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PURE STATES OF SIMPLE C*-ALGEBRAS

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Introduction

In [4] Powers studied uniformly hyperfinite (UHF) C*-algebras. He proved that factor states of such algebras can be characterized by a product decomposition property (Theorem 2.5 of [4]), and he found necessary and sufficient conditions that two factor representations be quasi-equivalent (Theorem 2.7 of [4]). Analogous results are also proved in [3]. In the present paper we shall derive the same type of results for pure states of simple C*-algebras with identity, thus indicating how properties of UHF-algebras may be extended to general C*-algebras.

A C*-algebra $\mathfrak{U}$ is called a CCR-algebra if every irreducible representation of $\mathfrak{U}$ maps $\mathfrak{U}$ into the completely continuous operators. If a C*-algebra $\mathfrak{U}$ has no non-zero CCR ideals, then we call $\mathfrak{U}$ an NGCR-algebra.

In lemma 4 of [2] Glimm proved that a separable NGCR-algebra with identity contains an ascending sequence of approximate matrix algebras of order $2,4,\ldots,2^n,\ldots$ with certain density properties, and we use these approximate matrix algebras to state our results.

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1. Definitions and simple consequences.

We use the notation and terminology developed by Glimm in [2]. We shall write 0_n for the n-tuple $(0,\ldots,0)$ and $[M]$ for the closed linear span of M, where M is a subset of a Hilbert space.

<u>Definition 1</u>. Let $V(a_1,\ldots,a_n)$, $a_i \in \{0,1\}$, and $B(n)$ be elements of a C^*-algebra, where n is a positive integer. We call

$$\{V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*, B(n) : a_i, b_i \in \{0,1\}\}$$

an <u>approximate matrix algebra</u> of order 2^n if the following axioms are satisfied:

(1) $V(a_1,\ldots,a_n)^*V(b_1,\ldots,b_n) = 0$ if $(a_1,\ldots,a_n) \neq (b_1,\ldots,b_n)$

(2) $V(0_n) \geq 0$ and $\|V(a_1,\ldots,a_n)\| = 1$

(3) $B(n) \geq 0$ and $\|B(n)\| = 1$

(4) $V(a_1,\ldots,a_n)^*V(a_1,\ldots,a_n)B(n) = B(n)$

<u>Definition 2</u>. For each $n = 1,2,\ldots$, let $V(a_1,\ldots,a_n)$, $a_i \in \{0,1\}$, and $B(n)$ be elements of a C^*-algebra $\mathfrak{U}$. We call

$$\{V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*, B(n) : a_i, b_i \in \{0,1\} \text{ and } n = 1,2,\ldots\}$$

an <u>approximate sequence of approximate matrix algebras</u> if the following properties are satisfied:

(1) We let $E(n) = \sum_{a_1,\ldots,a_n} V(a_1,\ldots,a_n)V(a_1,\ldots,a_n)^*$.

For each $S \in \mathfrak{U}$ and each $\epsilon > 0$ there exist an n and a linear combination T of elements of the form $V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*$ such that $\|E(n+1)(S-T)E(n+1)\| < \epsilon$.

(2) If $j \leq k$ and if $(a_1,\ldots,a_j) \neq (b_1,\ldots,b_j)$, then $V(a_1,\ldots,a_j)^*V(b_1,\ldots,b_k) = 0$.

(3) If $k \geq 2$, then $V(a_1,\ldots,a_k) = V(a_1,\ldots,a_{k-1})V(0_{k-1},a_k)$.

(4) If $j < k$, then $V(a_1,\ldots,a_j)^*V(a_1,\ldots,a_j)V(0_{k-1},a_k) = V(0_{k-1},a_k)$.

(5) $V(0_n) \geq 0$ and $\|V(a_1,\ldots,a_n)\| = 1$.

(6) $V(a_1,\ldots,a_n)^*V(a_1,\ldots,a_n)B(n) = B(n)$.

(7) $\|B(n)\| = 1$ and $B(n) \geq 0$.

The difference between the axioms of def.2 and those in lemma 4 of [2] is so small that lemma 5 of [2] remains valid for an approximate sequence of approximate matrix algebras. This latter lemma therefore tells us about the matrix structure for such a sequence. The next three lemmas establish some properties of approximate sequences of approximate matrix algebras which we shall need later.

<u>Lemma 1</u>. Let

$$\{V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*, B(n) : a_i, b_i \in \{0,1\} \text{ and } n = 1,2,\ldots\}$$

be an approximate sequence of approximate matrix algebras, and let $E(n)$ be defined as in def.2. Then the following are true:

(1) $\|E(n)\| = 1$ and $E(n) \geq 0$ for $n = 1,2,\ldots$.

(2) $V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*E(n+1) = \sum_{b=0,1} V(a_1,\ldots,a_n,b)V(b_1,\ldots,b_n,b)^*$.

(3) $E(n)E(m) = E(m)E(n) = E(m)$ when $n < m$.

(4) $V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*$ and $E(p)$ commute if $n < p$.

