}} \varphi(\xi) + \varphi(z + \xi) d[\frac{\rho(\xi)}{|\xi|^{2n}} w(\xi)] \]
\[ = \delta[\frac{\rho(\xi)}{|\xi|^{2n}} \varphi(\xi) + \frac{\rho(\xi)}{|\xi|^{2n}} d \varphi(\xi) \varphi(\xi) \wedge \frac{\rho(\xi)}{|\xi|^{2n}} w(\xi)] \]
\[ = (-1)^{n-1} \frac{n(n+1)}{2} (2i)^n \sum_{j=1}^n \delta \varphi(z + \xi) \frac{\partial \rho(\xi)}{\partial \xi^j} \wedge \frac{\rho(\xi)}{|\xi|^{2n}} d\xi^j \wedge d\xi^j \wedge \delta \varphi(z + \xi) \wedge \frac{\rho(\xi)}{|\xi|^{2n}} w(\xi). \]
\begin{align*}
= & \left(-1\right)^{n-1+n(n+1)} \frac{1}{2} (2i)^n \sum_{j=1}^{n} \frac{\partial}{\partial z^{j}} \left[ \frac{\rho(z)}{\xi^{2n}} \right] \xi^{j} \, d\xi^{j} + (-1)^{n-1} \frac{\rho(z)}{\xi^{2n}} \sum_{j=1}^{n} \frac{\partial}{\partial z^{j}} \left[ \frac{\rho(z)}{\xi^{2n}} \right] \xi^{j} \, d\xi^{j} \\
= & \left(-1\right)^{n-1+n(n+1)} \frac{1}{2} (2i)^n \left\{ \frac{\partial}{\partial z^{j}} (z+\xi) \frac{\rho(z)}{\xi^{2n}} \xi^{j} + \varphi(z+\xi) \tilde{h}(\xi) \right\} d\xi^{j} \\
= & \left(-1\right)^{n-1+n(n+1)} \frac{1}{2} (2i)^n \tilde{h}(\xi) \, d\xi^{j},
\end{align*}

where we have put \( \tilde{h}(\xi) = \frac{n-1+n(n+1)}{2} \). Putting all this together, we get

\[
\frac{d}{dz} \left( \varphi(z) \frac{\rho(z)}{\xi^{2n}} \right) = \left(-1\right)^{n+1} \frac{(n-1)!}{(2ni)^n}.
\]

In (6) above we now get, since \( A = (-1)^{n-1} \frac{(n-1)!}{(2ni)^n} \):

\[
\varphi(z) = \lim_{R \to 0} - \left(-1\right)^{n-1} \frac{n+n(n+1)}{2} (2i)^n \left\{ \sum_{j=1}^{n} \frac{\partial}{\partial z^{j}} (z+\xi) \frac{\rho(z)}{\xi^{2n}} \xi^{j} + \varphi(z+\xi) \tilde{h}(\xi) \right\} d\xi^{j}.
\]

Since the \( n \) first integrals all converge as \( R \to 0 \), then so does the last, and we get

\[
\varphi(z) = \left(-\frac{1}{n}\right)^{n-1} \left\{ \sum_{j=1}^{n} \frac{\partial}{\partial z^{j}} (z+\xi) \frac{\rho(z)}{\xi^{2n}} \xi^{j} d\xi^{j} + \varphi(z+\xi) \tilde{h}(\xi) d\xi^{j} \right\}.
\]

Let \( h_{j}(z) = \left(-\frac{1}{n}\right)^{n-1} \left\{ \varphi(z+\xi) \frac{\rho(z)}{\xi^{2n}} \xi^{j} d\xi^{j} \right\} \), and

\[
h(\xi) = \left(-\frac{1}{n}\right)^{n-1} \tilde{h}(\xi) = \left(-\frac{1}{n}\right)^{n-1} \left\{ \sum_{j=1}^{n} \frac{\partial}{\partial z^{j}} (z+\xi) \frac{\rho(z)}{\xi^{2n}} \xi^{j} \right\}.
\]
We then get, (supp \( \rho \) is compact):

\[
\varphi(z) = \sum_{j=1}^{n} \frac{\partial h_j}{\partial z_j}(z) + \int_{C^h} \varphi(z+\xi)h(\xi)dm(\xi).
\]

Here \( h_1, \ldots, h_j \in C_c^\infty(U) \), and also \( g \in C_c^\infty(U) \), where \( g(z) = \int_{C^h} \varphi(z+\xi)h(\xi)dm(\xi) \).

This is clear, since supp \( \varphi \subseteq U_\varepsilon \) and supp \( \rho \subseteq \{|\xi|<\varepsilon\} \).

If we further assume that \( \text{supp} \varphi \subseteq U_{2\varepsilon} \subseteq U_\varepsilon \), then supp \( g \subseteq U_{\varepsilon/2} \).

In fact, let \( z \in U-U_\varepsilon \). If \( |\xi| \geq \varepsilon \), then \( h(\xi) = 0 \) since \( \rho(\xi) = 0 \), and thus \( \varphi(z+\xi)h(\xi) = 0 \). If \( |\xi| < \varepsilon \), then \( z+\xi \in U-U_{2\varepsilon} \), and thus \( \varphi(z+\xi)h(\xi) = 0 \cdot h(\xi) = 0 \), which proves (8).

Let further \( \theta \in C_c^\infty(U) \) be such that \( \theta = 1 \) in a nbhd. of \( \overline{U_{\varepsilon/2}} \).

