

# ON THE PROBLEM OF ENUMERATING TWISTED CUBICS

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## 1. Introduction.

One of the problems in enumerative geometry treated by Schubert [Sch] is that of twisted cubic curves. Contrary to the case of plane conics, and more generally, arbitrary quadric varieties, this problem has not yet received a satisfactory "modern" treatment.

Some of Schubert's numbers not involving tangency conditions were verified by Cremona and Todd [T]. On the other hand, Alguneid [Al] gave a derivation of Schubert's 11 first order degenerations of complete twisted cubics using complete collineations. A similar study was done in [P], with the difference that the degenerations were taken to be flat specializations (thus distinguishing between different multiple structures and including possible embedded points) - thus working in Hilbert schemes rather than Chow schemes.

A main aspect of any enumerative problem is to find a compactification of the space parametrizing the objects in question, whose Chow ring can be computed. In principle this would be the solution of the enumerative problem (in addition to the verifi-

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cation that all solutions occur with multiplicity one). In the following we shall discuss some recent work which hopefully will lead towards a solution of the problem of enumerating twisted cubics.

## 2. The Hilbert scheme compactification.

Fix an algebraically closed field  $k$  of characteristic 0. A twisted cubic curve is a smooth, rational, proper curve of degree 3 in  $\mathbb{P}^3 = \mathbb{P}_k^3$ . The space,  $H_0$ , of twisted cubics can be identified with the homogeneous, affine, 12-dimensional space  $SL(4)/SL(2)$ : Let  $C \subset \mathbb{P}^3$  denote the standard twisted cubic, i.e., the image of the 3-fold Veronese embedding  $\mathbb{P}^1 \rightarrow \mathbb{P}^3$ . Then any other twisted cubic is obtained from  $C$  via an automorphism of  $\mathbb{P}^3$ , and two automorphisms give the same curve if they differ by an automorphism of  $\mathbb{P}^1$ .

The Hilbert polynomial of a twisted cubic is  $P(m) = 3m+1$ . Let  $H \subset \text{Hilb}^{3m+1}(\mathbb{P}^3)$  denote the closure of  $H_0$  in the Hilbert scheme. In [P-S] we proved the following result.

Theorem. (Piene-Schlessinger): The Hilbert scheme  $\text{Hilb}^{3m+1}(\mathbb{P}^3)$  has two irreducible components,  $H$  and  $H'$ , both smooth and rational, of dimensions 12 and 15. Moreover,  $H$  and  $H'$  intersect transversally, and  $H \cap H'$  is smooth, rational, and of dimension 11.

The proof proceeds in three steps (for details, see [P-S]):

Step 1.  $\text{Hilb}^{3m+1}(\mathbb{P}^3) = H \cup H'$ , where  $H'$  denotes the closure of the set of points in  $\text{Hilb}^{3m+1}(\mathbb{P}^3)$  corresponding to subschemes consisting of a plane, cubic curve union a point in  $\mathbb{P}^3$ . Clearly,  $H'$  has dimension 15. The intersection  $H \cap H'$  consists of points corresponding to plane, singular cubics which have an embedded point emerging from the plane at a singular point. In particular we show that no flat specialization, with an embedded point, of a twisted cubic can be schemetheoretically contained in a plane (or even a smooth surface). Any curve in  $H \cap H'$  specializes to one which is a line tripled in a plane, with an embedded point emerging from the plane.

Step 2. Computation of the tangent spaces.

In order to compute the tangent spaces to  $\text{Hilb}^{3m+1}(\mathbb{P}^3) = H \cup H'$ , we use the following theorem.

Comparison theorem: Let  $I = (f_1, \dots, f_r)$  be a homogeneous ideal in  $P = k[X_0, \dots, X_n]$ , with  $\deg f_i = d_i$ . Set  $X = \text{Proj}(P/I) \subset \mathbb{P}^n$ .

Assume

(1)  $(P/I)_{d_i} \rightarrow H^0(X, \mathcal{O}_X(d_i))$  is an isomorphism, for

$i = 1, \dots, r$ .

(2)  $(X_0, \dots, X_n) \notin \text{Ass}(I)$ .

Then, locally at  $I$  and  $X$ , the space of homogeneous deformations of  $I$  in  $P$  is the same as the Hilbert scheme parametrizing subschemes of  $\mathbb{P}^n$  with the same Hilbert polynomial as  $X$ .

This theorem is proved by comparing appropriate cotangent complexes and using [S, Prop. p. 153].

By considering the various types of curves  $C \in H \cup H'$ , or rather their homogeneous ideals  $I \subset P = k[X_0, \dots, X_3]$ , we show

$$\dim T_{H \cup H', C} = \begin{cases} 12 & \text{if } C \in H - H \cap H' \\ 16 & \text{if } C \in H \cap H' \\ 15 & \text{if } C \in H' - H \cap H' \end{cases}.$$

Hence  $H \cup H'$  is smooth outside  $H \cap H'$ .

