# Discrete series for locally compact groups

by

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Let $G$ be a locally compact group. By a representation of $G$ we mean a strongly continuous homomorphism of $G$ into the group of unitary operators on some Hilbert space $H_\pi$. In the present article we are concerned with the behavior of integrable and square-integrable representations under the inducing process, [9], together with their topological properties as points in the dual space $\hat{G}$ of $G$.

In [13] we proved if $\rho$ is an integrable cyclic representation of $K$, $K$ being a closed subgroup of $G$, then the induced representation $\mathrm{Ind}_K^G(\rho)$ is also integrable. The converse of this assertion does not hold (see e.g. Example (1.8)), however we show the following: If $\mathrm{Ind}_K^G(\rho)$ is completely integrable then $\rho$ is completely integrable; see Definition (1.1) for notation. Using the Mackey theory of induced representations we then derive that exponential Lie groups have no completely integrable irreducible representations.

By an [IN] group we mean a locally compact group possessing a compact neighborhood of the identity $e$ invariant under inner automorphisms. $G$ is an $[FC]^-$ group provided all of its conjugacy classes $\{yxy^{-1} : y \in G\}$ are relatively compact, [4]. We let $\hat{G}$ be the set of all equivalence classes, under unitary equivalence, of irreducible representations of $G$ endowed with the Fell-topology. This topology may be described as the inverse image of the hull-kernel topology on the space $\mathrm{Prim}(G)$ of all primitive ideals in the group $C^*$-algebra $C^*(G)$ , under the map $\pi \in C^*(G) \to \ker\pi \in \mathrm{Prim}(G)$ , [2]. Denote by $\hat{G}_r$ the reduced dual of $G$, i.e. $\hat{G}_r$ is the subspace of $\hat{G}$ consisting of all the representations that are weakly contained in the regular representation of $G$ ( $\pi$ is weakly contained in $\rho$ provided $\ker\pi \supseteq \ker\rho$ where $\pi$ and $\rho$ are regarded as representations of $C^*(G)$ , [2]). The following conjecture due to Dixmier is still open.

(*) Conjecture. Let $G$ be a locally compact group, $\pi \in \hat{G}$ square-integrable. Then $\{\pi\}$ is open in $\hat{G}_r$ where $\hat{G}_r$ is given the relative topology from $\hat{G}$ .

In Section 2 we study this question, succeeding to verify the conjecture for the class of separable [IN] groups. In fact we prove the somewhat stronger result that every square-integrable irreducible representation is an open point of $\hat{G}$ . This is known to be false in general. Our proofs depend on earlier results of the author for $[FC]^-$ groups [12], together with Mackey's little group method, [9]. It turns out that our method works for a somewhat larger class than [IN] , namely for all groups $G$ possessing an open normal $[FC]^-$ subgroup. Hence the conjecture (*) holds for many nilpotent Lie groups with compact centers.

1. Before beginning we establish some more notation. If $K$ is a closed subgroup of $G$ and $\rho$ a representation of $K$ we let $\mathrm{Ind}_K^G(\rho)$ be the unitary representation of $G$ induced by $\rho$ , [9]. For essentially bounded complex valued functions $\varphi$ on $G$ let $\|\varphi\|_G$ denote the essential supremum of $\varphi$ .

(1.1) Definition. A cyclic representation $\pi$ of $G$ is said to be integrable (resp. square-integrable) if there is a cyclic vector $v$ for $\pi$ such that the coordinate function $x \to (\pi(x)v,v)$ is integrable (resp. square-integrable) w.r.t. left Haar measure on $G$.
A representation $\pi$ of $G$ is completely integrable (resp. completely square-integrable) if all of its coordinate functions are in $L^1(G)$ (resp. $L^2(G)$) w.r.t. left Haar measure on $G$.
The discrete series of $G$ constitutes all the irreducible subrepresentations of the left regular representation of $G$ on the Hilbert space $L^2(G)$.

If $G$ is unimodular and $\pi \in \hat{G}$ then $\pi$ is completely square-integrable iff $\pi$ is square-integrable iff $\pi$ belongs to the discrete series of $G$. The first of these equivalences does not hold for non unimodular groups, see e.g. Example (1.7). For arbitrary locally compact $G$ we have $\pi \in \hat{G}$ is square-integrable implies $\pi$ is equivalent to a subrepresentation of the (left) regular representation (the proof given in [2] 14.1.1. for unimodular groups is valid in the general setting).