(5) $V(i)V(j)^*V(p)V(k)^*E(n+1) = \delta_{j,p}V(i)V(k)^*E(n+1)$ for all $i,j,k,p \in \{0,1\}^n$. ($\delta_{j,j} = 1$ and $\delta_{j,p} = 0$ if $j \neq p$)

(6) $V(a_1,\ldots,a_{n-1})V(b_1,\ldots,b_{n-1})^*E(n+1) = [V(a_1,\ldots,a_{n-1},0)V(b_1,\ldots,b_{n-1},0)^* + V(a_1,\ldots,a_{n-1},1)V(b_1,\ldots,b_{n-1},1)^*]E(n+1)$.

Proof:

(1) Since $V(a_1,\ldots,a_n)V(a_1,\ldots,a_n)^* \geq 0$ for all $(a_1,\ldots,a_n) \in \{0,1\}^n$, we have $E(n) \geq 0$.

$$V(b_1,\ldots,b_n)^*[V(b_1,\ldots,b_n)V(a_1,\ldots,a_n)^*]V(a_1,\ldots,a_n)B(n) = B(n)$$

is a consequence of axiom (6) in def.2. Since all the $V(a_a,\ldots,a_n)$ and $B(n)$ have norm one, we get from the Cauchy-Schwarz inequality that $\|V(b_1,\ldots,b_n)V(a_1,\ldots,a_n)^*\| = 1$ for $(a_1,\ldots,a_n),(b_1,\ldots,b_n) \in \{o,1\}^n$. This together with the fact that $V(a_1,\ldots,a_n)^*V(b_1,\ldots,b_n) = 0$ if $(a_1,\ldots,a_n) \neq (b_1,\ldots,b_n)$, implies that $\|E(n)\| = 1$.

(2) In the following we use without comment axioms 2, 3 and 4 of definition 2.

$$V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*V(c_1,\ldots,c_{n+1})V(c_1,\ldots,c_{n+1})^* = 0 \quad \text{if}$$
$$(b_1,\ldots,b_n) \neq (c_1,\ldots,c_n)$$

and

$$\begin{aligned}
&V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*V(b_1,\ldots,b_n,c_{n+1})V(b_1,\ldots,b_n,c_{n+1})^* \\
&= V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*V(b_1,\ldots,b_n)V(0_n,c_{n+1})V(b_1,\ldots,b_n,c_{n+1})^* \\
&= V(a_1,\ldots,a_n)V(0_n,c_{n+1})V(b_1,\ldots,b_n,c_{n+1})^* \\
&= V(a_1,\ldots,a_n,c_{n+1})V(b_1,\ldots,b_n,c_{n+1})^* .
\end{aligned}$$

From these equalities we can easily prove (2).

(3) From (2) we get $E(n)E(n+1) = E(n+1)$. Since $E(n)$ is self-adjoint for each n, it follows that $E(n+1)E(n) = E(n+1)$. We suppose $k > n$ and get

$$\begin{aligned}
E(n)E(k) &= E(n)E(n+1) \cdots E(k-1)E(k) = E(k) \\
&= E(k)E(k-1) \cdots E(n+1)E(n) = E(k)E(n) .
\end{aligned}$$

(4) We prove the assertion by induction with respect to the difference $p-n$. We suppose first that $p-n = 1$. From (2) and

$E(n) = E(n)^*$ it follows that

$$E(n+1)V(a_1,\ldots,a_n)V(c_1,\ldots,c_n)^* = [V(c_1,\ldots,c_n)V(a_1,\ldots,a_n)^*E(n+1)]^*$$
$$= [\sum_{b=0,1} V(c_1,\ldots,c_n,b)V(a_1,\ldots,a_n,b)^*]^*$$
$$= \sum_{b=0,1} V(a_1,\ldots,a_n,b)V(c_1,\ldots,c_n,b)^* = V(a_1,\ldots,a_n)V(c_1,\ldots,c_n)^*E(n+1) .$$

We suppose that the assertion is true for $p-n = s \geq 1$ and that $p-n = s+1$. From (2) and (3) we get

$$V(a_1,\ldots,a_n)V(c_1,\ldots,c_n)^*E(p) = V(a_1,\ldots,a_n)V(c_1,\ldots,c_n)^*E(n+1)E(p)$$
$$= \sum_{b=0,1} V(a_1,\ldots,a_n,b)V(c_1,\ldots,c_n,b)^*E(p)$$
$$= E(p) \sum_{b=0,1} V(a_1,\ldots,a_n,b)V(c_1,\ldots,c_n,b)^*$$
$$= E(p)V(a_1,\ldots,a_n)V(c_1,\ldots,c_n)^*E(n+1)$$
$$= E(p)E(n+1)V(a_1,\ldots,a_n)V(c_1,\ldots,c_n)^*$$
$$= E(p)V(a_1,\ldots,a_n)V(c_1,\ldots,c_n)^* .$$

(5) and (6) are proved in the same way as is lemma 5 in [2].

By a simple induction argument the next lemma follows from lemma 1.