In fact, this tangent space is equal to  $H^0(C, N_{C/P^3})$ , where  $N_{C/P^3}$  denotes the normal sheaf of  $C$  in  $P^3$ , which - by the Comparison theorem - is equal to  $\text{Hom}_P(I, P/I)_0$ , the graded homomorphisms of degree 0. The latter is computed via explicit resolutions of the various ideals  $I$ .

Step 3. Description of the deformation space.

Let  $I = (X_1 X_3, X_2 X_3, X_3^2, X_1^3) \subset P$  be the ideal of a curve  $C \in H \cap H'$  of the most degenerate kind. Via a basis for the 16-dimensional tangent space  $\text{Hom}_P(I, P/I)_0$  we construct "local equations" for  $H \cup H'$  in this space, and obtain an injective morphism

$$\mathbb{A}^{16}/(\text{equations}) = \mathbb{A}^{12} \cup \mathbb{A}^{15} \rightarrow H \cup H' = \text{Hilb}^{3m+1}(P^3),$$

which is a local (analytic) isomorphism, hence it is an embedding.

Hence  $H$  is a smooth compactification of the space  $H_0$  of twisted cubics, but its Chow ring has not yet been computed (see §3, however, for a report on work in progress).

The space  $H$  has a group,  $SL(4)$ , acting on it, with finitely many orbits. By a theorem of D. Luna [D-P, 7.2] the set of fixed points of a maximal torus of  $SL(4)$  is finite. Hence the

results of Bialynicki-Birula [B, §4] apply, to give a decomposition of  $H$  as a finite, disjoint union of affine spaces  $\mathbb{A}^{n_i}$ ,  $n_1 = 12 > n_2 > \dots > 0$  (all integers between 12 and 0 occur). By explicitly finding the fixed points of a given maximal torus, one could in principle determine the  $n_i$ 's and hence the Betti numbers of  $H$ . How one could proceed from there to find the Chow ring of  $H$  is not clear, however.

There is also a "stratification" of  $H$  given by the orbits of  $SL(4)$  (see [H]). In each  $\mathbb{A}^{12} \subset H$  constructed in the proof of the Theorem, equations for these strata can be given. One could also ask for a description of  $H - \mathbb{A}^{12}$ : it is easy to see that it contains the Schubert divisor consisting of curves meeting a certain line. What else it contains is less obvious - but there must be one more divisor because of the following result.

Proposition (Ellingsrud):  $\text{Pic } H = \mathbb{Z} \times \mathbb{Z}$ .

Since  $H$  contains exactly two degeneration divisors, namely  $H_\omega = \{\text{reducible curves}\}$  and  $H_\lambda = H \cap H'$ , this result is not surprising. It should be noted that  $H_\omega$  and  $H_\lambda$  do not form a generator set for  $\text{Pic } H$ , since Ellingsrud also proves  $\text{Pic } H_0 = \mathbb{Z}/2$ .

### 3. Other compactifications.

The space  $H$  is not the smallest smooth compactification of  $H_0$ . In a joint work with G. Ellingsrud [E-P] we show that there is a smooth compactification  $X$  of  $H_0$  and a morphism  $f: H \rightarrow X$

contracting the divisor  $H_\lambda = H \cap H'$  to a smooth 5-dimensional variety (equal to the point-plane incidence correspondence of  $\mathbb{P}^3$ ). The map  $f$  can be thought of as forgetting other than quadric generators of the homogeneous ideal of each curve. The variety  $X$  can also be realized as a quotient: there is a 23-dimensional variety  $P$  with an action of the group  $G = SL(2) \times SL(3)$ , such that  $X$  is the quotient by  $G$  of the stable points  $P_s \subset P$  (all semistable points are unstable for this action). The Chow ring of  $X$  is in principle computable, and hence so is the Chow ring of  $H$ .

Another direction, taken by I. Vainsencher (private communication), is to consider a compactification of a blow-up of  $H_0$ . Namely, fix a point in  $\mathbb{P}^3$  and let  $P$  denote the plane of lines through that point. Then consider  $F \rightarrow P$ , the Grassmann bundle whose fibers are  $\text{Grass}(1,6)$ , viewed as pencils of quadrics through the given line. Each such pencil determines a twisted cubic (or a degenerate one) as the residual base locus of the pencil, with the given line as a chord. The Chow ring of  $F$  is in principle computable. In this way Vainsencher was able to compute some of Schubert's numbers involving the condition for a cubic to pass through 5 given points.

One could of course hope that there would be no need to pass to a compactification of  $H_0$  in order to compute Schubert's numbers. De Concini and Procesi [D-P 2] have constructed a "universal Chow group" of a homogeneous space. So far, however, this group is known to be a ring only when the homogeneous space is a symmetric space.