The subset of $\hat{G}$ consisting of classes whose elements are square-integrable representations is denoted by $\hat{G}_s$. If $\pi \in \hat{G}$ is integrable we have $\pi \in \hat{G}_s$, since the functions $x \to (\pi(x)v,v)$ are bounded.

We come now to the principal result of this section.

(1.2) Theorem. Let $G$ be a separable locally compact group and $H$ a closed subgroup. Assume $G/H$ has a $G$-invariant measure. If $\rho$ is a cyclic representation of $H$ and $\pi = \mathrm{Ind}_H^G(\rho)$ then we have

(i) $\pi$ is cyclic

(ii) If $\rho$ is integrable then $\pi$ is integrable

(iii) If $\pi$ is completely integrable then $\rho$ is completely integrable.

<u>Proof.</u> (i) and (ii) follows as Proposition (1.6) in [13] since the proof given there for unimodular groups may be adapted to this more general situation. Note only that the "cyclic" function $f$ in that proof may be chosen from $L^1(G)$ rather than from $C_c(G)$ (there is an inaccuracy at that point in [13]). Also, $f$ may be chosen such that $f(x^{-1}) = \overline{f(x)}$ ; see [5] Section 3. After these remarks the proof in [13] goes through.

To show (iii) assume all the coordinate functions $x \to (\pi(x)v,v)$ are integrable. This is equivalent to the following: The linear functionals

$$F_v : L^\infty(G) \cap L^1(G) \to \mathbb{C} ; \quad f \to (\pi(f)v,v)$$

are continuous in the norm $\|\cdot\|_G$ ; all $v \in H_\pi$ . That is, given $v \in H_\pi$ there is a constant $C_v$ such that

$$|F_v(f)| = |(\pi(f)v,v)| \leq C_v\|f\|_G ,$$

for all $f \in L^\infty(G) \cap L^1(G)$ , (see [2] 14.6.1).

Let $u \in H_\rho$ be arbitrary and put

$$\varphi_u(k) = (\rho(k)u,u) \quad ; \quad \text{all } k \in L^\infty(H) \cap L^1(H) .$$

We wish to prove that $\varphi_u$ is a continuous linear functional on $L^\infty(H) \cap L^1(H)$ in the norm $\|\cdot\|_H$ . This will imply integrability of the corresponding coordinate function

$$h \to (\rho(h)u,u) \quad ; \quad H \to \mathbb{C} .$$

Now, since $\pi = \mathrm{Ind}_H^G(\rho)$ we have from Blattner's theory of positive definite measures and induced representations that

$$f \to \mu(f) = \varphi_u(f|H) \quad ; \quad f \in C_c(G),$$

is a measure associated to $\pi$. Hence we may assume $\pi$ is constructed from $\mu$ in the usual way (see e.g. [13] the proof of (1.6)). By [5] Theorem 3.1 there is an $f \in L^1(G)$ such that $[f]$ is a cyclic vector for $\pi$ and in addition

$$\mu(\psi) = (\pi(\psi)[f],[f])_\mu \,, \quad \text{all} \quad \psi \in C_c(G) .$$

If $k \in C_c(H)$ let $\tilde{k}$ be any extension of $k$ to a continuous function with compact support on $G$ such that $\|\tilde{k}\|_G = \|k\|_H$ (Tietze's extension theorem). Then

$$\begin{aligned} |\varphi_u(k)| &= |\varphi_u(\tilde{k}|H)| = |\mu(\tilde{k})| \\ &= |(\pi(\tilde{k})[f],[f])_\mu| \leq C_f\|\tilde{k}\|_G \\ &= C_f\|k\|_H \quad ; \quad \text{all} \quad k \in C_c(H) , \end{aligned}$$

where $C_f$ is a constant depending only on $f$.
Thus $\varphi_u$ is a continuous linear functional on $L^\infty(H) \cap L^1(H)$.

QED.

In view of the easy fact that every cyclic representation of a compact group is integrable the following result is clear.

<u>(1.3) Corollary</u>. Let $\pi$ be a representation of the separable group $G$ and assume $\pi = \mathrm{Ind}_K^G(\rho)$ where $K$ is a compact subgroup of $G$ and $\rho$ is a cyclic representation of $K$. Then $\pi$ is integrable.