<u>Lemma 2</u>. Let

$$\{V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*, B(n) : a_i, b_i \in \{0,1\} \text{ and } n = 1,2,\ldots\}$$

be an approximate sequence of approximate matrix algebras in a C^*-algebra $\mathfrak{A}$. For each n we let $\mathfrak{B}_n$ be the $*$-algebra generated by all $V(a_1,\ldots,a_m)V(c_1,\ldots,c_m)^*$ such that $0 < m \leq n$ and $(a_1,\ldots,a_m),(c_1,\ldots,c_m) \in \{0,1\}^m$.

Then for each $x \in \mathfrak{B}_n$ there exist complex numbers $a_{(a_1,\ldots,a_n),(c_1,\ldots,c_n)}$ such that

$$xE(n+1) = \sum_{\substack{(a_1,\dots,a_n)\\(c_1,\dots,c_n)}} a_{(a_1,\dots,a_n),(c_1,\dots,c_n)} V(a_1,\dots,a_n)V(c_1,\dots,c_n)^*E(n+1) .$$

We illustrate the proof by an example. We let $x = V(1,1)V(0,0)^*V(0,0,1)V(1,1,1)^*$, and it follows that

$$xE(4) = V(1,1)V(0,0)^*E(3)E(4)V(0,0,1)V(1,1,1)^*$$
$$= [V(1,1,0)V(0,0,0)^* + V(1,1,1)V(0,0,1)^*]V(0,0,1)V(1,1,1)^*E(4)$$
$$= V(1,1,1)V(1,1,1)^*E(4) .$$

Lemma 3.

$\{V(a_1,\dots,a_n)V(b_1,\dots,b_n)^*, B(n) : a_i, b_i \in \{0,1\}$ and $n = 1,2,\dots\}$ and $\mathcal{B}_n$ are defined in lemma 2.

Then for each $y \in \mathcal{U}$ we have

$$z = E(n+1)[\sum_{(a_1,\dots,a_n)} V(a_1,\dots,a_n)V(0_n)^*y\,V(0_n)V(a_1,\dots,a_n)^*]E(n+1) \in \mathcal{B}_n^c$$

where $\mathcal{B}_n^c$ is the commutant to $\mathcal{B}_n$ in $\mathcal{U}$.

Proof: In this proof we use without comment the axioms of definition 2 and the results in lemma 1. We have for $j,k \in \{0,1\}^n$

$$V(j)V(k)^*z = E(n+1)V(j)V(k)^*V(k)V(0_n)^*y\,V(0_n)V(k)^*E(n+1)$$
$$= V(j)V(k)^*V(k)V(0_n)^*E(n+1)y\,V(0_n)V(k)^*E(n+1)$$
$$= V(j)V(0_n)^*E(n+1)y\,V(0_n)V(k)^*E(n+1)$$
$$= E(n+1)V(j)V(0_n)^*y\,V(0_n)V(j)^*V(j)V(k)^*E(n+1)$$
$$= E(n+1)V(j)V(0_n)^*y\,V(0_n)V(j)^*E(n+1)V(j)V(k)^*$$
$$= z\,V(j)V(k)^* .$$

We let $x \in \mathcal{B}_n$. By lemma 2 there exist complex numbers $a_{i,j}$, $i,j \in \{0,1\}^n$, such that

$$xE(n+1) = \sum_{i,j\in\{0,1\}^n} a_{i,j} V(i)V(j)^* E(n+1) .$$

This implies that $xz = \sum_{i,j} a_{i,j} V(i)V(j)^* z$

and $zx = \sum_{i,j} a_{i,j} z V(i)V(j)^*$. It follows now that $xz = zx$, and we have $z \in \mathcal{B}_n^c$.

2. Two variations of Glimm's lemma.

We need two small variations on the fundamental lemma 4 of Glimm in [2].

<u>Lemma 4</u>. Let $\mathfrak{U}$ be a simple, separable NGCR - algebra with identity, and let f be a pure state. Then $\mathfrak{U}$ contains an approximate sequence of approximate matrix algebras such that $f(B(n)) = 1$ for all n.

<u>Proof</u>: We let $S_0, S_1, \ldots$ be a dense subset of the self-adjoint elements in $\mathfrak{U}$. We change the proof of lemma 4 in [2] such that we in addition get $f(B(n)) = 1$ for all n. The induction step in the proof need be changed in only two places.

First, in the seventh line from the top of page 577 in [2], we let $\mu = f$. This is possible since $f(B(n)) = 1$.

The other change is in lines 11 - 13 of page 578. There we let $\varphi = \varphi_f$ and $y = x_f$. This is possible since $\varphi_f(B_\sigma)$ is non-compact, because $\mathfrak{U}$ is simple, and since $\varphi_f(B_\sigma)x_f = x_f$ (line 10, page 578).

From the 13th line from the bottom of page 579 in Glimm's proof it follows that $\varphi_f(B(n+1))x_f = x_f$. This implies that $f(B(n+1)) = 1$.

We have now found elements $V(a_1,\dots,a_n)$ and $B(n)$ such that the axioms 2) - 7) in definition 2 are satisfied and elements $T_n \in \mathfrak{M}(n)$ ($\mathfrak{M}(n)$ is the linear span of elements of the form $V(a_1,\dots,a_n)V(b_1,\dots,b_n)^*$) such that $\|E(n+1)(S_n-T_n)E(n+1)\| < \frac{1}{n}$.