#### 4. Complete twisted cubics.

In order to obtain proper intersections in the space parametrizing the objects of an enumerative problem one usually has to pass to so-called complete objects. For example, the space of smooth conics has  $\mathbb{P}^5$  as a compactification, but in order to get proper intersections of the hypersurfaces representing conics tangent to a given curve one passes to  $\tilde{\mathbb{P}}^5$ , the blow-up of  $\mathbb{P}^5$  along the Veronese surface corresponding to double lines. In order to obtain proper intersections when considering higher order tangency conditions one would have to blow up  $\tilde{\mathbb{P}}^5$  further.

Points of  $\tilde{\mathbb{P}}^5$  are called complete conics - they correspond to pairs consisting of a conic and its dual conic, and degenerations of such pairs. This notion of "completeness" thus refers to the kind of conditions considered (in this case simple tangencies) and is not an absolute one.

Similarly, for the enumerative problems of twisted cubics considered by Schubert, a complete twisted cubic would involve three aspects: the points of the curve, its tangent lines, and its osculating planes (see [Sch], [A1, 2], [P]). To consider other conditions one should include other aspects: chords, intersections of osculating planes, etc. ([Sch], [A1, 2]).

In the case of conics there is no difference whether one views degenerations as cycles or as flat specializations, but for twisted cubics these notions are different. The space  $\bar{T}$  of complete twisted cubics used (implicitly) by Alguneid (op. cit.) lives in a product of Chow schemes, whereas the space  $T$  of complete twisted cubics constructed in [P] lives in a product of Hilbert schemes (here degenerations are viewed as flat speci-

alizations). Since both  $\bar{T}$  and  $T$  are compactifications of the affine space  $H_0 = SL(4)/SL(2)$ , the complements  $\bar{T}-H_0$  and  $T-H_0$  are both purely of codimension one. But  $T-H_0$  contains more components (hence more "first order degenerations" than  $\bar{T}-H_0$  (see the lemma below), and this might explain Alguneid's problem, that "... certain degenerations of higher order do not appear to be specializations of those of the first order ..." [A1, p. 208]. In particular, if degenerations are viewed as flat specializations and not as cycles, there are more first order degenerations of complete twisted cubics than the 11 given by Schubert. However, given Alguneid's work, it does seem reasonable that  $\bar{T}$  contains only 11 first order degenerations, so that  $\text{Pic } \bar{T}$  has rank 11.

Lemma.  $\text{rk Pic } T > \text{rk Pic } \bar{T}$ .

Proof. We use the method of "projection" [P] to construct a first order degeneration of a complete twisted cubic in  $T$  which is not of first order in  $\bar{T}$ , i.e., such that the set of degenerations of this type has dimension 11 in  $T$ , but dimension less than 11 (in fact 9) in  $\bar{T}$ .

Let  $C \subset \mathbb{P}^3$  be the standard twisted cubic,  $A$  the tangent line  $X_2 = X_3 = 0$ ,  $B$  the tangent line  $X_0 = X_1 = 0$ . Consider the 1-parameter family of curves  $C_a$  obtained by "projecting"  $C$  from  $A$  to  $B$ ;  $C_a$  is given parametrically by  $X_0 = au^3$ ,  $X_1 = au^2v$ ,  $X_2 = uv^2$ ,  $X_3 = v^3$ . The corresponding family of ideals is given by

$$I_1 = (aX_0X_2 - X_1^2, X_1X_3 - aX_2^2, X_0X_3 - X_1X_2).$$



For  $a = 0$  we obtain

$$I_0 = (X_1^2, X_1X_3, X_0X_3 - X_1X_2),$$

hence the corresponding curve  $C_0$  is the union of a simple and a double line, contained in a smooth quadric. As in [P] one computes the ideal of the corresponding degeneration of the tangent curve,  $\Gamma_0$ , and finds that  $\Gamma_0$  is the union of a simple and a triple line; the dual curve  $C_0^*$  is of the same type as  $C_0$  (the triple  $(C_0, \Gamma_0, C_0^*)$  is selfdual). By counting parameters, one sees that the curves of type  $C_0$  form a set of dimension 9 in  $H$  (see e.g. [H]). The triple line of  $\Gamma_0$  is determined by  $C_0$  (and so is  $C_0^*$ ), but the simple line varies according to the way of passing to the limit - hence  $\Gamma_0$  gives 2 more parameters, and we obtain 11. (One could also argue "backwards": this type of degeneration is obviously not a flat specialization of any of Schubert's eleven.) The dimension of this set of triples viewed as elements of  $\bar{T}$  is  $7+2 = 9$ . As a triple of cycles, it is a degeneration of a triple of type  $\kappa$  (see [P]) viewed as a triple of cycles.

Note that the space  $T$ , which is a modification (blow-up) of  $H$ , has lost one property that  $H$  had: Under the action of  $SL(4)$  on  $T$  there are infinitely many orbits. This follows, because one of the first order degeneration types,  $\eta$  (see [P]), involves the cross-ratio of four points on a line, hence there will be infinitely many orbits of objects of this type.

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