In what follows we shall assume the reader is familiar with Mackey's little group method, [9]. If $H$ is a closed normal subgroup of $G$ and $\rho \in \hat{H}$, $G$ acts on $\rho$ by inner automorphisms:

$$x \cdot \rho(h) = \rho(x^{-1}hx) \ , \quad \text{all} \ \ x \in H \ , \ \ h \in H \ .$$

If $\pi \in \hat{G}$ and the restriction $\pi|H$ is a multiple of a direct integral $\int_{G\cdot\rho}^{\oplus} \sigma(s)\, d\mu(s)$ over some $G$-orbit $G\cdot\rho$ in $\hat{H}$ we shall say that <u>$\pi$ lies over (the orbit of) $\rho$</u>. We denote by $\hat{G}_{\rho,H}$ the set of all $\pi \in \hat{G}$ such that $\pi$ lies over $\rho$, where $\rho \in \hat{H}$. The isotropy group of $\rho$ is denoted by $G(\rho)$. Thus $G(\rho) = \{x \in G : x\cdot\rho \simeq \rho\}$. In case $\pi$ lies over $\rho \in \hat{H}$ it follows from Mackey's theory that $\pi = \mathrm{Ind}_{G(\rho)}^{G}(\sigma)$ for some $\sigma \in \widehat{G(\rho)}_{\rho,H}$, ([9] Theorem 8.1).

Next we use Mackey's little group method to derive some consequences of Theorem (1.2) (iii). The following lemma will be helpful.

<u>(1.4) Lemma.</u> Let $H$ be a closed normal subgroup of $G$ and let $\pi \in \hat{G}$ be integrable.

(1) If $\rho \in \hat{H}$ and $\rho$ is a subrepresentation of the restriction $\pi|_H$ then $\rho$ is integrable.

(2) If $\rho \in \hat{H}$ is integrable then all the representations $x\cdot\rho$ in the $G$-orbit of $\rho$ are integrable.

<u>Proof.</u> The proofs of Lemma (1.1) and Lemma (1.2) given in [13] are valid for integrable representations.

<u>(1.5) Proposition.</u> Suppose $N$ is a closed normal subgroup of $G$. Fix $\omega \in \hat{N}$ and let $G(\omega)$ be the isotropy subgroup of $\omega$. Assume $G/G(\omega)$ has a $G$-invariant measure, $\pi \in \hat{G}$ is completely integrable, and $\pi$ lies over $\omega$. Then $G\cdot\omega$ consists entirely of integrable representations.

Proof. Suppose $\pi \in \hat{G}$ is completely integrable and lies over $\omega \in \hat{N}$. By virtue of [9] Theorem 8.1 $\pi = \mathrm{Ind}_{G(\omega)}^{G}(\sigma)$ for some $\sigma \in G(\omega)^{\wedge}_{\omega,N}$. By Theorem (1.2) $\sigma$ is completely integrable, and since $\omega$ is a subrepresentation of $\sigma|N$ it follows from Lemma (1.4) (1) that $\rho$ is integrable. Thus each $x\cdot\omega$ ; $x \in G$, is integrable (Lemma (1.4) (2)).

QED.

(1.6) Corollary. Let $N$ be a closed normal subgroup of $G$ and assume $N$ is type I and regularly embedded in $G$. Suppose $G/G(\omega)$ has a $G$-invariant measure for all $\omega \in \hat{N}$. If $N$ has no integrable irreducible representations then there is no completely integrable irreducible representation of $G$.

Proof. If $\pi \in \hat{G}_{\omega,N}$ were completely integrable then $\omega$ were integrable by Proposition (1.5).

QED.

A (solvable) Lie group is said to be exponential provided its exponential map is a bijection from the Lie algebra to the group.

(1.7) Corollary. Let $G$ be an exponential Lie group. Then $G$ has no completely integrable representations.

Proof. The nilradical $N$ of $G$ is regularly embedded, [10] Corollary 2, and $N$ has no integrable irreducible representations, being simply connected and nilpotent. Thus $G$ has no completely integrable irreducible representations (1.6)

QED.

In order to illustrate the results above we shall give some examples.

(1.8) Example. Let $G$ be the "$ax+b$" group:

$G = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0,\ b \in \mathbb{R}\right\}$ with the usual topology and matrix multiplication. $G$ is an exponential Lie group and there is exactly one square-integrable $\pi \in \hat{G}$. It may be verified that $\pi$ is not completely integrable (in fact, $\pi$ is not even completely square-integrable), [7].

(1.9) Example. Consider the group

$$G = \left\{\begin{pmatrix} 2^n & e^{2\pi i\theta} \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z},\ \theta \in \mathbb{R}\right\}$$

with multiplication

$$\begin{pmatrix} 2^n & e^{2\pi i\theta} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 2^m & e^{2\pi i\tau} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{n+m} & e^{2\pi i\, 2^n\tau+\theta} \\ 0 & 1 \end{pmatrix}$$

and the obvious topology from $\mathbb{Z}$ and the circle $\mathbb{T}$.

Let $\chi : e^{2\pi i\theta} \to e^{2\pi i\theta}$ be the generating character of the circle group $\mathbb{T}$. $\mathbb{T}$ may be identified with a closed normal subgroup $N$ of $G$:

$$N = \left\{\begin{pmatrix} 1 & e^{2\pi i\theta} \\ 0 & 1 \end{pmatrix} : \theta \in \mathbb{R}\right\}$$

The isotropy group of $\chi$ under the action of $G$ on $\hat{N}$ by inner automorphisms is seen to equal $N$. Hence $\pi = \mathrm{Ind}_N^G(\chi)$ is an irreducible representation of $G$. $\pi$ may be realized on a space of functions from $G$ into the complex numbers as follows.

$$\pi\begin{pmatrix} 2^n & e^{2\pi i\theta} \\ 0 & 1 \end{pmatrix} f(2^m) = e^{2\pi i\, 2^{-n}\theta} f(2^{m+n}) ,$$

where $\sum_{m=-\infty}^{+\infty} |f(2^m)|^2 < \infty$.

Letting $f \geq 0$ be such that $\sum_{-\infty}^{\infty} f(2^m) < +\infty$ it may be seen that the coordinate function

$$\begin{pmatrix} 2^n & e^{2\pi i\theta} \\ 0 & 1 \end{pmatrix} \to \langle \pi \begin{pmatrix} 2^n & e^{2\pi i\theta} \\ 0 & 1 \end{pmatrix} f, f \rangle$$

is integrable on $G$ . Thus $\pi$ is integrable.
Letting $f \in L^2(G)$ , $f \geq 0$ , be such that $\sum_{-\infty}^{\infty} f(2^m) = +\infty$ one may verify that the corresponding coordinate function is not integrable. Thus $\pi$ is not completely integrable. (Note that $G$ is unimodular.)
Hence Theorem (1.2) (ii) is as good as possible: $\chi$ is completely integrable without $\pi = \operatorname{Ind}_N^G(\chi)$ being completely integrable.

2. In this section we study square-integrable irreducible representations. Our main result states that for $G \in [IN]$ each $\pi \in \hat{G}_s$ forms an isolated point of $\hat{G}$ , thus justifying the name "discrete series" for this class of groups. All our groups will be separable.

(2.1) Proposition. Let $G$ be unimodular and assume $G$ contains an open normal subgroup $N$ . If $\pi \in \hat{G}_s$ then $\pi \in \hat{G}_{\rho,N}$ , for some $\rho \in \hat{N}$ , where $\hat{G}_{\rho,N}$ consists of only a finite number of elements; all of them being square-integrable. Moreover, $\pi = \operatorname{Ind}_{G(\rho)}^G(\sigma)$ for some $\sigma \in G(\rho)^\wedge_{\rho,N}$ and we have $|G(\rho)/N| < \infty$ .

Proof. Since $\pi \in \hat{G}_s$ and $G$ is unimodular, all the coordinate functions of $\pi$ are in $L^2(G)$ , [2], and it follows since $N$ is open, that $\pi|N$ splits into a discrete direct sum of square-integrable irreducible representations of $N$ (Kunze [8] Cor. to Thm. 2). Now $N$ is normal in $G$ and the usual arguments give that $\pi|N$ is concentrated on exactly one $G$-orbit in $\hat{N}$:

$$\pi|N = m\cdot \bigoplus_{G/G(\rho)} \dot{g}\cdot\rho \quad , \text{ for some } \rho \in \hat{N} \, .$$

Hence $\pi \simeq \mathrm{Ind}_{G(\rho)}^{G}(\sigma)$ for some $\sigma \in G(\rho)^{\wedge}_{\rho,N}$, [9] Theorem 8.1. Moreover, $\sigma$ is on the form $\sigma \simeq \tilde{\rho} \otimes \gamma'$ where $\gamma$ is some multiplier representation of $G(\rho)/N$ (say $\alpha$-representation), and $\tilde{\rho}$ is an extension of $\rho$ to an $\bar{\alpha}$-representation of $G(\rho)$, [9].

Since $N$ is open $\sigma$ is a subrepresentation of $\pi|N$ (Mackey's subgroup theorem, or direct verification). Thus $\sigma$ is square-integrable, and this gives easily

$$\int_{G(\omega)/N} |(\gamma(\dot{x})v,v)|^2 d\dot{x} < \infty \; ; \quad \text{all } v \in H_\gamma \, ,$$

i.e., $\gamma$ is a square-integrable multiplier-representation of the discrete group $G(\omega)/N$. By [12] Lemma (2.1) $G(\omega)/N$ must then be finite. Now each element of $\hat{G}_{\rho,N}$ is induced from a representation of $G(\rho)$ on the form $\tilde{\rho} \otimes \tau'$ where $\tau$ is an irreducible $\alpha$-representation of $G(\rho)/N$. By the finiteness of $G(\rho)/N$ there is only a finite number of nonequivalent $\tilde{\rho} \otimes \tau'$, and this yields the finiteness of $\hat{G}_{\rho,N}$.

QED.

We now assume $G$ is an [IN] group. Let $N$ be the set of all $x \in G$ such that the conjugacy class $\{yxy^{-1} : y \in G\}$ is relatively compact. $N$ is obviously a normal subgroup of $G$, and since $G$ contains a compact neighbourhood of $e$ invariant under inner automorphisms $N$ is open. By abuse of notation $N$ is called the FC-subgroup of $G$. Let $\pi \in \hat{G}_s$. By the above result there is a $\sigma \in \hat{N}_s$ such that $\pi \in \hat{G}_{\sigma,N}$ and $\pi = \mathrm{Ind}_{G(\sigma)}^{G}(\rho)$ where $|G(\sigma)/N| < \infty$ and $\rho \in G(\sigma)^{\wedge}_{\sigma,N}$.

Since $\pi|N$ splits into a direct sum of irreducible square-integrable representations, $\hat{N}_s \neq \emptyset$ so that $N$ is type I ([12] Theorem (2.3)). We have proved the following result.

(2.2) Corollary. Let $G$ be an [IN] group, $N$ the open normal FC-subgroup. If $\pi \in \hat{G}$ is square-integrable then $N$ is type I, and we have $\pi \in \hat{G}_{\sigma,N}$ for some $\sigma \in \hat{N}_s$. Also, the stability group $G(\sigma)$ is type I, and $\pi = \mathrm{Ind}_{G(\sigma)}^{G}(\rho)$ for some $\rho \in G(\sigma)^{\wedge}_{\sigma,N}$.

Let $G$ be an [IN] group and $N$ its FC-subgroup. If $\pi \in \hat{G}_s$ we may pick a $\rho \in \hat{N}_s$ such that $\pi \in \hat{G}_{\rho,N}$ (2.2). Write $\pi|N \simeq m \cdot \bigoplus_{G/G(\rho)} \dot{x}\cdot\rho$. By (2.2) $N$ is a type I $[FC]^-$ group and satisfies thereby an exact sequence of topological groups

$$(e) \to K \to N \to \mathbb{R}^n \to (e)$$

where $K$ is compact, [11] the note preceeding Theorem (1.3). Now $K$ is invariant under all the automorphisms of $N$, in particular it is normal in $G$, [4] Theorem 3.16, (1).
Since $K$ is compact $\pi \in \hat{G}_{\omega,K}$ for some $\omega \in \hat{K}$, and in addition $\hat{G}_{\omega,K}$ is open in $\hat{G}$ ([11] Lemma (1.2)).

The proof of our next theorem requires the following lemma.

(2.3) Lemma. Let $G$ be an [IN] group, $N$ the FC-subgroup of $G$. Suppose $\hat{G}_s \neq \emptyset$ and let $K$ be a maximal compact subgroup of $N$ ($K$ exists since the hypothesis yields $N$ is type I). Then $K$ is normal in $G$.
Let $\pi \in \hat{G}_s$ and pick $\omega \in \hat{K}$ such that $\pi \in \hat{G}_{\omega,K}$. If $\gamma \in \hat{G}_{\omega,K}$ and $\gamma \neq \pi$ then $\gamma$ lies over the discrete $G$-orbit of some $\sigma \in \hat{N}_s$, and we have

$$\gamma = \mathrm{Ind}_{G(\sigma)}^{G}(\varphi)$$

for some $\varphi \in G(\sigma)^{\wedge}_{\sigma,N}$.

<u>Proof</u>. Assume $\gamma \in \hat{G}_{\omega,K}$ and $\gamma \neq \pi$. We have $\gamma|K = n\cdot \bigoplus_{G/G(\omega)} \dot{x}\cdot\omega$, for some $n \in \{1,2,3,\ldots,\aleph_0\}$. Putting $\lambda = \gamma|N$, $\lambda$ has a direct integral decomposition (centraldecomposition)

$$\lambda = \int_{\hat{N}}^{\oplus} k(s)\lambda(s)d\mu(s)$$

for some standard Borel measure $\mu$ on $\hat{N}$, where each $\lambda(s)$ is an irreducible representation of $N$ and $k(s)$ its multiplicity $(0 \leq k(s) \leq \aleph_0)$, [2] 8.4.2.

Hence

$$\lambda|K = \int_{\hat{N}}^{\oplus} k(s)(\lambda(s)|K)d\mu(s) .$$

Now each $\lambda(s)$ is irreducible so that $\lambda(s) \in \hat{N}_{\omega(s),K}$ for some $\omega(s) \in \hat{K}$.

$N/K$ acts continuously on $\hat{K}$ by inner automorphisms, and since $\hat{K}$ is discrete and $N/K \simeq \mathbb{R}^n$ is connected it follows that the orbits in $\hat{K}$ are singletons; in particular the stability group $N(\omega(s))$ equals $N$. Thus

$$\lambda(s)|K \simeq m(s)\omega(s) , \quad \text{where} \quad 1 \leq m(s) \leq \aleph_0 .$$

Writing $n(s) = k(s)m(s)$ we have

$$\lambda|K \simeq \int_{\hat{N}}^{\oplus} n(s)\omega(s)d\mu(s)$$

Also

$$\lambda|K \simeq \gamma|K \simeq n\cdot \bigoplus_{G/G(\omega)} \dot{x}\cdot\omega .$$

Comparing these two decompositions of $\lambda|K$ it follows that a.a. $\omega(s)$ equals some $\dot{x}\cdot\omega$. Hence $\lambda(s)|K \simeq n(s)\dot{x}\cdot\omega$ for some $\dot{x} \in G/G(\omega)$ (a.a. $s \in \hat{N}$), that is, $\lambda(s) \in \hat{N}_{\dot{x}\cdot\omega,K}$.

Since $\rho \in \hat{N}_{\omega,K}$ we have

$$\rho \simeq \tilde{\omega} \otimes \tau'$$

where $\tau$ is some multiplier representation (say $\alpha$-representation)

of $N/K$ and $\tau'$ its inflation to $N$, and where $\tilde{\omega}$ is some extension of $\omega$ to an $\tilde{\alpha}$-representation of $N$ ([9] Theorem 8.1). Clearly $\dot{x}\cdot\tilde{\omega}$ is an irreducible $\tilde{\alpha}$-representation of $N$ extending $\dot{x}\cdot\omega$, so that $(\dot{x}\cdot\tilde{\omega}) \otimes \tau' \in \hat{N}_{\dot{x}\cdot\omega,K}$.
Now, since $\rho \simeq \tilde{\omega} \otimes \tau'$ is square-integrable one easily verifies that $(\dot{x}\tilde{\omega}) \otimes \tau'$ is square-integrable. By [11] Theorem (2.4), (4) (in view of the maximality of $K$, $P = K$) we have $\hat{N}_{\dot{x}\cdot\omega,K}$ consists only of the singleton $(\dot{x}\cdot\tilde{\omega}) \otimes \tau'$. Hence

$$(\dot{x}\cdot\omega) \otimes \tau' \simeq \lambda(s) \ , \quad \text{a.a.} \quad s \in \hat{N} \ ;$$

thus $\lambda(s) \in \hat{N}_s$, a.a. $s \in \hat{N}$. By [11] Theorem (2.4) a.a. $\lambda(s)$ is open in $\hat{N}$, and this gives $\mu(\{\lambda(s)\}) > 0$, a.a. $s \in \hat{N}$, [3] Theorem 3.2, so that the direct integral

$$\gamma|N \simeq \int_{\hat{N}}^{\oplus} k(s)\lambda(s)d\mu(s)$$

decomposes into a discrete sum, say

$$\gamma|N \simeq \bigoplus_i \lambda_i \qquad , \quad \lambda_i \in \hat{N}_s \ .$$

Now since $N$ is normal in $G$ one sees by the usual arguments that all the $\lambda_i$'s belong to the same $G$-orbit, say $G\cdot\sigma$, in $\hat{N}_s$. Hence $\gamma \in \hat{G}_{\sigma,N}$ and it follows that

$$\gamma \simeq \mathrm{Ind}_{G(\sigma)}^{G} (\varphi) \qquad , \quad \text{for some} \quad \varphi \in G(\sigma)^{\wedge}_{\sigma,N} \ .$$

QED.

We are now in a position to prove the main result of this section.

<u>(2.4) Theorem</u>. Let $G$ be a separable [IN] group and suppose $\pi \in \hat{G}$ is square-integrable. Then the singleton $\{\pi\}$ is open in $\hat{G}$.

<u>Proof</u>. Assume $\pi \in \hat{G}_s$ and let $\langle \gamma_n \rangle_{n=1}^{\infty}$ be a sequence from $\hat{G}$ con-

verging to $\pi$ (we may restrict ourself to sequences since $\hat{G}$ is separable). Fix $\omega \in \hat{K}$ such that $\pi \in \hat{G}_{\omega,K}$ where $K$ is a maximal compact subgroup of the FC-subgroup $N$, as in Lemma (2.3). In view of the fact that $\hat{G}_{\omega,K}$ is open in $\hat{G}$ we may assume each $\gamma_n \in \hat{G}_{\omega,K}$. Let $v \in H_\pi$ be arbitrary and pick a sequence $\langle v_k \rangle_{k=1}^{\infty}$ of vectors $v_k \in H_{\rho_k}$, such that

$$(\rho_k(x)v_k, v_k) \underset{k}{\rightarrow} (\pi(x)v, v)$$

uniformly on compacta in $G$, [2]. In particular the above converges uniformly on compacta in $N$. Hence $\pi|N$ is weakly contained in the set $\{\rho_k|N : n = 1,2,3,\ldots\}$. If we write

$$\gamma_n|N = k(n)\cdot \bigoplus_{G/G(\sigma_n)} \dot{x}\cdot\sigma_n \quad , \quad k(n) \leq \aleph_0 \quad , \; n = 1,2,3,\ldots,$$

it follows that the orbit $G\cdot\sigma$ of $\pi$ in $\hat{N}$ is weakly contained in the collection of orbits $S = \{G\cdot\sigma_n : n = 1,2,3,\ldots\}$, see e.g. [3] Theorem 4.5. (here the orbit space $\hat{N}/G$ is provided with the quotient topology from $\hat{N}$). Thus $G\cdot\sigma \in \bar{S}$.
Now each element $x\cdot\sigma$ in the orbit of $\sigma$ is square-integrable ([13] Lemma (1.2)) and is therefore open in $\hat{N}$, $N$ being an $[FC]^-$ group ([12] Theorem (2.3)). Hence the orbit $G\cdot\sigma$ is an open point in $\hat{N}/G$. Since $G\cdot\sigma_n \underset{n}{\rightarrow} G\cdot\sigma$ in $\hat{N}/G$ we have $G\cdot\sigma = G\cdot\sigma_n$ for $n$ greater than a certain $n_o$. Hence $\gamma_n \in \hat{G}_{\sigma,N}$ for all $n \geq n_o$ and since $\hat{G}_{\sigma,N}$ is discrete in the relative topology from $\hat{G}$ we must have $\gamma_n \simeq \pi$ from a certain $n$ on, and $\{\pi\}$ is open in $\hat{G}$.

QED.

A closer inspection of the results obtained so far in this section reveals that they are valid under slightly more general conditions. In fact, let $G$ be a separable locally compact group and suppose that $N$ is an open normal subgroup. Assume also that all the conjugacy classes $C_n = \{m n m^{-1} : m \in N\}$ of $N$ are relatively compact, i.e. $N \in [FC]^-$ . If $\pi \in \hat{G}$ is square-integrable we have $\hat{N}_s \neq \emptyset$ since $N$ is open (see the proof of Proposition (2.1)). Thus $N$ is type I ([12] Theorem (2.3)), and we may pick a maximal compact subgroup $K$ of $N$, invariant under all automorphisms of $N$. It follows that all the arguments used in the proofs of (2.2), (2.3), and (2.4) are valid even in the present situation, and we have the following improvement of Theorem (2.4).

(2.5) Theorem. Let $G$ be a separable locally compact group and assume there is an open normal $[FC]^-$ subgroup $N$ of $G$. Then the points of $\hat{G}_s$ are open in $\hat{G}$.

As a consequence we immediately obtain the following result.

(2.6) Corollary. Let $G$ be a nilpotent Lie group and $G_o$ its identity component. Suppose $G_o/K \simeq \mathbb{R}^m$ where $K = G_o \cap Z(G)$, and $Z(G)$ is the center of $G$. If $\hat{G}_s \neq \emptyset$ then the points of $\hat{G}_s$ are open in $\hat{G}$.

Proof. If $\hat{G}_s \neq \emptyset$ we must have $Z(G)$ is compact, thus $G_o$ is on the form $(e) \to K \to G_o \to \mathbb{R}^m \to (e)$ where $K$ is compact; and the commutator group $[G_o, G_o]$ must have compact closure. Hence $G_o \in [FC]^-$ and the corollary follows from (2.5).

QED.

We illustrate the theory by an example.

(2.7) Example. Let $H_1$ be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ isomorphic to the m-th order Heisenberg algebra, i.e. there is a basis $X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z$

for $\mathfrak{g}$ such that $[X_i, Y_j] = Z$ and all other brackets are zero. Let $L$ be a discrete subgroup in the one-dimensional center of $H_1$ isomorphic to $\mathbb{Z}$. Put $H = H_1/L$. We may realize $H$ as a "matrix" group as follows. $H$ consists of all $(m+1) \times (m+1)$ matrices on the form

$$\begin{pmatrix} 1 & x_1 & \cdots & x_m & e^{2\pi i\theta} \\ & 1 & & O & y_m \\ & & \ddots & & \vdots \\ & O & & 1 & y_1 \\ & & & & 1 \end{pmatrix}$$

where the entries $x_i, y_j, (1 \leq i, j \leq m)$ are real and $\theta$ is real. The multiplication in $H$ is the obvious modification of ordinary matrix multiplication. $H$ has one-dimensional compact center: $Z(H) \approx \mathbb{T}$, and $H/Z(H) \approx \mathbb{R}^{2m}$.

The collection of all infinite dimensional irreducible representations of $H$ (equivalence classes) constitutes the discrete series (these are the irreducible representations lying over nontrivial characters of the center). Clearly $H$ is an $[FC]^-$ group and the points of $\hat{H}_s$ are open by (2.4). The same conclusion holds for groups $G$ with identity component isomorphic to $H$, see (2.6).

Having noted the close relationship between [IN] groups and certain nilpotent groups it would be natural to include a complete discussion of Dixmier's conjecture for the latter class. Using Mackey theory and induction on the dimension of the group we have verified the conjecture for connected nilpotent Lie groups. We omit the proof, noting that results of Auslander, Kostant, Moore, and Pukanszky combined with the fact that for nilpotent groups $G$

the "Kirillov correspondence" between $\hat{G}$ and the orbit space $\mathfrak{g}^*/G$ under the action of the coadjoint representation of G on the real dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G is a homéomorphism, (Brown [11]).

(2.8) Proposition. Let G be a connected nilpotent Lie group. Then the points of $\hat{G}_s$ are open in $\hat{G}$.

It should be noted that R. Lipsman has verified the conjecture (*) for split-rank one semisimple Lie groups (Dual topology for principal and discrete series, Trans. Amer. Math. Soc. 152 (1970) 399-417).

We hope to study the conjecture for a larger class of groups on a later occasion.

## REFERENCES


[1] I Brown, Dual topology of a nilpotent Lie group, Ann. Sci. École norm. sup., IV. Sér. 6 (1973) 407-411.

[2] J. Dixmier, Les C*-algèbres et leurs représentations, Gauthier-Villars, Paris 1964.

[3] J. Fell, Weak containment and induced representations, Can. J. Math. 14 (1962) 237-268.

[4] S. Grosser and M. Moskowitz, Compactness conditions in topological groups, J. Reine angew. Math. 246 (1971) 1-40.

[5] A. Hulanicki and T. Pytlik, On commutative approximate identities and cyclic vectors of induced representations, Studia Math. 48 (1973) 189-199.

[6] A. Kirillov, Unitary representations of nilpotent Lie groups, Russian Math. Surveys 17 (1962) no.4, 53-104.

[7] A. Kleppner and R. Lipsman, The Plancherel formula for group extensions, I and II, Ann. Sci. École norm. super. T. 5 (1972) 189-199, and T. 6 (1973), 103-132.

[8] R. Kunze, A note on square-integrable representations, J. Func. Anal. 6 (1970) 454-459.

[9] G. Mackey, Unitary representations of group extensions I, Acta Math. 99 (1958) 265-311.

[10] L. Pukanszky, On the unitary represenations of exponential groups, J. Func. Anal. 2 (73-113) 1968.

[11] T. Sund, Duality theory for locally compact groups with precompact conjugacy classes II, the dual space, preprint.

[12] T. Sund, On the dual topology of [FC]$^-$ groups, preprint.

[13] T. Sund, Square-integrable representations and the Mackey theory, Trans. Amer. Math. Soc., 194 (1974) 131-139.


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