We let $\epsilon > 0$ and $S \in \mathfrak{U}$ be arbitrary. There exist self-adjoint elements S' and S'' such that $S = S' + i S''$. We choose k_1 and k_2 such that $\|S' - S_{k_1}\| < \frac{\epsilon}{4}$, $\|S'' - S_{k_2}\| < \frac{\epsilon}{4}$, $\frac{1}{k_1} < \frac{\epsilon}{4}$ and $\frac{1}{k_2} < \frac{\epsilon}{4}$. Since $\|E(n)\| = 1$ and $E(n)E(m) = E(m)$ if $n < m$, it follows by an $\frac{\epsilon}{4}$- argument that

$$\|E(p+1)[S - (T_{k_1} + i T_{k_2})]E(p+1)\| < \epsilon ,$$

where $p = \max(k_1,k_2)$. By lemma 2 there is a $T \in \mathfrak{M}(p)$ such that $(T_{k_1} + i T_{k_2})E(p+1) = TE(p+1)$. This implies that $\|E(p+1)(S - T) E(p+1)\| < \epsilon$, and we are done.

<u>Lemma 5</u>. Let $\mathfrak{U}$ be a simple NGCR - algebra with identity. Let f_1 and f_2 be two pure states such that f_1 and f_2 are not unitary equivalent. Let

$$\{V(a,\dots,a_n)V(b_1,\dots,b_n)^*, B(n) : a_i, b_i \in \{0,1\}\}$$

be an approximate matrix algebra such that $f_1(B(n)) = 1$. Then there exists an approximate matrix algebra

$$\{V(a_1,\dots,a_{n+1})V(b_1,\dots,b_{n+1})^*, B(n+1) : a_i, b_i \in \{0,1\}\}$$

such that $f_1(B(n+1)) = 1$ and $f_2(E(n+1)) = 0$, where

$$E(n+1) = \sum_{(a_1,\dots,a_{n+1})} V(a_1,\dots,a_{n+1})V(a_1,\dots,a_{n+1})^* ,$$

and such that

$$(1)\quad V(a_1,\dots,a_{n+1}) = V(a_1,\dots,a_n)V(0_n,a_{n+1})$$

and

$$(2)\quad V(a_1,\dots,a_n)^*V(a_1,\dots,a_n)V(0_n,a_{n+1}) = V(0_n,a_{n+1}) \text{ .}$$

Proof: The proof is analogous to the proof of the induction step in lemma 4 of [2]. We make some small changes.

We let φ_i and x_i respectively be the induced representation and induced vector of f_i . We let H_i be the Hilbert space on which φ_i acts. The elements $D_o, D_1, B_\sigma, B_{2\sigma}$ and V , which we mention in the following proof, are defined on page 578 in Glimm's proof, and the function f_σ is defined on page 577.

First, in the seventh line from the top of page 577 we let $\mu = f_1$. This is possible since $f_1(B(n)) = 1$.

In lines 10 - 18 on page 578 we make the following changes. let $\varphi = \varphi_1$ (line 11). This is possible since $\varphi_1(B_\sigma)x_1 = x_1$ and $\mathfrak{U}$ is simple, hence $\varphi_1(B_\sigma)$ is non-compact. We let $y = x_1$. This is possible since $\varphi_1(B_\sigma)x_1 = x_1$, which implies that $x_1 \in \text{Range } \varphi_1(B_\sigma)$.

We define N by

$$(2.1)\quad N = [\varphi_2(V(i)^*)x_2 : i \in \{0,1\}^n] \text{ ,}$$

which is a finite dimensional subspace of H_2 . We require in addition of C_o and U in the lines 14 and 17 that

$$(2.2.)\quad \varphi_2(C_o)(B_{2\sigma}N) = \{0\}$$

and that

$$(2.3.)\quad \varphi_2(U^*)(f_\sigma(D_1)N) \subset N \text{ .}$$

This is possible by an application of theorem 2.8.3 in [1], since $\dim[f_\sigma(D_1)N] \leq \dim N < \infty$, and since f_1 and f_2 are not unitarily equivalent.

By making these changes in the induction step of Glimm's proof we find an approximate matrix algebra

$$\{V(a_1,\ldots,a_{n+1})V(b_1,\ldots,b_{n+1})^*, B(n+1) : a_i, b_i \in \{0,1\}\}$$

such that (1) and (2) are satisfied. It remains to prove that our changes imply that $f_1(B(n+1)) = 1$ and $f_2(E(n+1)) = 0$.

By (2.2.) we have $\varphi_2(D_o)(N) = \{0\}$, and hence $\varphi_2(V)(N) = \{0\}$. Since $V^* = f_\sigma(D_o)U^*f_\sigma(D_1)$, by (2.3) we have $\varphi_2(V^*)(N) = \{0\}$. From the definition of $V(0_n,1)$ and $V(0_{n+1})$ we get $\varphi_2(V(0_n,1)^*)(N) = \{0\}$ and $\varphi_2(V(0_{n+1}))(N) = \{0\}$.
(2.2.) implies now that

$$\varphi_2(V(0_n,1)^*V(a_1,\ldots,a_n)^*x_2 = 0 \quad \text{and}$$

$$\varphi_2(V(0_{n+1})^*V(a_1,\ldots,a_n)^*x_2 = 0 \quad \text{for all} \quad (a_1,\ldots,a_n) \in \{0,1\}^n .$$

This implies that $\varphi_2(E(n+1))x_2 = 0$, and hence $f_2(E(n+1)) = 0$.

From line 13 from the bottom of page 579 we get $\varphi(B(n+1))y = y$. Since we have chosen $\varphi = \varphi_1$ and $y = x_1$, we then get $\varphi_1(B(n+1))x_1 = x_1$ and hence $f_1(B(n+1)) = 1$.

We suppose we have two approximate matrix algebras which satisfy (1) and (2) in lemma 5. Then, in the same way as in the proof of lemma 5 of [2], we can show the following: $\mathfrak{M}(n)$ is the set of all finite linear combinations of elements of the form $V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*$. For each representation φ of $\mathfrak{U}$,

$$\varphi(\mathfrak{M}(n))\big|_{[\text{range } \varphi(E(n+1))H\varphi]}$$

is a $2^n \times 2^n$ matrix algebra with matrix units

$$\varphi(V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*\big|_{[\text{range } \varphi(E(n+1))H\varphi]} .$$

This justifies definition 1 of an approximate matrix algebra.

3. Main results.

We prove in theorem 1 that pure states of a simple separable C*-algebra with identity hava a product decomposition property. Moreover, we prove in theorem 2 that two pure states of a simple C*-algebra with identity are unitarily equivalent if and only if they are asymptotically equal. The following result is well known, and is stated without proof.

Lemma 6. Let $\mathfrak{U}$ be a simple C*-algebra with identity. Then either $\mathfrak{U}$ is an NGCR-algebra or else $\mathfrak{U}$ is *-isomorphic with an $n \times n$ matrix algebra, where n is finite.

Theorem 1. Let $\mathfrak{U}$ be a simple separable C*- algebra with identity. We suppose that $\mathfrak{U}$ is not *-isomorphic with any $n \times n$ matrix algebra such that n is finite. Let f be a pure state of $\mathfrak{U}$.

Then $\mathfrak{U}$ contains an approximate sequence of approximate matrix algebras

$$\{V(a_1,\dots,a_n)V(b_1,\dots,b_n)^*, B(n) : a_i, b_i \in \{0,1\}^n \text{ and } n = 1,2,\dots\}$$

such that the following are satisfied:

We let $\mathfrak{A}$ be the C*-algebra generated by $\{V(a_1,\dots,a_n)V(b_1,\dots,b_n)^* : a_i, b_i \in \{0,1\} \text{ and } n = 1,2,\dots\}$, and we let $\mathfrak{M}(n)$ be the set of all linear combinations of $V(a_1,\dots,a_n)V(b_1,\dots,b_n)^*$. Then for each $\epsilon > 0$ and each $x \in \mathfrak{A}$, there is an n such that

$$|f(xy) - f(x)f(y)| < \epsilon\|y\| \quad \text{for } y \in \mathfrak{M}(n)^c .$$

($\mathfrak{M}(n)^c$ is the commutant of $\mathfrak{M}(n)$ in $\mathfrak{U}$.)

Proof: In this proof we use the axioms of definition 2 and lemma 1 without comment.

By lemma 6 $\mathfrak{U}$ is an NGCR algebra. We use lemma 4 and choose an approximate sequence of approximate matrix algebras such that $f(B(n)) = 1$ for all n.

$$E(n)B(n) = E(n)V(0_n)V(0_n)B(n)$$
$$= \sum_{(a_1,\dots,a_n)} V(a_1,\dots,a_n)V(a_1,\dots,a_n)^*V(0_n)V(0_n)B(n)$$
$$= V(0_n)V(0_n)V(0_n)V(0_n)B(n) = B(n) .$$

Since $f(B(n)) = 1$ and $\|B(n)\| = 1$, we have

$$(\varphi_f(B(n))x_f, x_f) = 1 = \|\varphi_f(B(n))x_f\| \cdot \|x_f\| .$$

Thus $\varphi_f(B(n))x_f$ is proportional to x_f, and so is equal to x_f. Since $E(n)B(n) = B(n)$, we have

(3.1) $\varphi_f(E(n))x_f = x_f$ and $f(E(n)) = 1$ for $n = 1,2,3 \dots$.

We have now to prove the following assertion:

$f|_{\mathfrak{A}}$ is a pure state.

We prove first that $f|_{\mathfrak{A}}$ has a unique extension to $\mathfrak{U}$. Suppose then that g is a pure state such that $f|_{\mathfrak{A}} = g|_{\mathfrak{A}}$. In the same way as we prove $\varphi_f(B(n))x_f = x_f$, we prove that $\varphi_g(E(n))x_g = x_g$ for $n = 1,2,\dots$. From this and (3.1) we get

(3.2) $f(\cdot) = f(E(n)\cdot E(n))$ and $g(\cdot) = g(E(n)\cdot E(n))$ for $n = 1,2,\dots$

We let $S \in \mathfrak{U}$ and $\epsilon > 0$ be arbitrary and choose n and $T \in \mathfrak{A}$ such that $\|E(n)(T-S)E(n)\| < \epsilon$. By (3.2) it follows that

$$|f(S) - g(S)| = |f(T) - g(T) + f(S-T) - g(S-T)|$$
$$= |(f-g)(E(n)(S-T)E(n))| < 2\epsilon .$$

Since $\varepsilon > 0$ was arbitrary, we have $f(S) = g(S)$.
Next we prove that $f|\mathcal{Q}$ is pure. We suppose $f|\mathcal{Q} = \frac{1}{2}(h+g)$, where h and g are states of $\mathcal{Q}$. We extend h and g to $\mathcal{U}$ and call the extensions h' and g'. Since we have just proved that $f|\mathcal{Q}$ has a unique extension to $\mathcal{U}$, it follows that $f = \frac{1}{2}(h'+g')$. f is pure, hence $f = h' = g'$, and we have proved the assertion.

We let $\mathcal{B}_n$ be the *-algebra generated by $\{V(a_1,\dots,a_k)V(b_1,\dots,b_k)^* : a_i, b_i \in \{0,1\},\ k \leq n\}$. Since $\mathcal{Q} = \overline{\bigcup_{n=1}^{\infty} \mathcal{B}_n}^{\text{norm}}$, it is sufficient to prove the theorem for each $x \in \bigcup_{n=1}^{\infty} \mathcal{B}_n$.

We let $x \in \mathcal{B}_n$ and $\varepsilon > 0$ be given. We choose $\delta > 0$ such that

$$\|x\| \cdot \delta + \delta\,|f(x)| + \delta(1+\delta) < \varepsilon .$$

$\{\mathcal{B}_n\}_{n=1}^{\infty}$ is an ascending sequence of *-algebras such that $\mathcal{Q} = \overline{\bigcup_{n=1}^{\infty} \mathcal{B}_n}^{\text{norm}}$, and $f|\mathcal{Q}$ is a pure state, in particular a factor state.
We copy the proof of theorem 2.5 i) → ii) in [4] and find $m > n$ such that

$$(3.3) \qquad |f(xy) - f(x)f(y)| \leq \delta\|y\| \quad \text{for all} \quad y \in \mathcal{B}_m^c \cap \mathcal{Q} .$$

We let $y \in \mathcal{M}(m)^c$, and we suppose without loss of generality that $\|y\| = 1$. We need now the following assertion:

> For each $\delta > 0$ and each $S \in \mathcal{U}$ there exist k and $T \in \mathcal{B}_k$ such that $\|T\| \leq \|S\| + \delta$ and $\|E(k+1)(S-T)E(k+1)\| < \delta$.

We choose p and T' such that $\|E(p+1)(S-T')E(p+1)\| < \delta$. We define $k = p+1$ and $T = E(p+1)T'E(p+1)$.

$$\|T\| = \|E(p+1)T'E(p+1)\|$$
$$\leq \|E(p+1)(S-T')E(p+1)\| + \|E(p+1)S\,E(p+1)\| < \delta + \|S\| ,$$

since $\|E(p+1)\| = 1$. We get

$$\|E(p+2)(S-T)E(p+2)\|$$
$$= \|E(p+2)(E(p+1)S\,E(p+1) - E(p+1)T'E(p+1))E(p+2)\|$$
$$\leq \|E(p+2)\|\cdot\|E(p+1)(S-T')E(p+1)\|\cdot\|E(p+2)\| < \delta$$

and we have proved the assertion.

By the assertion we can find $k > \max(m,n)$ and $z \in \mathcal{B}_k$ such that

$$(3.4) \quad \|z\| < 1+\delta \quad \text{and} \quad \|E(k+1)(z-y)E(k+1)\| < \frac{\delta}{2^m} .$$

Since $\|V(a_1,\dots,a_m)V(0_m)^*\| = 1$, we have by (3.4)

$$(3.5) \quad \|V(a_1,\dots,a_m)V(0_m)^*E(k+1)(z-y)E(k+1)V(0_m)V(a_1,\dots,a_m)^*\| < \frac{\delta}{2^m}$$

for all $(a_1,\dots,a_m) \in \{0,1\}^m$.

$$\sum_{(a_1,\dots,a_m)} V(a_1,\dots,a_m)V(0_m)^*E(k+1)(y-z)E(k+1)V(0_m)V(a_1,\dots,a_m)^*$$
$$= \sum_{(a_1,\dots,a_m)} E(k+1)y\,V(a_1,\dots,a_m)V(0_m)^*V(0_m)V(a_1,\dots,a_m)^*E(k+1)$$
$$- E(k+1)(E(m+1) \sum_{(a_1,\dots,a_m)} V(a_1,\dots,a_m)V(0_m)^*zV(0_m)V(a_1,\dots,a_m)^*E(m+1))E(k+1)$$
$$= E(k+1)y\,E(m)E(k+1) - E(k+1)z'E(k+1)$$
$$= E(k+1)(y-z')E(k+1) ,$$

where z' is defined by

$$(3.6) \quad z' = E(m+1) \sum_{(a_1,\dots,a_m)} V(a_1,\dots,a_m)V(0_m)^*z\,V(0_m)V(a_1,\dots,a_m)^*E(m+1) .$$

We add the inequalities in (3.5) and get

$$(3.7) \quad \|E(k+1)(y-z')E(k+1)\| < \delta .$$

From (3.4) it follows that

$$\|V(a_1,\dots,a_m)V(0_m)^* z\, V(0_m)V(a_1,\dots,a_m)^*\| < 1+\delta .$$

Since

$$V(a_1,\dots,a_m)^* V(b_1,\dots,b_m) = 0 \quad \text{if} \quad (a_1,\dots,a_m) \neq (b_1,\dots,b_m) ,$$

we get

$$\| \sum_{(a_1,\dots,a_m)} V(a_1,\dots,a_m)V(0_m)^* z\, V(0_m)V(a_1,\dots,a_m)^*\| < 1+\delta .$$

By (3.6) this gives

$$(3.8) \qquad \|z'\| < 1+\delta .$$

By lemma 3 we have $z' \in \mathcal{B}_m^c \cap \mathcal{Q}$. This implies by (3.3) and (3.8) that

$$(3.9) \qquad |f(xz') - f(x)f(z')| < \delta\|z'\| \leq \delta(1+\delta) .$$

Since $x \in \mathcal{B}_n$, $z' \in \mathcal{B}_{m+1}$ and $k+1 > \max(k,n)$, we have by (3.2) and (3.7) that

$$(3.10) \qquad |f(xz') - f(xy)| \leq \|x\|\delta ,$$

because

$$\begin{aligned} &|f(xz') - f(xy)| \\ &= |f(xE(k+1)z'E(k+1)) - f(xE(k+1)yE(k+1))| \\ &\leq \|xE(k+1)z'E(k+1) - xE(k+1)yE(k+1)\| \leq \|x\|\cdot\delta . \end{aligned}$$

Moreover, we have by (3.2) and (3.7) that

$$(3.11) \qquad |f(z') - f(y)| = |f(E(k+1)(z'-y)E(k+1))| \leq \delta .$$

(3.9), (3.10) and (3.11) imply

$$|f(xy) - f(x)f(y)| \leq \|x\|\cdot\delta + \delta\cdot|f(x)| + \delta(1+\delta) < \varepsilon ,$$

and we are done.

<u>Theorem 2</u>. Let $\mathfrak{U}$ be a simple C*-algebra with identity. We suppose that $\mathfrak{U}$ is not *-isomorphic with any $n \times n$ matrix algebra such that n is finite. Let f_1 and f_2 be two pure states of $\mathfrak{U}$. Then the following are equivalent:

(1) f_1 and f_2 are unitarily equivalent.

(2) There is an approximate matrix algebra
$\{V(a_1)V(b_1)^*, B(1) : a_1, b_1 \in \{0,1\}\}$
such that
$f_1(B(1)) = 1$ and $\|(f_1-f_2)|_{\mathfrak{M}(1)^c}\| = 0$.

(3) There is an approximate matrix algebra
$\{V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*, B(n) : a_i, b_i \in \{0,1\}\}$
such that
$f_1(B(n)) = 1$ and $\|(f_1-f_2)|_{\mathfrak{M}(n)^c}\| < 1$.

$\mathfrak{M}(n)$ is the linear span of the elements $V(a_1,\ldots,a_n)V(b_1,\ldots,b_n)^*$, and $\mathfrak{M}(n)^c$ is the commutant of $\mathfrak{M}(n)$ in $\mathfrak{U}$.

<u>Proof</u>: By lemma 6, $\mathfrak{U}$ is a simple NGCR-algebra with identity. 1) → 2): We suppose $f_1 \sim f_2$. We define $\pi = \pi_{f_1}$. If π is a one-dimensional representation, the theorem is trivially satisfied. We suppose that π is at least two-dimensional, that $f_1(\cdot) = (\pi(\cdot)x_1, x_1)$, that $f_2(\cdot) = (\pi(\cdot)x_2, x_2)$, and that $x_2 = \lambda x_1 + \mu z$ where $x_1 \perp z$, $\|z\| = 1$ and $\lambda, \mu \in \mathbb{C}$. By theorem 2.8.3 in [1] there exist elements D and U of $\mathfrak{U}$ such that $D \geq 0$, $\|D\| = 1$, $\pi(D)x_1 = x_1$, $\pi(D)z = 0$, U is unitary, and $\pi(U)x_1 = z$.

For each $\varepsilon > 0$ in $(0,1)$ we let f_ε be the function defined by: $f_\varepsilon((-\infty, 1-\varepsilon]) = 0$, $f_\varepsilon([1-\frac{\varepsilon}{2}, \infty)) = 1$, and f_ε is linear on $[1-\varepsilon, 1-\frac{\varepsilon}{2}]$. We define

$$V = f_{\frac{1}{2}}(I-D)U f_{\frac{1}{2}}(D) .$$

We prove now that $f_{\frac{1}{2}}(I-D)f_{\frac{1}{2}}(D) = 0$. We define g by $g(t) = f_{\frac{1}{2}}(1-t)f_{\frac{1}{2}}(t)$. Since $f_{\frac{1}{2}} = 0$ on $[0,\frac{1}{2}]$ and $sp(D) \subset [0,1]$, it follows that $g = 0$ on $sp(D)$. This implies $g(D) = 0$. Since $f_{\frac{1}{2}}(I-D)f_{\frac{1}{2}}(D) = 0$, it follows that $V^2 = 0$.

We have

$$(13.12) \quad \pi(V)x_1 = z \text{ and } \pi(V^*)z = x_1 .$$

We define

$$V(1) = Vk(V^*V) \text{, where } k(t) = (f_{\frac{1}{2}}(t)t^{-1})^{\frac{1}{2}} , \; k(0) = 0 ,$$

$$V(0) = f_{\frac{1}{2}}(V^*V) \text{, and}$$

$$B(1) = f_{1/4}(V^*V) .$$

Next we want to prove that

$$\{V(i)V(j)^*, B(1) : i,j \in \{0,1\}\}$$

is an approximate matrix algebra. $V(1)^*V(0) = 0$, since $(V^*)^2 = 0$. Moreover, $V(0)^*V(1) = 0$, since $V^2 = 0$. This means that axiom (1) in definition 1 is satisfied. Axioms (2) and (3) are trivially satisfied. Since

$$V(1)^*V(1) = k(V^*V)V^*Vk(V^*V) = f_{\frac{1}{2}}(V^*V) ,$$

it follows that $V(1)^*V(1)B(1) = B(1)$, because $f_{1/2}f_{1/4} = f_{1/4}$. Since $f_{1/2}f_{1/4} = f_{1/4}$, it follows that $V(0)^*V(0)B(1) = B(1)$, and axiom 4 is satisfied. Thus we have proved that

$$\{V(i)V(j)^*, B(1) : i,j \in \{0,1\}\}$$

is an approximate matrix algebra.

We define G by

$$G = \lambda V(0)V(0)^* + \mu V(1)V(0)^* .$$

From (13.12) we get

$$\pi(V(0)V(0)^*)x_1 = \pi([f_{\frac{1}{2}}(V^*V)]^2)x_1 = x_1$$

$$\pi(V(1)V(0)^*)x_1 = \pi(Vk(V^*V)f_{\frac{1}{2}}(V^*V))x_1 = z$$

$$\pi(V(0)V(1)^*)z = \pi(f_{\frac{1}{2}}(V^*V)k(V^*V)V^*)z = x_1 ,$$

and hence

$$\pi(G)x_1 = \lambda x_1 + \mu z = x_2 .$$

We get

$$\pi(V(0)V(0)^*)z = \pi(V(0)V(0)^*V(1)V(0)^*)x_1 = 0$$

and

$$\pi(V(0)V(1)^*)x_1 = \pi(V(0)V(1)^*V(0)V(1)^*)z = 0 .$$

This implies

$$\pi(G^*)(\lambda x_1+\mu z) = (\bar{\lambda}V(0)V(0)^* + \bar{\mu}V(0)V(1)^*)(\lambda x_1+\mu z)$$

$$= (|\lambda|^2+|\mu|^2)x_1 = 1\cdot x_1 = x_1 .$$

We get

$$\pi(G^*G)x_1 = x_1 .$$

We let $A \in \mathfrak{M}(1)^c$.

We get

$$f_2(A) = (\pi(A)x_2,x_2) = (\pi(A)\pi(G)x_1,\pi(G)x_1)$$

$$= f_1(G^*AG) = f_1(AG^*G) = f_1(A)$$

since G and A commute and $\pi(G^*G)x_1 = x_1$.

2) → 3) is trivial.

3) → 1): We suppose $f_1 \not\sim f_2$, and we let

$$\{V(a_1,\dots,a_n)V(b_1,\dots,b_n)^*, B(n) : a_i,b_i \in \{0,1\}\}$$

be an approximate matrix algebra such that $f_1(B(n)) = 1$. By lemma 5 we choose an approximate matrix algebra

$$\{V(a_1,\dots,a_{n+1})V(b_1,\dots,b_{n+1})^*, B(n+1) : a_i, b_i \in \{0,1\}\}$$

such that (1) and (2) in lemma 5 are satisfied and such that $f_1(B(n+1)) = 1$ and $f_2(E(n+1)) = 0$. $f_1(B(n+1)) = 1$ implies $f_1(E(n+1)) = 1$, since $B(n+1) \leq E(n+1)$. In the same way as in the proof of lemma 1, (1) and (4), we get $E(n+1) \in \mathfrak{M}(n)^c$ and $\|E(n+1)\| = 1$. Since we have $|(f_1-f_2)(E(n+1)| = 1$, it follows that

$$\|(f_1-f_2)|_{\mathfrak{M}(n)^c}\| \geq 1 .$$

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