Then the function

\[
C^h \ni \xi \rightarrow \langle \theta T, \xi \rangle_c = \langle \theta T, \tau^{V}_{\xi} \rangle_c
\]

is translation, \( \tau^{V}_{\xi} \) is translation, \( \tau^{V}_{\xi}(z) = h(-z) \) and \( (\theta T)_z \) means that \( \theta T \) acts w.r.t. \( z \).

That \( \langle \theta T, \tau^{V}_{\xi} \rangle_c \) is well defined, follows since \( \text{supp} \theta T \) is compact and the \( C^\infty \) statement follows as in part I) of the proof of our first lemma. By (8) we have \( \langle T, g \rangle = \langle \theta T, g \rangle \), and further

\[
\langle T, g \rangle = \langle \theta T, g \rangle = \langle \theta T, \int_{C^h} \varphi(z+\xi)h(\xi)dm(\xi) \rangle
\]

\[
= \langle \theta T, \int_{U_{2\varepsilon}} \varphi(z+\xi)h(\xi)dm(\xi) \rangle = \int_{U_{2\varepsilon}} \langle \theta T, \tau^{V}_{\xi} h \rangle \varphi(\xi)dm(\xi) \tag{10}
\]

We must prove the last equality, and after that we will quickly finish the proof of the theorem. Consider \( \int_{C^h} h(\xi-z)\varphi(\xi)dm(\xi) \) as a limit of Riemann sums of the form \( \sum_{\alpha} h_{\alpha}(\xi_{\alpha}-z)\varphi(\xi_{\alpha})m(S_{\alpha}) \), where \( m(S_{\alpha}) \) is the measure of a rectangle \( S_{\alpha} \) containing \( \xi_{\alpha} \). More generally, if \( D \) is a differentiation monomial of order \( p \) in the
real components of \( z \in \mathbb{C}^n \), then we have 
\[
D \int_{\mathbb{C}^n} h(\xi - z) \varphi(\xi) dm(\xi)
\]
\[= \lim_{\alpha} \sum_{\alpha} (-1)^{P(\alpha)} h(\xi_{\alpha} - z) \varphi(\xi_{\alpha}) m(S_{\alpha}) \] . These sums converge uniformly w.r.t. \( z \) on compact sets, and thus \( \lim(z \to \infty h(\xi_{\alpha} - z) \varphi(\xi_{\alpha}) m(S_{\alpha})) \)
\[= (z \to \int_{\mathbb{C}^n} h(\xi - z) \varphi(\xi) dm(\xi)) \] in the space \( C_c^{\infty}(\mathbb{C}^n) \). By continuity of \( \theta T \) on this space we get 
\[\lim(\theta T, z \to \infty h(\xi_{\alpha} - z) \varphi(\xi_{\alpha}) m(S_{\alpha})) = \langle \theta T, \int_{\mathbb{C}^n} h(\xi - z) \varphi(\xi) dm(\xi) \rangle \] . The left hand side of this equals
\[
\lim_{\alpha} \sum_{\alpha} \langle \theta T, h(\xi_{\alpha} - z) \varphi(\xi_{\alpha}) m(S_{\alpha}) \rangle . \] Since this is a Riemann sum for 
\[\int \langle \theta T, h(\xi - z) \varphi(\xi) dm(\xi) \rangle, \] (by (9) above the integrand is \( C_c^{\infty} \) with support in \( \text{supp} \varphi \)), we get 
\[\langle \theta T, \int_{\mathbb{C}^n} h(\xi - z) \varphi(\xi) dm(\xi) \rangle = \int_{\mathbb{C}^n} \langle \theta T, \tau_{\xi} h \rangle h(\xi) dm(\xi), \text{ for } \varphi \in C_c^{\infty}(U_{2\varepsilon}) . \] Thus (10) above is proved.

By (7) and (10) above we get for \( \varphi \in C_c^{\infty}(U_{2\varepsilon}) \), using the fact that \( \partial T / \partial z^j = 0 \) for \( j = 1, \ldots, n \) :
\[\langle T, \varphi \rangle = \langle T, \sum_{j=1}^{n} \partial \varphi / \partial z_j + \int_{\mathbb{C}^n} \varphi(z + \xi) h(\xi) dm(\xi) \rangle
\]
\[= \sum_{j=1}^{n} \langle T, \partial \varphi / \partial z_j \rangle + \langle T, \int_{\mathbb{C}^n} \varphi(z + \xi) h(\xi) dm(\xi) \rangle
\]
\[= \sum_{j=0}^{n} \langle \partial T / \partial z^j, h_j \rangle + \langle T, \int_{\mathbb{C}^n} \varphi(z + \xi) h(\xi) dm(\xi) \rangle
\]
\[= \int_{U_{2\varepsilon}} \langle \theta T, \tau_{\xi} h \rangle h(\xi) dm(\xi), \text{ and here } \xi \to \langle \theta T, \tau_{\xi} h \rangle, \]
which is independent of \( \varphi \), is a \( C_c^{\infty} \)-function, by (9) above. Thus \( T \) equals the \( C_c^{\infty} \)-function \( \xi \to \langle \theta T, \tau_{\xi} h \rangle \) in \( U_{2\varepsilon} \). Since this holds for all \( \varepsilon > 0 \), we have that \( T \) is a \( C_c^{\infty} \)-function, and thus holomorphic.

\[\text{QED.}\]
References:


